Instituto Superior Técnico

Calculus of variations and Optimal Control

Problem Series nº 1

P1. Consider the linear SISO continuous system described by the following transfer function

$$G(s) = \frac{s - 0.5}{s^2 + 0.1s + 1}$$

a) Define a state and write the state-space equations in the standard form.
b) Using SIMULINK simulate the state-space model. Show time plots of the state and the output when the input is a step function and under zero initial conditions.
c) Using the SIMULINK block diagram of b), show a 3 plot of trajectories in state-space for various initial conditions and under zero input.

P2. Consider the system formed by two coupled sliding masses shown in fig. P2-1. The spring coupling the two masses acts as follows: If the masses are a distance d_0 apart, the force exerted by the spring is zero. If the spring is stretched from this position by a distance Δd, an attracting force of module $k\Delta d$ is exerted. If the spring is compressed by a distance Δd, a repelling force of module is exerted.

Taking as input the force u, as state the position and velocities of the masses and as output the position of mass 2, write a state space model that represents the system. Use Newton's Laws of motion.

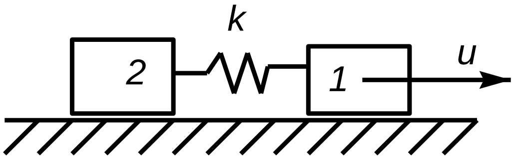

Fig. P2-1 – Problem P2. Modelling sliding masses.

P3. Find e^{At} for each of the following systems:

$$\text{a)}\quad A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \qquad\qquad \text{b)}\quad B=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$$

P4. Find the extremal for the following basic calculus of variations problem by solving the Euler-Lagrange equation

$$\text{Minimize } J(y)=\int_1^2 y'(1+x^2y')dx$$

$$\text{Subject to } y(1)=0,\ y(2)=1$$

Hints:

A) Observe that if α is a function of x, then $\frac{d}{dx}\alpha(x)=0$ implies that $\alpha(x)=C$ for some constant C.

B) If y is a function of x, then the solution of the differential equation $y'=\alpha(x)$ is $y(x)=y(x_0)+\int_{x_0}^{x}\alpha(s)ds$.

P5. The flour factory *Confeitaria Rainha Regional*, is a well known company (that since 1890 manufactures the delicious *Farinha 33,* to which the students of IST with classes at 8 o'clock are so much in debt for many and so nutritive breakfasts) is considering the optimal investment policy in its production lines. After deep studies of its skillful managers, it has been concluded that the production rate, P, is related to the investment rate I (a time function) therough the model

$$\frac{dP}{dt}=-0.1P+0.5I \qquad\qquad P(0)=1$$

Where the time unit is 1 year.

The production line will operate for 15 years, and will then be sold by a price that is proportional to its production rate at that time. The total value of the production line is thus

$$J = P(15) + \int_0^{15} [P(t) - I(t)] dt$$

The investment rate is always positive and may not exceed a maximum value $I_{\max}$, meaning that

$$0 \le I(t) \le I_{\max}$$

Using Pontryagin's Principle, find the optimal investment policy $I(t)$, $0 \le t \le 15$ that maximizes J.

An useful aid:

The solution of the differential equation

$$\dot{x}(t) + ax(t) = b,$$

with a and b constant, is given by

$$x(t) = \frac{b}{a} + ce^{-at},$$

where c is a constant that depends on the initial conditions.

P6. *Proof of the formula of variation of constants*.

Consider the system described by the linear state equation

$$\frac{dx}{dt} = Ax + bu \text{, with initial condition } x(0),$$

where $x \in R^n$ (column vector) is the state, $u \in R$ (scalar), is the input, $t \in R$ denotes time, and $A \in R^{n \times n}$ and $b \in R^n$ are matrices of parameters. Using the change of variables

$$x(t) = e^{At} z(t),$$

where $z \in R^n$ is a new state variable, get an expression for the solution of the state equation as a function of t, of the initial condition, the input and of the matrices that define the system.

Aids: $\frac{d}{dt} e^{At} = Ae^{At}$, $\frac{d}{dt}(M(t)N(t)) = \dot{M}N + M\dot{N}$