Duration: **30** minutes

- Write your number and name below.
- Add your answers to this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and TWO QUESTIONS. The total of points is 5.0.

Number: **Name**:

1. Twenty-four vehicles have completed a 74 000 km scheduled run at an automobile company's proving grounds. During the test eight radiator hoses failed (and were replaced) at kilometers: 2 760, 3 700, 7 100, 17 220, 29 500, 48 400, 52 600, 65 000. Assume that the times to failure are i.i.d. exponentially r.v. **(1.5)**

Compute a 95% confidence interval for the probability that a radiator hose is not replaced during a 24 000 km warranty period.

- **Failure times**
  T_i = failure time of a radiator hose $i \stackrel{i.i.d.}{\sim}$ exponential(λ), $\quad i = 1,\ldots,n \quad (n = 20)$
  $\lambda > 0$ (UNKNOWN)
- **Censored data**
  Since the test lasted for $t_0 = 74\,000$ hours and none of the $r = 8$ failed radiator hoses was replaced, we are dealing with a Type I/item censored testing with replacement.
- **Total time on test**

$$\begin{aligned}\tilde{t} &\stackrel{Table\ 5.11}{=} n\,t_0 \\ &= 24 \times 74\,000 \\ &= 1\,776\,000\end{aligned}$$

- **Unknown parameter**
  $P(T > 24\,000) = e^{-24\,000\times\lambda}$
- **Confidence interval for λ**
  Since we are dealing with a Type I/item censored testing with replacement, we get from Table 5.16

$$CI_{(1-\alpha)\times 100\%}(\lambda) = \left[\frac{F^{-1}_{\chi^2_{(2r)}}(\alpha/2)}{2\times\tilde{t}}, \frac{F^{-1}_{\chi^2_{(2r+2)}}(1-\alpha/2)}{2\times\tilde{t}}\right] = [\lambda_L, \lambda_U]$$

$$\begin{aligned}CI_{95\%}\left(e^{-24\,000\times\lambda}\right) &= \left[e^{-24\,000\times\lambda_U}, e^{-24\,000\times\lambda_L}\right] = \left[e^{-24\,000\times\frac{F^{-1}_{\chi^2_{(2\times 8+2)}}(1-0.05/2)}{2\times\tilde{t}}}, e^{-24\,000\times\frac{F^{-1}_{\chi^2_{(2\times 8)}}(0.05/2)}{2\times\tilde{t}}}\right] \\ &= \left[e^{-24\,000\times\frac{31.53}{2\times 1\,776\,000}}, e^{-24\,000\times\frac{6.908}{2\times 1\,776\,000}}\right] \\ &\simeq [0.808123, 0.954397].\end{aligned}$$

2. An engineer uses a g-chart to monitor the proportion p of incompletely filled low-voltage liquid crystal display units in a high-yield process. The associated control statistic X has c.d.f. $F_p(x) = 1-(1-p)^{x+1}$, $x \in \mathbb{N}_0$, where p may shift from its target value $p_0 = 10^{-3}$ to $\rho \times p_0$, with $0 < \rho < 1/p_0$. The adopted control limits are $LCL_\alpha = \frac{\ln(1-\alpha/2)}{\ln(1-p_0)}$ and $UCL_\alpha = \frac{\ln(\alpha/2)}{\ln(1-p_0)}$, where $\alpha = 0.005$ is the *acceptable risk of false alarm.* **(2.0)**

Determine and comment on its ARL values when $\rho = 1, 1.1$.

