Instituto Superior Técnico

Calculus of variations and Optimal Control

Problem Series nº 3

P1. Basic CV problem

Find the extremal for the following fixed end points basic CV problem

$$J(y) = \int_0^{\pi} (y'^2 + 2y \sin x) dx$$

$$y(0) = y(\pi) = 0$$

P2. Find the extremal for the following fixed end points basic CV problem

$$J(y) = \int_a^b [y^2 + 2kyy' + (y')^2]\, dx$$

$$y(a) = y_0,\ y(b) = y_1$$

With a, b, y_0 and y_1 generic and k a constant.

P3. Find the extremal for the following fixed end points basic CV problem

$$J(y) = \int_0^{\pi} \frac{(y')^2}{x^3} dx$$

$$y(1) = 0,\ y(2) = 3$$

Take advantage of the fact that this is special case 1 (“no y”) of the EL equation.

P4. Free end point. Find the curve that links the point $x = 0, y(0) = 5$ with the circle defined in the plane (x, y) by $y^2 + (x - 5)^2 - 4 = 0$, and such that the curve is an extremal to the length.

In other words, consider the problem with free end point and free end “time”

Minimize $J(y) = \int_0^{x_f} \sqrt{1 + y'^2}\, dx$

s. t. $y(0) = 5$

The end points must lie on the circle

$y^2 + (x - 5)^2 - 4 = 0,$

with x_f free.

Draw a sketch to interpret the solution.

P5. This problem shows how a variational technique based on the Euler-Lagrange equation can be used to model conservative mechanical systems. This technique is based on the description of physical system using the so called *generalised coordinates*. The vector of generalised coordinates will be represented by q and exists in the *configuration space* of the system. For instance, a mass point in the plane is described by the Cartesian coordinates describing its position, $q_1 = x$ and $q_2 = y$.

From the dynamical point of view, a mechanical system may be seen as a set of interconnected particles. These interconnections impose constraints on the system behaviour.

The basis for modelling is Hamilton's Principle. In order to understand it, imagine a mass point in the plane that is thrown with an initial velocity from point 1 at instant t_1 and reaches point 2 at time t_2 (fig. P5-1).

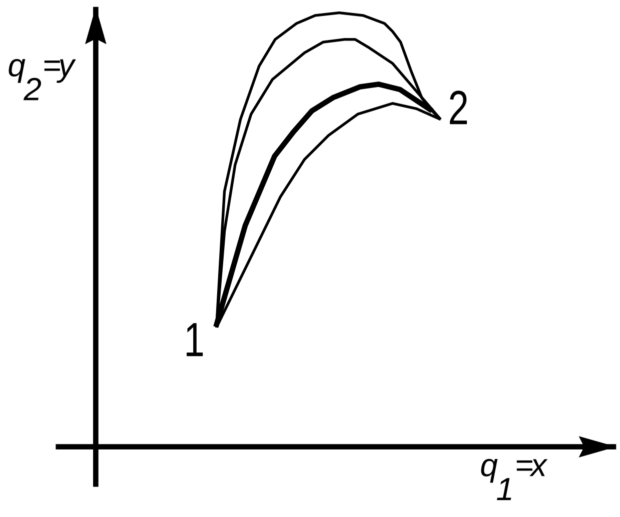

Fig. P5-1. A mass point moving between two points.

The mass point follows a unique and well defined trajectory, shown in bold in fig. P5-1. We may however imagine several other trajectories. Actually, there is an infinite number of them connecting points 1 and 2. Hamilton's Principle characterizes the trajectory that is actually followed. For that sake, define the *Lagrangian function* L as the difference between the kinetic energy T and the potential energy V:

$$L = T - V$$

The Lagrangian function is a function of the generalized coordinates q and of their first derivatives $\dot{q}$:

$$L = L(q, \dot{q})$$

Consider the following integral:

$$I = \int_{t_1}^{t_2} (T - V)\,dt$$

From all values of I for all possible trajectories (of which there is an infinite number), the one that corresponds to the actual trajectory is the one that remains invariant to a small perturbation (is an extremal of I). This statement is Hamilton's Principle that for a system made from interacting particles reads as follows:

> *Hamilton's Principle*: From the whole set of admissible conditions that a system may assume when evolving from one configuration in a given time instant to another configuration in a successive time instant, the one that is actually followed is the one that is an extremal of the integral of the Lagrangian function in that time interval.

