CDI1-2224-25 - LEIC-T, LEE, LETI, LEGI
Primitivação - Soluções

1.

a) $x^2 + x^3 + x^4$, b) $2\sqrt{x} + \ln x - \frac{1}{x}$, $x > 0$,

c) $\int \left(\frac{x^2 - x + 1}{\sqrt{x}}\right)dx = \int \left(x^{\frac{3}{2}} - x^{\frac{1}{2}} + x^{-\frac{1}{2}}\right)dx = \frac{2}{5}x^{\frac{5}{2}} - \frac{2}{3}x^{\frac{3}{2}} + 2x^{\frac{1}{2}} = \frac{2}{5}\sqrt{x^5} - \frac{2}{3}\sqrt{x^3} + 2\sqrt{x}$.

d) $-\frac{3}{4}\sqrt[3]{(1-x)^4}$, e) $\int \left(\frac{\sqrt[3]{x^2} + \sqrt{x^3}}{x}\right)dx = \int \left(x^{-\frac{1}{3}} + x^{\frac{1}{2}}\right)dx = \frac{3}{2}\sqrt[3]{x^2} + \frac{2}{3}\sqrt{x^3}$,

f) $\frac{5}{6}\sqrt[5]{(x^2-1)^6}$, g) $\frac{1}{2}\ln(1 + 2e^x)$, h) $\ln(1 + \operatorname{sen} x)$, i) $-\frac{1}{2}\cos(2x)$,

j) $\int \left(\frac{\operatorname{sen}(2x)}{1 + \operatorname{sen}^2 x}\right)dx = \int \left(\frac{2\operatorname{sen} x \cos x}{1 + \operatorname{sen}^2 x}\right)dx = \ln(1 + \operatorname{sen}^2 x)$,

k) $\int \left(\cos^2 x\right)dx = int\left(\frac{\cos(2x) + 1}{2}\right)dx = \frac{1}{4}\operatorname{sen}(2x) + \frac{x}{2}$, l) $\operatorname{tg} x$, m) $e^{\operatorname{tg} x}$,

n) $\frac{1}{2}\operatorname{sen}(x^2 + 2)$, o) $-\cos(e^x)$, p) $\frac{1}{4}\sqrt[3]{(1 + x^3)^4}$, q) $-\frac{1}{1 + e^x}$,

r) $-\operatorname{arctg}(\cos x)$, s) $\int \left(\frac{1}{\sqrt{1 - 4x^2}}\right)dx = \int \left(\frac{1}{\sqrt{1 - (2x)^2}}\right)dx = \frac{1}{2}\operatorname{arcsen}(2x)dx$,

t) $\int \left(\frac{x + 1}{\sqrt{1 - x^2}}\right)dx = \int \left(\frac{x}{\sqrt{1 - x^2}}\right)dx + \int \left(\frac{1}{\sqrt{1 - x^2}}\right)dx = -\sqrt{1 - x^2} + \operatorname{arcsen} x$,

u) $\int \left(\frac{x^3}{(1 + x^4)^2}\right)dx = -\frac{1}{4(1 + x^4)}$, v) $\int \left(\cos^3 x \sqrt{\operatorname{sen} x}\right)dx =$

$= \int \left(\cos x(1 - \operatorname{sen}^2 x)\sqrt{\operatorname{sen} x}\right)dx = \int \left(\cos x(\sqrt{\operatorname{sen} x} - \operatorname{sen}^{\frac{5}{2}} x)\right)dx = \frac{2}{3}\operatorname{sen}^{\frac{3}{2}} x - \frac{2}{7}\operatorname{sen}^{\frac{7}{2}} x$,

w) $\int (\operatorname{tg}^2 x)dx = \int (\sec^2 x - 1)dx = \operatorname{tg} x - x$, x) $\frac{1}{7}x^7 + \frac{3}{5}x^5 + x^3 + x$; z) e^{x+3};

aa) $\frac{1}{\ln 2}2^{x-1}$; ab) $\int \left(\frac{1}{\sqrt[5]{1 - 2x}}\right)dx = -\frac{1}{2}\int \left(-2(1 - 2x)^{-\frac{1}{5}}\right)dx = -\frac{5}{8}(1 - 2x)^{\frac{4}{5}}$;

