2. Background on finite and infinite dimensional optimization.

Objective: *Prepare the ground by introducing (reviewing) optimality conditions in finite dimensional problems that will then be generalized to infinite dimensional problems and later applied to CV problems.*

Ref.: [L2012] ch. 1, pp. 3-25.

Slides reviewed 2019

Finite dimensional optimization

- [L2, pp.012], pp. 3-17
- [F1968], Ch. 3 (*Differential Calculus of* $\mathbb{R}^n$), pp. 77-167. Section 3.14 is devoted to *Extrema for* $\mathbb{R}^n - \mathbb{R}$ *functions*. This material may also be found in many Vector Calculus books. Select your favorite one and give a look. What is peculiar in Fadell's book are its intuitive and clear, yet rigorous, explanations supported on nice pictures. Unfortunately it is an old book that is a bit hard to find

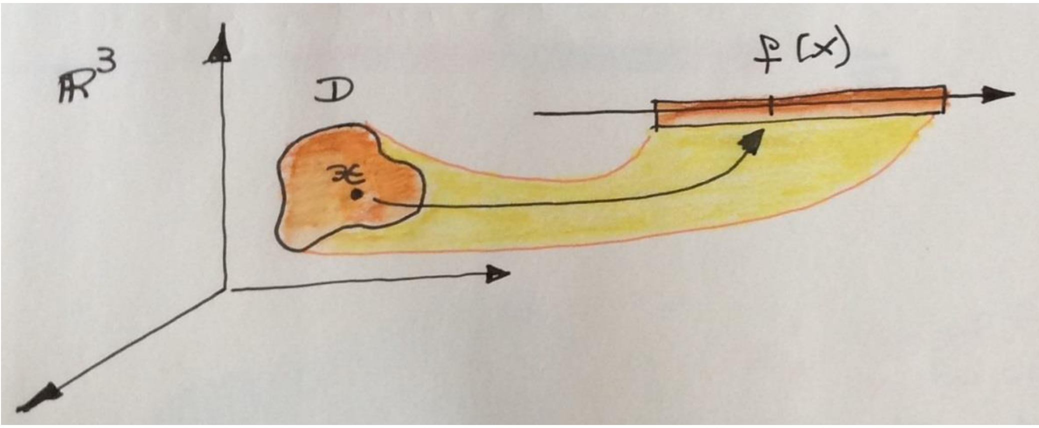
[F1968] A. G. Fadell. *Vector Calculus and Differential equations*, Van Nostrand 1968.

Functions A function $f: \mathbb{R}^n \to \mathbb{R}$

The domain of the function, D a subset of $\mathbb{R}^n$ for which the function is defined.

D might be the entire $\mathbb{R}^n$. Each element of D is a point

Example in $\mathbb{R}^3$:



An example is the Euclidean norm of each point of $\mathbb{R}^n$

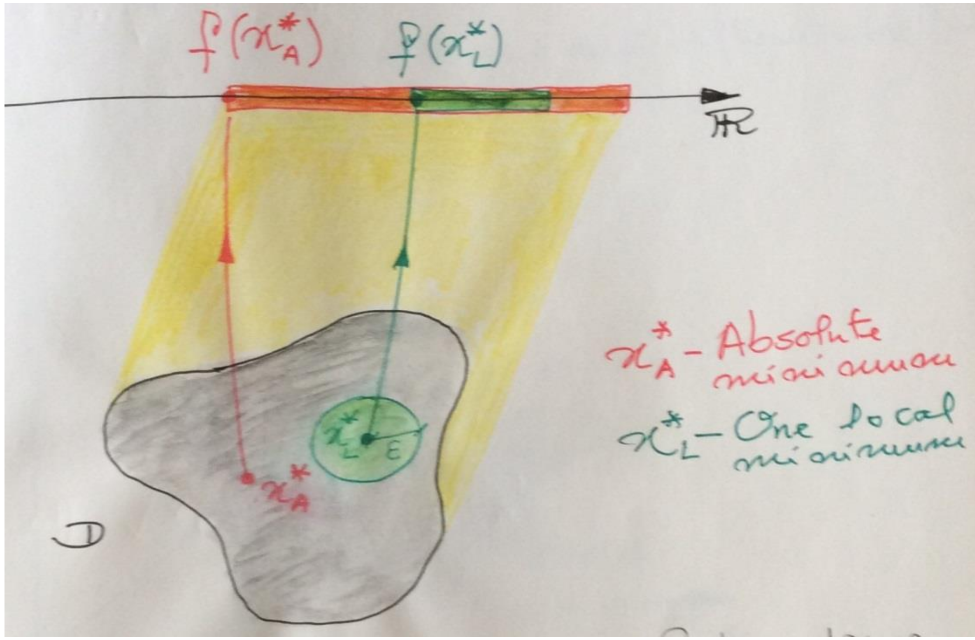
$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Minima

