Instituto Superior Técnico

Calculus of variations and Optimal Control

Problem Series nº 4

P1. A CV problem with an integral constraint

Find the extremals for the following functional

$$J(y) = \int_0^1 (y'^2 + x^2)dx$$

that satisfy the boundary conditions

$$y(0) = y(1) = 0$$

and the constraint

$$\int_0^1 y^2\, dx = 2$$

P2. Boltzmann distribution from the Principle of Maximum Entropy

For a gas-filled rectangular box consider the question: *What is the distribution of the kinetic energies E of the molecules that form the gas in the state of maximal disorder?*

Let $p(E)$ be the probability density function of the energy of the molecules of the gas, given an average energy $\bar{E}$ of the set of molecules.

Clearly, $p(E) = 0$ for $E < 0$.

According to the Principle of Maximum Entropy, the energy distribution is obtained by solving the following variational problem:

$$\max_p H(p) = \int_0^\infty -p(E)\log p(E)\, dE$$

Subject to

$$\int_0^\infty p(E)dE = 1 \qquad \text{(C1)}$$

$$\int_0^\infty Ep(E)dE = \bar{E} \qquad \bar{E} \text{ given} \qquad \text{(C2)}$$

Solve this problem that yields the Boltzmann distribution.

Hints:

1) Write the EL equation in the presence of constraints and solve it to obtain an expression for $p(E)$ as a function of the two Lagrange multipliers γ (associated to the constraint (C1)), and λ (associated to the constraint (C2)).

   Impose the constraint (C1) and eliminate γ. Use the integral

$$\int_{\alpha}^{\beta} e^{ax} dx = \frac{e^{ax}}{a} |_{\alpha}^{\beta}$$

2) Impose the constraint (C1) and eliminate λ. Use the integral

$$\int_{\alpha}^{\beta} x e^{ax} dx = \frac{e^{ax}}{a^2} (ax - 1) |_{\alpha}^{\beta}$$