ac) $\int (\operatorname{cotg} x)dx = \int \left(\frac{\cos x}{\operatorname{sen} x}\right)dx = \ln|\operatorname{sen} x|$;

ad) $\int \left(3^{\operatorname{sen}^2 x}\operatorname{sen} 2x\right)dx = \int \left(3^{\operatorname{sen}^2 x} 2\operatorname{sen} x \cos x\right)dx = \int \left(3^{\operatorname{sen}^2 x}(\operatorname{sen}^2 x)'\right)dx = \frac{1}{\ln 3}3^{\operatorname{sen}^2 x}$;

ae) $\int \left(\frac{\operatorname{tg}\sqrt{x}}{\sqrt{x}}\right)dx = \int \left(\frac{1}{2\sqrt{x}}\operatorname{tg}\sqrt{x}\right)dx = 2\int \left(\left(\sqrt{x}\right)' \operatorname{tg}\sqrt{x}\right)dx = -2\ln|\cos\sqrt{x}|$;

af) $\operatorname{arcsen} e^x$; ag) $\int (\cos x \cos 2x)dx = \int \Big(\cos x(1 - 2\operatorname{sen}^2 x)\Big)dx = \int \Big(\cos x - 2\cos x \operatorname{sen}^2 x\Big)dx$

$= \operatorname{sen} x - \frac{2}{3}\operatorname{sen}^3 x$;

ah) $\int \Big(\operatorname{sen}^3 x \cos^4 x\Big)dx = \int \Big(\operatorname{sen} x(1-\cos^2 x)\cos^4 x\Big)dx = \int \Big(\operatorname{sen} x(\cos^4 x - \cos^6 x)\Big)dx =$

$= -\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x$ai) $\int (\operatorname{tg}^3 x + \operatorname{tg}^4 x)dx = \int ((\sec^2 x - 1)\operatorname{tg} x)dx + \int ((\sec^2 x - 1)\operatorname{tg}^2 x)dx =$

$\int (\sec^2 x \operatorname{tg} x)dx - \int (\operatorname{tg} x)dx + \int (\sec^2 x \operatorname{tg}^2 x)dx - \int (\operatorname{tg}^2 x)dx =$

$\frac{1}{2}\operatorname{tg}^2 x + \ln|\cos x| + \frac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x.$

aj) $\sqrt{2x^3}$, ak) $-3\cos x + \frac{2}{3}x^3$, al) $-\frac{1}{2}e^{-x^2}$, am) $\frac{3}{1+\cos x}$, an) $\frac{1}{3}\left(1+x^2\right)^{3/2}$,

ao) $\frac{1}{2}e^{2\operatorname{sen} x}$, ap) $-\ln(1+e^{-x})$, aq) $-\ln|\cos x|$,

ar) $\frac{1}{3}\sec^3 x$, as) $\frac{1}{4}\operatorname{sen}^4 x - \frac{1}{6}\operatorname{sen}^6 x$,

at) $\ln|\operatorname{arctg} x|$, au) $2\operatorname{arctg}(\sqrt{x})$,

av) $\frac{1}{2}\operatorname{arctg}\left(\frac{1}{2}e^x\right)$, aw) $\frac{2}{3}\sqrt{(\operatorname{arcsen} x)^3}$,

ax) $\frac{1}{2\sqrt{2}}\operatorname{arcsen}\left(\sqrt{2}x^2\right)$, ay) $\operatorname{sen}(\ln x)$, az) $\ln(\ln x)$,

2. a) Calculamos primeiro uma primitiva de $\frac{1}{4+9x^2}$:

$$\int \left(\frac{1}{4+9x^2}\right)dx = \frac{1}{4}\int \left(\frac{1}{1+\left(\frac{3}{2}x\right)^2}\right)dx = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x.$$

Temos então, para $x \in \mathbb{R}$, $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + c$, com $c \in \mathbb{R}$. Para determinar c temos $f(0) = c = 1$, logo $f(x) = \frac{1}{6}\operatorname{arctg}\frac{3}{2}x + 1$.

b) $\int ((\frac{1}{x-1}) = \ln|x-1|dx$, para $x \neq 1$. Temos então

$$g(x) = \begin{cases} \ln(x-1) + c_1, & \text{se } x > 1 \\ \ln(1-x) + c_2, & \text{se } x < 1. \end{cases}$$

com $c_1, c_2 \in \mathbb{R}$. Para determinar as constantes, temos $g(0) = \ln 1 + c_2 = 0$, logo $c_2 = 0$, e $g(2) = \ln 1 + c_1 = 3$, logo $c_2 = 3$.