Def. (p.3): $x^* \in D$ is a local minimum over D if

$\exists \varepsilon > 0 : \ \forall x \in D$ such that if $\|x - x^*\| < \varepsilon$ then $f(x^*) \leq f(x)$

x^* is a local minimum if in some ball around it, f does not attain a value smaller than $f(x^*)$.



Strict local minimum: $f(x^*) < f(x)$

Global minimum:

$$\forall x \in D, f(x^*) \leq f(x)$$

First order necessary conditions for optimality p. 4, 5

f a C^1 (continuous, with continuous derivatives) function

x^* a local minimum

Pick a vector d

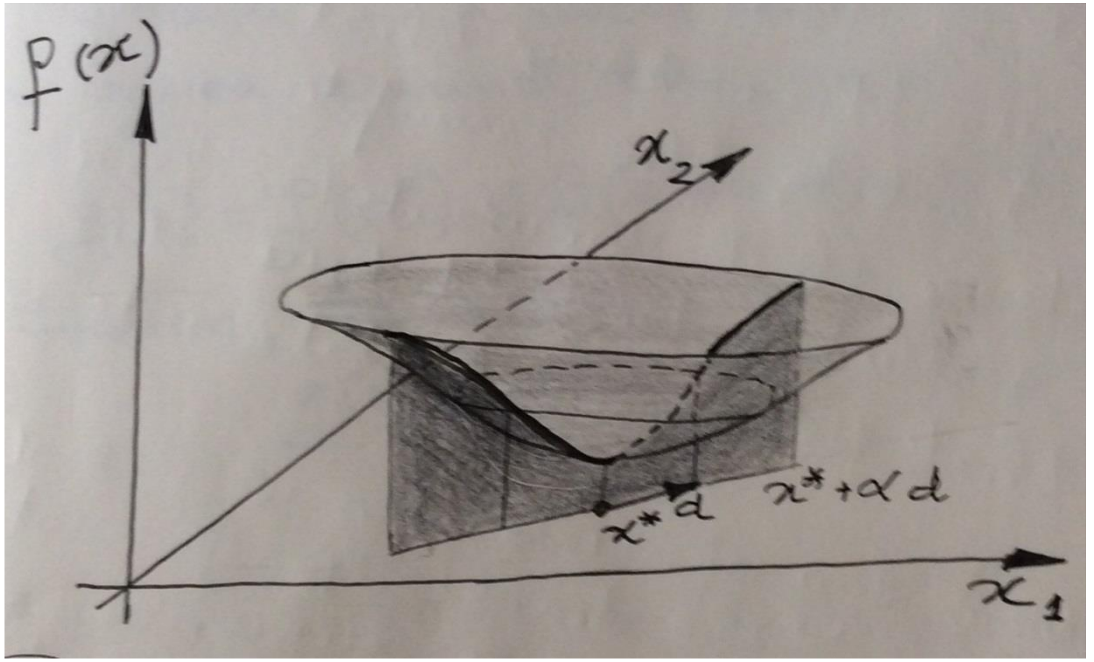
Define the scalar function g by

$$g(\alpha) \coloneqq f(x^* + \alpha d)$$

Since x^* is a local minimum of f,

then $\alpha = 0$ is a local minimum of

$g(\alpha)$. (prove!)