- **Control limits**

$$LCL_\alpha = \frac{\ln(1-\alpha/2)}{\ln(1-p_0)} = \frac{\ln(1-0.005/2)}{\ln(1-0.001)} = 2.501878$$
$$UCL_\alpha = \frac{\ln(\alpha/2)}{\ln(1-p_0)} = \frac{\ln(0.005/2)}{\ln(1-0.001)} \simeq 5988.468315$$

- **Probability of a signal and ARL function**

$$\begin{aligned}
\xi_\alpha(\rho) &= P_{\rho\times p_0}(X \not\in [LCL_\alpha, UCL_\alpha]) = 1 - \left[F_{\rho\times p_0}(UCL_\alpha) - F_{\rho\times p_0}(LCL_\alpha^-)\right] \\
&= 1 - \left[F_{\rho\times p_0}(5988.468315) - F_{\rho\times p_0}(2.501878^-)\right] = 1 - F_{\rho\times p_0}(5988) + F_{\rho\times p_0}(2) \\
&= 1 - \left[1-(1-\rho\times p_0)^{1+5988}\right] + \left[1-(1-\rho\times p_0)^{1+2}\right] \\
&= 1-(1-\rho\times 0.001)^3 + (1-\rho\times 0.001)^{5989}, \quad 0<\rho<1/p_0 \\
ARL_\alpha(\rho) &= 1/\xi_\alpha(\rho), \quad 0<\rho<1/p_0
\end{aligned}$$

- **Requested ARL values and comments**

$$ARL_\alpha(1) = \frac{1}{1-(1-0.001)^3+(1-0.001)^{5989}} \simeq 181.961;$$
$$ARL_\alpha(1.1) = \frac{1}{1-(1-1.1\times 0.001)^3+(1-1.1\times 0.001)^{5989}} \simeq 214.210.$$

Note that for this g-chart:

  - $ARL_\alpha(1) = 181.961 < \frac{1}{\alpha} = 200$, i.e., the in-control ARL (does not coincide with and) is smaller than the reciprocal of the *acceptable risk of false alarm* — an undesirable property;
  - regretfully $ARL_\alpha(1.1) = 214.210 > ARL_\alpha(1) = 181.961$, i.e., it takes longer, in average, to trigger a valid signal in the presence of a 10% increase in p than to trigger a false alarm — another unwelcome property.

3. The high-voltage output of a certain power supply used in a copy machine is assumed to have a normal distribution with nominal mean and standard deviation equal to $\mu_0 = 350$ and $\sigma_0 = 2.0$ (V dc at 20 milliamps). Samples of $n = 9$ power supply units have been inspected every half-hour. The process mean and standard deviation have increased and the magnitudes of the associated shifts are $\delta = \sqrt{n}(\mu-\mu_0)/\sigma_0 = 0.25$ and $\theta = \sigma/\sigma_0 = \sqrt{23.574603/21.95}$. The out-of-control ARL of the $\bar{X}-$chart with 3$-$sigma limits and the upper one-sided S^2-chart are $ARL_\mu(\delta,\theta) \simeq 1/0.004839$ and $ARL_\sigma(\theta) = 1/0.005009$. **(1.5)**

Compute the probability that the first 10 samples are associated with at least two valid signal triggered by the joint $(\bar{X}, S^2)$-scheme.

- **Probability of a signal**

$$\begin{aligned}
\xi_{\mu,\sigma}(\delta,\theta) &= \xi_\mu(\delta,\theta) + \xi_\sigma(\theta) - \xi_\mu(\delta,\theta)\times\xi_\sigma(\theta) \\
&= ARL_\mu^{-1}(\delta,\theta) + ARL_\sigma^{-1}(\theta) - ARL_\mu^{-1}(\delta,\theta)\times ARL_\sigma^{-1}(\theta) \\
&\simeq 0.004839 + 0.005009 - 0.004839\times 0.005009 \\
&\simeq 0.009824.
\end{aligned}$$

- **Auxiliary r.v.**

W = number of valid signals triggered by the joint scheme in the first 10 samples

$W \sim \text{binomial}\left(10, \xi_{\mu,\sigma}(\delta,\theta) \simeq 0.009824\right)$

$P(W=w) \simeq \binom{10}{w} 0.009824^w (1-0.009824)^{10-w}, \quad w = 0,1,\ldots,10$

- **Requested probability**

$$\begin{aligned}
P(W\geq 2) &= 1 - P(W\leq 1) \\
&\simeq 1-(1-0.009824)^{10} - 10\times 0.009824\times(1-0.009823860)^{10-1} \\
&\simeq 0.004121.
\end{aligned}$$