c) O domínio da secante, $\sec x = \dfrac{1}{\cos x}$, é $\mathbb{R} \setminus \{\frac{\pi}{2} + k\pi \ : \ k \in \mathbb{Z}\}$. Neste conjunto temos $\int (\sec^2 x)dx = \operatorname{tg} x$, e portanto para $x \in \left]\frac{\pi}{2} + (k-1)\pi, \frac{\pi}{2} + k\pi\right[$, para cada $k \in \mathbb{Z}$, temos $h(x) = \operatorname{tg} x + c_k$. Como $k\pi \in \left]\frac{\pi}{2} + (k-1)\pi, \frac{\pi}{2} + k\pi\right[$, temos que $0 + c_k = k$, ou seja, $c_k = k$.

3.
- $\int (x \operatorname{sen}(x^2)) = \frac{1}{2}\cos(x^2)dx$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \frac{1}{2}\cos(x^2) + C$, com $C \in \mathbb{R}$.
  - a) $F(0) = 0 \Leftrightarrow \frac{1}{2} + C = 0$, logo $C = -\frac{1}{2}$.
  - b) $\lim_{x\to+\infty} F(x)$ não existe, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas.
- $P(\frac{e^x}{2+e^x}) = \ln(2+e^x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \ln(2+e^x)+C$, com $C \in \mathbb{R}$.
  - a) $F(0) = 0 \Leftrightarrow \ln 3 + C = 0$, logo $C = -\ln 3$.
  - b) $\lim_{x\to+\infty} F(x) = +\infty$, para qualquer $C \in \mathbb{R}$, logo não existe uma primitiva nas condições dadas. .
- $\int (\frac{1}{(1+x^2)(1+\operatorname{arctg}^2 x)}) = \operatorname{arctg}(\operatorname{arctg} x)$, $x \in \mathbb{R}$, logo a forma geral das primitivas é $F(x) = \operatorname{arctg}(\operatorname{arctg} x) + C$, com $C \in \mathbb{R}$.
  - a) $F(0) = 0 \Leftrightarrow C = 0$.
  - b) $\lim_{x\to+\infty} F(x) = \lim_{x\to+\infty} \operatorname{arctg}(\operatorname{arctg} x) + C = \operatorname{arctg}\frac{\pi}{2} + C$, logo $C = -\operatorname{arctg}\frac{\pi}{2}$.

4.

a) $\displaystyle\int\left(\frac{1}{1-x}\right)dx = -\ln|1-x|$, b) $\displaystyle\int\left(\frac{1}{(x-3)^3}\right)dx = -\frac{1}{2(x-3)^2}$,

c) $\displaystyle\int\left(\frac{x+1}{x^2+1}\right)dx = \frac{1}{2}\ln(x^2+1) + \operatorname{arctg} x$,

d) $\displaystyle\int\left(\frac{x}{1+(x-1)^2}\right)dx = \frac{1}{2}\ln(1+(x-1)^2) + \operatorname{arctg}(x-1)$,

e) $\displaystyle\int\left(\frac{2x+1}{x^2+4}\right)dx = \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)$, f) $\displaystyle\int\left(\frac{1}{x^2+2x+2}\right)dx = \operatorname{arctg}(x+1)$

g) $\displaystyle\frac{1}{4}\ln(3+x^4)$, h) $\displaystyle\int\left(\frac{x}{1+x^2}\right)dx = \frac{1}{2}\int\left(\frac{2x}{1+x^2}\right)dx = \frac{1}{2}\ln(1+x^2) = \ln\sqrt{1+x^2}$;

i) $\displaystyle\int\left(\frac{x^3}{x^8+1}\right)dx = \frac{1}{4}\int\left(\frac{4x^3}{(x^4)^2+1}\right)dx = \frac{1}{4}\operatorname{arctg}(x^4)$;

j) $\displaystyle\frac{1}{2(1-\alpha)}\frac{1}{(1+x^2)^{\alpha-1}}$, se $\alpha \neq 1$, $\ln\sqrt{1+x^2}$, se $\alpha = 1$; k) $\displaystyle\frac{1}{3}\ln\left|1+x^3\right|$,

l) $\displaystyle\frac{1}{\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}$, m) $\displaystyle\frac{1}{2}\operatorname{arctg}\left(x^2\right)$, n) $\displaystyle\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\sqrt{3}x\right)$, o) $x + 3\ln|x-2|$