Exercise 1

Let $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and x^* a local minimum in the interior of D.

Define the function g by

$$g(\alpha) \coloneqq f(x^* + \alpha d),$$

where d is a constant.

Show that $\alpha = 0$ is a local minimum of g.

Solution

Since x^* is a local minimum of f (by definition of local minimum), in a region around this point

$$g(0) = f(x^*) \leq f(x^* + \alpha d) = g(\alpha).$$

Hence, for $\alpha \neq 0$, sufficiently small,

$$g(0) \leq g(\alpha).$$

q.e.d.

$$g(\alpha) \coloneqq f(x^* + \alpha d),$$

First order expansion of g around $\alpha = 0$

$$g(\alpha) = g(0) + g'(0)\alpha + o(\alpha)$$

The higher order terms $o(\alpha)$ verify (by definition)

$$\lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0$$

We claim that

$$g'(0) = 0$$

The proof is made by <u>contradiction</u>.

Method of proof by contradiction

- Assume that the contrary of what you want to prove holds true. This is the so-called absurd assumption.
- From this assumption show that the fact that contradicts the assumptions of the sentence that you want to prove can be concluded to hold true.
- This means that the original statement must be true.

This logic way of reasoning is valid although it has the drawback of not being constructive.

It is remarked that, because of this fact, proof by contradiction is not accepted by the intuitionist school of Mathematics. Indeed, for the intuitionists, the claim that an object with certain properties exists is equivalent to claim that an object with those properties can be constructed. But this is another story…

Proof by contradiction that $g'(0) = 0$ in $g(\alpha) = g(0) + g'(0)\alpha + o(\alpha)$

Suppose (absurd assumption) that $g'(0) \neq 0$.

$$g(\alpha) - g(0) = g'(0)\alpha + o(\alpha)$$

There exists $\varepsilon > 0 : \forall |\alpha| < \varepsilon$, then $\left|\frac{o(\alpha)}{\alpha}\right| < |g'(0)|$

Hence

$$g(\alpha) - g(0) < g'(0)\alpha + |g'(0)\alpha|$$

Further restrict α to have the negative sign to $g'(0)$. For these values of α

$$g(\alpha) - g(0) < 0 \quad \text{or} \quad g(\alpha) < g(0)$$

that contradicts the assumption that $\alpha = 0$ is a minimum. Therefore, it must be

$$g'(0) = 0$$ *q.e.d.*

Re-express now this result in terms of the original function f.

By the chain rule of derivatives

$$g'(\alpha) = \frac{d}{d\alpha} f(x^* + \alpha d) =$$

$$= \frac{\partial f}{\partial x_1} d_1 + \frac{\partial f}{\partial x_2} d_2 + \cdots + \frac{\partial f}{\partial x_n} d_n = \nabla f(x^* + \alpha d).\, d$$