Remark that applying Hamilton's Principle requires the solution of an optimization problem in an infinite dimensional space. This means that the integral I is a function that takes real values but its argument is itself a continuous function (hence requiring an infinite set of numbers to be described). This problem may no longer be solved with the basic optimization technique of "equating the derivative to zero". Its solution is yielded by other methods, called "variational" because they rely on performing variations of the optimal trajectory and relating them with the corresponding variation in I. They are studied within the realm of Variational Calculus theory.

Under certain conditions that are assumed satisfied in the cases to consider, it is a necessary condition for the integral I to be minimum that the Laggrangian satisfies the *Euler-Lagrange equation* that reads:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F$$

where F is the vector of generalized forces (moments in the case of rotating movements) that act positively along the direction of coordinate q.

a) Consider the mass-spring system shown in fig. P5-2. This system is made of a mass m suspended by a spring. In the absence of an outside F applied to the mass, the spring is stretched up to a length where the elastic force equilibrates the gravity force.

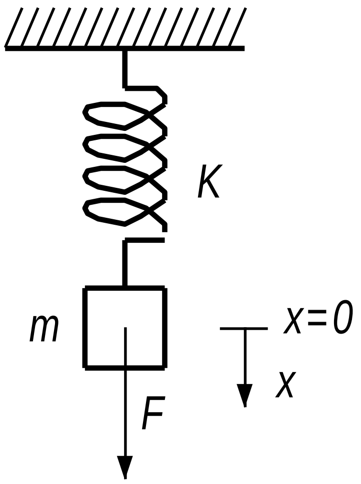

Fig. P5-2 – Mass-spring system to model.

Beyond this equilibrium point, that corresponds to the coordinate $x = 0$, when the elongation x increases, the spring applies to the mass a force in the opposite sense, with modulus Kx. At position x, with velocity $\dot{x}$, the system has a kinetic energy given by

$$T = \frac{1}{2} m\dot{x}^2 ,$$

and a potential energy given by

$$V = \frac{1}{2} Kx^2 .$$

Take as generalized coordinate $q = x$. Write the Euler-Lagrange equation for this system, obtaining thereby an ordinary differential equation for x. Compare this equation with the equation yielded by Newton's Law of motion.

b) Consider now (fig. P5-3) a satellite of mass m moving in a gravity force field that varies with $-k/r^2$. The satellite is equipped with actuators that exert radial oriented forces, along u_1, and tangential, along u_2.

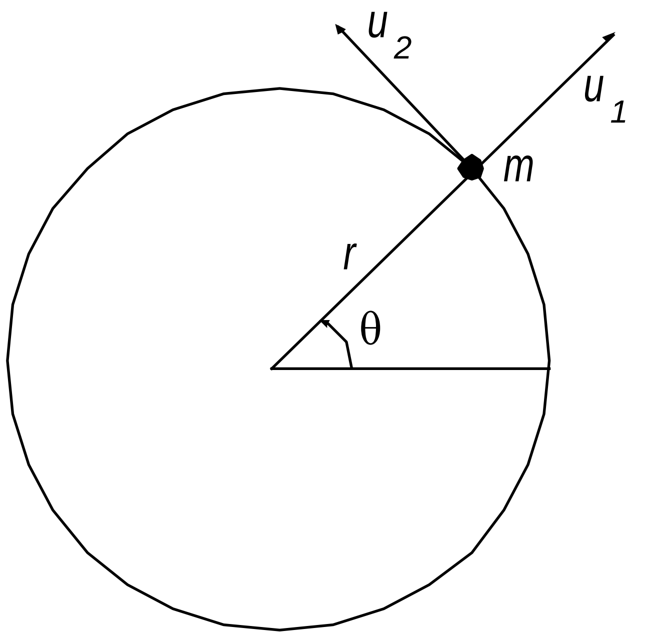

Fig. P5-3. Satellite model.

The kinetic energy is

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 \right).$$

The potential energy is

$$V = -\frac{k}{r}$$

Take as generalized coordinates $q_1 = r,\, q_2 = \theta$. Remark that in this case the Euler-Lagrange equation yields two equations (one for each generalized coordinate). Write this equations with respect to $\ddot{r}$ and $\ddot{\theta}$. Write a nonlinear state model for the satellite movement using as state $x = [r \quad \dot{r} \quad \theta \quad \dot{\theta}]^T$.