5. a) $\int\left(\frac{1}{x^2+x}\right)dx = \int\left(\frac{1}{x(x+1)}\right)dx$. Usando a decomposição em fracções simples $\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1}$ temos

$$\frac{1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = \frac{Ax + A + Bx}{x(x+1)} = \frac{(A+B)x + A}{x(x+1)}$$

logo $A + B = 0$ e $A = 1$, ou seja, $A = 1$ e $B = -1$. Temos então

$$P\left(\frac{1}{x^2+x}\right) = P\left(\frac{1}{x} - \frac{1}{x+1}\right) = \ln|x| - \ln|x+1| = \ln\left|\frac{x}{x+1}\right|.$$

b) Usando a decomposição em fracções simples $\frac{x+1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, temos

$$\begin{aligned}\frac{x+1}{x(x-1)^2} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}\\ &= \frac{A(x-1)^2 + Bx(x-1) + Cx}{x(x-1)^2}\\ &= \frac{Ax^2 - 2Ax + A + Bx^2 - Bx + Cx}{x(x-1)^2}\\ &= \frac{(A+B)x^2 + (-2A - B + C)x + A}{x(x-1)^2}\end{aligned}$$

logo $A + B = 0$, $-2A - B + C = 1$, $A = 1$, ou seja, $A = 1$, $B = -1$, $C = 2$. Temos então

$$\int\left(\frac{x+1}{x(x-1)^2}\right)dx = \int\left(\frac{1}{x} - \frac{1}{x-1} + \frac{2}{(x-1)^2}\right)dx = \ln|x| - \ln|x-1| - \frac{2}{x-1}.$$

c) Usando a decomposição em fracções simples $\frac{x^2+x-4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$, temos

$$\begin{aligned}\frac{x^2+x-4}{x(x^2+4)} &= \frac{A}{x} + \frac{Bx+C}{x^2+4}\\ &= \frac{Ax^2 + 4A + Bx^2 + Cx}{x(x^2+4)}\\ &= \frac{(A+B)x^2 + 4A + Cx}{x(x^2+4)}\end{aligned}$$

logo $A + B = 1$, $C = 1$ e $4A = -4$, ou seja, $A = -1$, $B = 2$, $C = 1$. Temos então

$$\int\left(\frac{x^2+x-4}{x(x^2+4)}\right)dx = \int\left(-\frac{1}{x} + \frac{2x+1}{x^2+4}\right)dx = -\ln|x| + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right).$$

d) $2\ln|x-1| - \ln|x| + \dfrac{1}{x}$, e) $\dfrac{x^4}{4} + \dfrac{x^2}{2} + \dfrac{1}{2}\ln|x^2-1|$,

f) $\ln\left|\dfrac{x+2}{x+1}\right| - \dfrac{2}{x+2}$, g) $\dfrac{x^2}{2} + \ln|x+1| + \dfrac{1}{x+1}$,

h) $x + \dfrac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\operatorname{arctg} x$, i) $\dfrac{1}{2}\ln(x^2+4) + \operatorname{arctg}\left(\dfrac{x}{2}\right) + \dfrac{1}{2}\ln\left|\dfrac{x-2}{x+2}\right|$.

6. a) O domínio de $\frac{1}{x^2+x}$ é $\mathbb{R} \setminus \{-1, 0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x+1) + C_1, & \text{se } x > 0, \\ \ln(-x) - \ln(x+1) + C_2, & \text{se } -1 < x < 0, \\ \ln(-x) - \ln(-x-1) + C_3, & \text{se } x < -1, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

b) O domínio de $\frac{x+1}{x(x-1)^2}$ é $\mathbb{R} \setminus \{0, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x-1) - \frac{2}{x-1} + C_1, & \text{se } x > 1, \\ \ln x - \ln(-x+1) - \frac{2}{x-1} + C_2, & \text{se } 0 < x < 1, \\ \ln(-x) - \ln(-x+1) - \frac{2}{x-1} + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

c) O domínio de $\frac{x^2+x-4}{x(x^2+4)}$ é $\mathbb{R} \setminus \{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} -\ln x + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_1, & \text{se } x > 0, \\ -\ln(-x) + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_2, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

d) O domínio de $\frac{x^2+1}{x^2(x-1)}$ é $\mathbb{R} \setminus \{0, 1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} 2\ln(x-1) - \ln x + 1/x + C_1, & \text{se } x > 1, \\ 2\ln(1-x) - \ln x + 1/x + C_2, & \text{se } 0 < x < 1, \\ 2\ln(1-x) - \ln(-x) + 1/x + C_3, & \text{se } x < 0, \end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