Thus, since $g'(0) = 0$, it follows that $\nabla f(x^*).\, d = 0$

Since this equality is valid for all d, we conclude that

$$\nabla f(x^*) = 0$$

First order necessary conditions for optimality

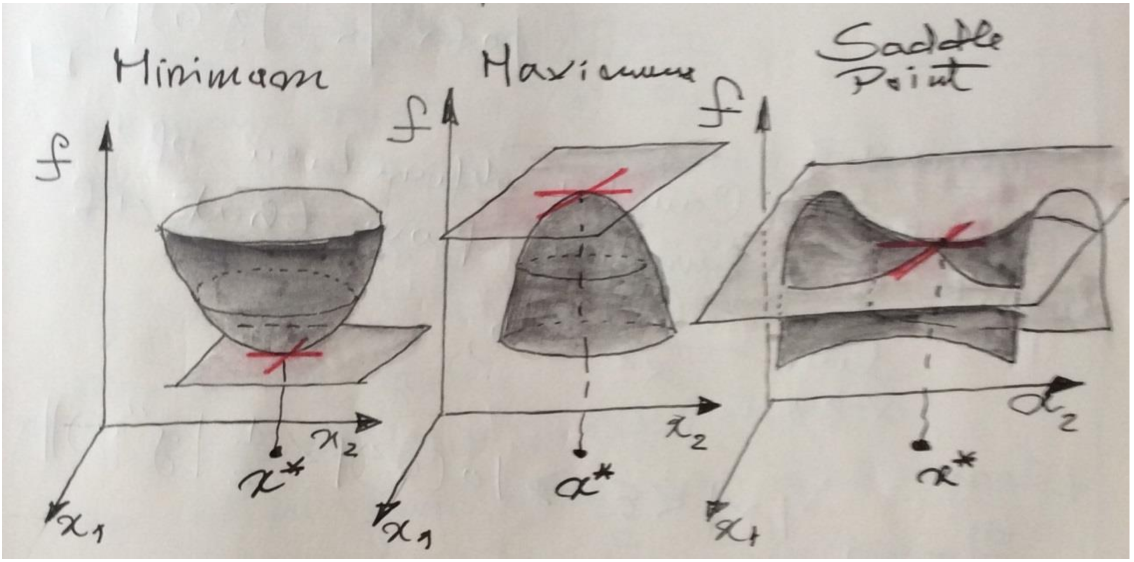
x^* an interior point of D

$f \in C^1$

$$\nabla f(x^*) = 0$$

Stationary points

Points satisfying $\nabla f(x^*) = 0$ are called stationary points



Stationary points are not just minima. They comprise minima, maxima and saddle/inflection points.

Second-order necessary condition for optimality p. 6, 7

$f \in C^2$ (twice continuous differentiable)

x^* a local minimum in the interior of D

$$g(\alpha) := f(x^* + \alpha d),$$

Second order Taylor expandion

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2)$$

$$\lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0$$

Caychy definition of limit of a real function of real variable

$$f : \mathbb{R} \to \mathbb{R}$$

We say that the limit of $f(x)$ is b when $x \to a$ iff

$$\forall \delta > 0 \ \exists \varepsilon > 0 : \ |x - a| < \varepsilon \ \Rightarrow |f(x) - b| < \delta$$

An alternative definition of limit of a function is the Heine definition. A function f has a limit b when $x \to a$ iff for every sequence $\{x_n\}$, that has a limit a, the sequence $\{f(x_n)\}$ has the limit b. The Heine and Cauchy definitions of limit of a function are equivalent.

END DETOUR

Exercise 2

Prove that, if

$$\lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0,$$

then

$$\exists \varepsilon > 0 : \ |\alpha| < \varepsilon \ \Rightarrow \ |o(\alpha^2)| < \frac{1}{2} |g''(0)| \alpha^2$$

Hint: Use the Cauchy definition of limit of a function and make a suitable choice for the δ appearing on it.

$$\lim_{\alpha \to 0} \frac{o(\alpha^2)}{\alpha^2} = 0,$$

By definition of limit

$$\forall_{\delta>0}\ \exists_{\varepsilon>0}\colon\ |\alpha| < \varepsilon\ \Rightarrow \frac{|o(\alpha^2)|}{\alpha^2} < \delta$$

In particular, if we choose

$$\delta = \frac{1}{2}|g''(0)|,$$

then,

$$\exists_{\varepsilon>0}\colon\ |\alpha| < \varepsilon\ \Rightarrow \frac{|o(\alpha^2)|}{\alpha^2} < \frac{1}{2}|g''(0)| \quad \text{or} \quad |o(\alpha^2)| < \frac{1}{2}|g''(0)|\alpha^2.$$

q.e.d.

$$g(\alpha) \coloneqq f(x^* + \alpha d),$$

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2}g''(0)\alpha^2 + o(\alpha^2)$$