7. Sendo $\int \left(\frac{1}{1+x}\right) = \ln(x+1)dx$, para todo o $x \in]-1, +\infty[$, temos

$$\psi'(x) = \ln(x+1) + C_1.$$

A condição $\psi'(0) = 1$, resulta em $C_1 = 1$. Usando primitivação por partes (verifique!) temos

$$P\left(\ln(x+1)+1\right) = (x+1)\ln(x+1),$$

ou seja $\psi(x) = (x+1)\ln(x+1) + C_2$. Dado que $\psi(0) = 1$, obtém-se o resultado

$$\psi(x) = (x+1)\ln(x+1) + 1.$$

8.

a) $\int (xe^x)dx = xe^x - \int (e^x)dx = (x-1)e^x$,

b) $\int (x \operatorname{arctg} x)dx = \dfrac{x^2}{2}\operatorname{arctg} x - \int \left(\dfrac{x^2}{2}\dfrac{1}{1+x^2}\right)dx$

$= \dfrac{x^2}{2}\operatorname{arctg} x - \dfrac{1}{2}P\left(1 - \dfrac{1}{1+x^2}\right) = \dfrac{1}{2}\left(-x + (x^2+1)\operatorname{arctg} x\right)$,

c) $P(\operatorname{arcsen} x) = x \operatorname{arcsen} x - P\left(x\frac{1}{\sqrt{1-x^2}}\right) = x\operatorname{arcsen} x + \sqrt{1-x^2}$,

d) $P(x \operatorname{sen} x) = -x\cos x + P(\cos x) = -x\cos x + \operatorname{sen} x$,

e) $P(x^3 e^{x^2}) = P(x^2 \cdot xe^{x^2}) = x^2\frac{e^{x^2}}{2} - P\left(2x\frac{e^{x^2}}{2}\right) = (x^2-1)\frac{e^{x^2}}{2}$,

f) $P(\ln^3 x) = x\ln^3 x - P(3\ln^2 x) = x(\ln^3 x - 3\ln^2 x) + P(6\ln x) =$

$x(\ln^3 x - 3\ln^2 x + 6\ln x) - P(6) = x(\ln^3 x - 3\ln^2 x + 6\ln x - 6)$,

g) $P(x^n \ln x) = \frac{1}{n+1}x^{n+1}\ln x - P\left(\frac{1}{n+1}x^{n+1}\frac{1}{x}\right) = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1}$,

h) $P\left(\frac{x^7}{(1-x^4)^2}\right) = P\left(x^4\frac{x^3}{(1-x^4)^2}\right) = x^4\frac{1}{4(1-x^4)} - P\left(4x^3\frac{1}{4(1-x^4)}\right) =$

$\frac{x^4}{4(1-x^4)} + \frac{1}{4}\ln(1-x^4)$.

i) $P(\operatorname{arcsen}^2 x) = x\operatorname{arcsen}^2 x - P\left(\frac{2x\operatorname{arcsen} x}{\sqrt{1-x^2}}\right)$

$P\left(\frac{2x\operatorname{arcsen} x}{\sqrt{1-x^2}}\right) = -\sqrt{1-x^2}\operatorname{arcsen} x + P\left(\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}}\right)$,

j) $e^x(e^x + x - 1) - e^{2x}/2$, k) $e^x(\operatorname{sen} x - \cos x)/2$,

l) $-e^{-x^2}(x^2+1)/2$, m) $x\operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)$,

n) $\frac{2}{3}x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right)$ o) $\frac{1}{4}(1+x^2)^2\operatorname{arctg} x - x/4 - x^3/12$,

p) $\frac{2}{3}x^3\sqrt{1+x^3} - \frac{4}{9}(1+x^3)^{3/2}$, q) $x\ln|1/x+1| + \ln|x+1|$,

r) $\frac{x^3}{3}\ln^2 x - \frac{2}{9}x^2\ln x + \frac{2}{27}x^3$, s) $x\ln^2 x - 2x\ln x + 2x$,

t) $-\frac{1}{x}\operatorname{sen}\frac{1}{x} - \cos\frac{1}{x}$, u) $\frac{1}{2}\operatorname{sen}(2x)\ln(\operatorname{tg} x) - x$,

v) $-(1-x^2)^{3/2}\operatorname{arcsen} x + x - x^3/3$, x) $-\frac{\ln x}{1+x} + \ln\left|\frac{x}{1+x}\right|$,

w) $\frac{1}{2}(\operatorname{sh} x\cos x + \operatorname{ch} x \operatorname{sen} x)$, y) $\frac{1}{1+\ln^2 3}3^x(\operatorname{sen} x + \ln 3\cos x)$,

z) $\frac{x}{2}(\cos(\ln x) + \operatorname{sen}(\ln x))$, zz) $-\frac{1}{2}\frac{x}{1+x^2} + \frac{1}{2}\operatorname{arctg} x$.

9.

a) $\frac{1}{2}e^{2x} - \frac{1}{2}\ln\left(e^{2x}+1\right)$, b) $\frac{3}{2}\operatorname{arctg}\sqrt[3]{x^2}$, c) $2\sqrt{x-1} - 2\operatorname{arctg}\sqrt{x-1}$,

d) $\frac{6}{7}x\sqrt[6]{x} - \frac{6}{5}\sqrt[6]{x^5} - \frac{3}{2}\sqrt[3]{x^2} + 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} - 3\ln\left|1+\sqrt[3]{x}\right| + 6\operatorname{arctg}\sqrt[6]{x}$,

e) $\frac{1}{4}\ln\left|\frac{e^x-1}{e^x+1}\right| - \frac{1}{2(1+e^x)}$, f) $-2\operatorname{arctg}\sqrt{1-x}$,

g) $\ln|\cos x| + \ln|\operatorname{tg} x + 1|$, h) $\ln|\ln x - 1| - \frac{1}{\ln x - 1}$,

i) $3\ln(\sqrt[3]{x}+1)$,

10. a) Fazendo a substituição $\sqrt{x} = t \Leftrightarrow x = t^2$, com $x > 0$, $x \neq 16$, e $t > 0$, $t \neq 4$, temos

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right) = P\left(\frac{1+t}{t^2(4-t)}\,2t\right) = 2P\left(\frac{1+t}{t(4-t)}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2+2t}{t(4-t)} = \frac{A}{t} + \frac{B}{4-t}$$

temos $A = \frac{1}{2}$, $B = \frac{5}{2}$, logo

$$2P\left(\frac{1+t}{t(4-t)}\right) = \frac{1}{2}P\left(\frac{1}{t} + \frac{5}{4-t}\right) = \frac{1}{2}\ln\left|\frac{t}{(4-t)^5}\right|$$

e assim,

$$P\left(\frac{1+\sqrt{x}}{x(4-\sqrt{x})}\right) = \frac{1}{2}\ln\left|\frac{\sqrt{x}}{(4-\sqrt{x})^5}\right|.$$

b) Fazendo a substituição $\sqrt[4]{1+x} = t \Leftrightarrow x = t^4 - 1$, com $x > -1$ e $t > 0$, temos

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = P\left(\frac{1}{(t^4-1)t}\,4t^3\right) = P\left(\frac{4t^2}{t^4-1}\right).$$

Usando a decomposição em fracções simples:

$$\frac{4t^2}{t^4-1} = \frac{4t^2}{(t-1)(t+1)(t^2+1)} = \frac{A}{t-1} + \frac{B}{t+1} + \frac{Ct+D}{t^2+1},$$

temos $A = 1$, $B = -1$, $C = 0$, $D = 2$. Logo,

$$P\left(\frac{4t^2}{t^4-1}\right) = P\left(\frac{1}{t-1} - \frac{1}{t+1} + \frac{2}{t^2+1}\right) = \ln\left|\frac{t-1}{t+1}\right| + 2\operatorname{arctg} t$$

e assim,

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = \ln\left|\frac{\sqrt[4]{1+x}-1}{\sqrt[4]{1+x}+1}\right| + 2\operatorname{arctg}\sqrt[4]{1+x}.$$

c) Fazendo a substituição $e^{2x} = t \Leftrightarrow x = \frac{1}{2}\ln t$, com $x \in \mathbb{R}$ e $t > 0$, temos

$$P\left(\frac{1}{1+e^{2x}}\right) = P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right).$$

Usando a decomposição em fracções simples:

$$\frac{1}{(1+t)2t} = \frac{A}{1+t} + \frac{B}{t}$$

temos $A = -\frac{1}{2}$, $B = \frac{1}{2}$, logo

$$P\left(\frac{1}{1+t}\cdot\frac{1}{2t}\right) = P\left(-\frac{1}{2(1+t)} + \frac{1}{2t}\right) = \frac{1}{2}\ln\left|\frac{t}{1+t}\right|$$

e assim,

$$P\left(\frac{1}{1+e^{2x}}\right) = \frac{1}{2}\ln\left|\frac{e^{2x}}{1+e^{2x}}\right|.$$

d) Fazendo a substituição $e^x = t \Leftrightarrow x = \ln t$, com $x \in \mathbb{R}\setminus\{0\}$ e $t > 0$, $t \neq 1$, temos

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = P\left(\frac{t^3}{(1+t^2)(t-1)^2}\,\frac{1}{t}\right) = P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{t^2}{(1+t^2)(t-1)^2} = \frac{At+B}{1+t^2} + \frac{C}{t-1} + \frac{D}{(t-1)^2}$$

temos $A = -\frac{1}{2}$, $B = 0$, $C = D = \frac{1}{2}$, logo

$$\begin{aligned}P\left(\frac{t^2}{(1+t^2)(t-1)^2}\right) &= \frac{1}{2}P\left(-\frac{t}{1+t^2} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right)\\ &= -\frac{1}{4}\ln(1+t^2) + \frac{1}{2}\ln|t-1| - \frac{1}{2}\frac{1}{t-1}\end{aligned}$$

e assim

$$P\left(\frac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right) = -\frac{1}{4}\ln(1+e^{2x}) + \frac{1}{2}\ln|e^x-1| - \frac{1}{2}\frac{1}{e^x-1}.$$

e) Fazendo a substituição $\ln x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+\setminus\{1, e\}$ e $t \in \mathbb{R}\setminus\{0,1\}$, temos

$$P\left(\frac{2\ln x - 1}{x\ln x(\ln x - 1)^2}\right) = P\left(\frac{2t-1}{e^t t(t-1)^2}\,e^t\right) = P\left(\frac{2t-1}{t(t-1)^2}\right).$$

Usando a decomposição em fracções simples:

$$\frac{2t-1}{t(t-1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{C}{(t-1)^2}$$

temos $A = -1$, $B = C = 1$, logo

$$P\left(\frac{2t-1}{t(t-1)^2}\right) = P\left(-\frac{1}{t} + \frac{1}{t-1} + \frac{1}{(t-1)^2}\right) = \ln\left|\frac{t-1}{t}\right| - \frac{1}{t-1}$$

e assim

$$P\left(\frac{2\ln x - 1}{x\ln x(\ln x - 1)^2}\right) = \ln\left|\frac{\ln x - 1}{\ln x}\right| - \frac{1}{\ln x - 1}.$$

f) Fazendo a substituição $\operatorname{sen} x = t \Leftrightarrow x = \operatorname{arcsen} t$, obtem-se (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x \cos x}\right) = -\frac{1}{\operatorname{sen} x} + \frac{1}{2}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|.$$

11.

a) $\frac{1}{2}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|$, b) $\sqrt{1-\frac{1}{x^2}}$, c) $\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\operatorname{arcsen} x$,

d) $\ln\left|1+\operatorname{tg}\frac{x}{2}\right|$, e) $-\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{3/2}$, f) $-2\operatorname{arcsen}\sqrt{1-e^x}$,

g) $-x+\operatorname{tg} x + \sec x$, h) $2\operatorname{arcsen}\sqrt{x}$, i) $\ln|\frac{1+2\operatorname{sen} x}{1-\operatorname{sen} x}|$,

j) $\frac{1}{4}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \frac{1}{4(1-\operatorname{sen} x)} - \frac{1}{4(1+\operatorname{sen} x)} = \frac{1}{2}\ln\left|\frac{1+\operatorname{sen} x}{\cos x}\right| + \frac{\operatorname{sen} x}{2\cos^2 x}$
$= \frac{1}{2}\ln\left|\sec x + \operatorname{tg} x\right| + \frac{1}{2}\sec x \operatorname{tg} x$, k) $\ln|x+\sqrt{x^2+1}|$,

l) $\ln\left|\frac{\operatorname{sen} x}{1+\operatorname{sen} x}\right|$, m) $\ln\left|\frac{\sqrt{1-x^2}-1}{\sqrt{1-x^2}+1}\right|$, n) $\ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|$,

o) $2\ln\left|\sqrt{1+\left(\frac{x}{2}\right)^2}+\frac{x}{2}\right| + x\sqrt{1+\left(\frac{x}{2}\right)^2}$,

p) $\frac{\sqrt{x^2-1}}{2}(x-2) + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|$.

12. a) $\frac{1}{2}x|x|$,

b) $\frac{x^2}{2}\operatorname{arcsen}\frac{1}{x} + \frac{1}{2}x\sqrt{1-\frac{1}{x^2}}$, (por partes, por ex.)

c) $\frac{x}{2}\operatorname{sen}(\ln x + 1) - \frac{x}{2}\cos(\ln x + 1)$, (por partes, por ex.)

d) $\frac{x}{8} - \frac{1}{32}\operatorname{sen} 4x$,

e) $\frac{2}{3}x^{3/2}\operatorname{arctg}\sqrt{x} - \frac{1}{3}x + \frac{1}{3}\ln(1+x)$, (por partes, por ex.)

f) $-\ln x + 2\ln|1+\ln x| + \frac{\ln^2 x}{2}$, (substituição $t = \ln x$, por ex.)

g) $\frac{x}{2} - \frac{1}{2}e^{-x} - \frac{1}{4}\ln(e^{2x} - 2e^{x} + 2)$, (substituição $t = e^x$, por ex.)

h) $\frac{2}{3}\sqrt{x^3} - x + 4\sqrt{x} - 4\ln(\sqrt{x}+1)$, (substituição $t = \sqrt{x}$, por ex.)

i) $\operatorname{sen} x - \frac{1}{3}\operatorname{sen}^3 x$,

j) $\frac{3}{8}x + \frac{1}{4}\operatorname{sen} 2x + \frac{1}{8}\operatorname{sen} 4x$,

k) $\frac{1}{2}(x^2-1)\ln\left|\frac{1-x}{1+x}\right| - x$,

l) $\frac{1}{2}\ln\left|\frac{(x-1)(x+3)}{(x+2)^2}\right|$,

m) $\frac{1}{2}\ln^2(\ln x)$,

n) $x\ln(x+\sqrt{x}) - x + \sqrt{x} - \ln(1+\sqrt{x})$, (substituição $t = \sqrt{x}$ e por partes, por ex.)

o) $-\left(\frac{1}{x}+1\right)e^{\frac{1}{x}}$, (por partes, por ex.)

p) $\operatorname{sen} x\ln(1+\operatorname{sen}^2 x) - 2\operatorname{sen} x + 2\operatorname{arctg}(\operatorname{sen} x)$,

q) $\ln x\ln(\ln x) - \ln x$,

r) $\frac{x^2+1}{2}\operatorname{arctg}^2 x - x\operatorname{arctg} x + \frac{1}{2}\ln(1+x^2)$,

s) $2\sqrt{1+x}(\ln(1+x) - 2)$,

t) $\ln\left|\frac{\operatorname{sen} x}{\cos x + 1}\right|$,

u) $-\frac{x}{\operatorname{sen} x} + \ln\left|\frac{\operatorname{sen} x}{\cos x + 1}\right|$,

v) $-\frac{\sqrt{3}}{3}\operatorname{arctg}\left(\sqrt{3}\cos x\right)$,

w) $-\frac{1}{2}\ln^2(\cos x)$,

x) $\ln\left|\frac{\sqrt{x+2}-1}{\sqrt{x+2}+1}\right|$ (substituição $t = \sqrt{x+2}$, por ex.),

y) $x(\arccos x)^2 - 2\sqrt{1-x^2}\arccos x + 2x$ (por partes),

z) $\frac{1}{4}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \frac{1}{2(1-\operatorname{sen} x)}$ (substituição $t = \operatorname{sen} x$, por ex.).