By the first order necessary conditions, $g'(0) = 0$:

$$g(\alpha) = g(0) + \frac{1}{2}g''(0)\alpha^2 + o(\alpha^2)$$

We claim that

$$g''(0) \geq 0$$

Proof by contradiction

$$g(\alpha) = g(0) + \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2) \quad \rightarrow \quad g(\alpha) - g(0) = \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2) \quad (*)$$

We have shown that $\exists_{\varepsilon>0}$: $|\alpha| < \varepsilon$ implies $|o(\alpha^2)| < \frac{1}{2}|g''(0)|\alpha^2$

For these values of α, (*) yields

$$g(\alpha) - g(0) < \frac{1}{2} g''(0)\alpha^2 + \frac{1}{2}|g''(0)|\alpha^2 = \frac{1}{2}\alpha^2(g''(0) + |g''(0)|)$$

Assume that $g''(0) < 0$ (absurd assumption). Then it would be

$$g(\alpha) - g(0) < 0 \quad \text{or} \quad g(\alpha) < g(0)$$

A conclusion that contradicts that $\alpha = 0$ is a minimum. Therefore, $g''(0) < 0$ may not be true, and the conclusion follows.

q.e.d.

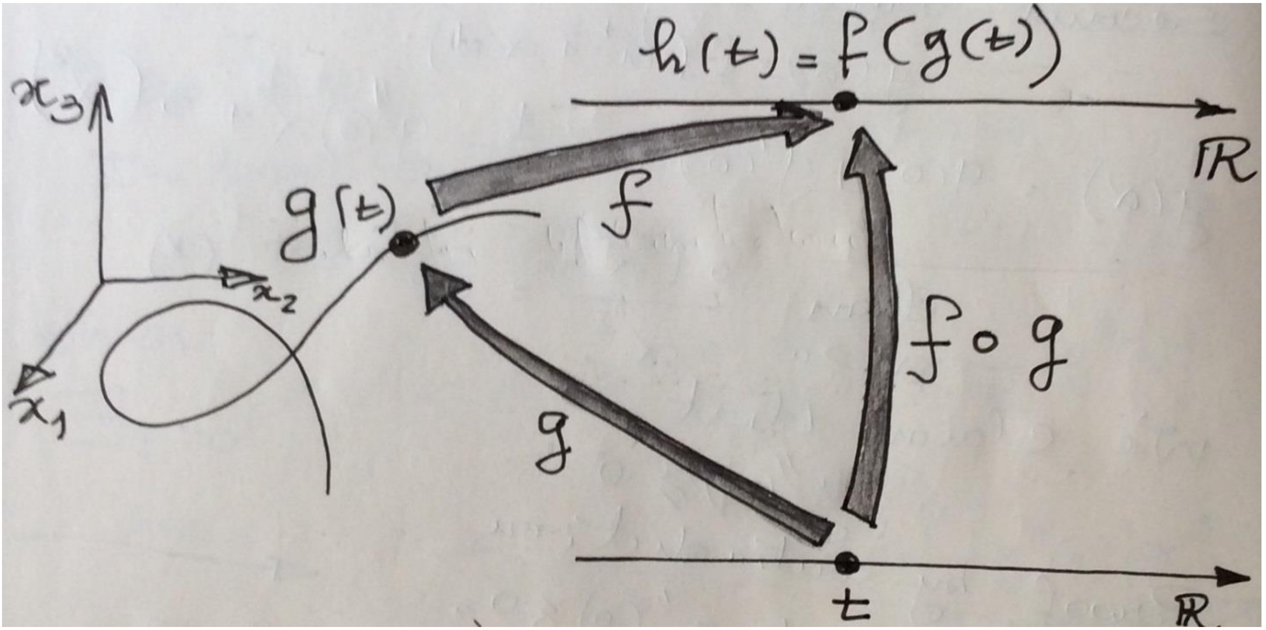
The general chain rule for derivatives [F1968] p. 127, 128

$g: \mathbb{R} \rightarrow \mathbb{R}^n$

$f: \mathbb{R}^n \rightarrow \mathbb{R}$

$h(t) = f(g(t))$

$h: \mathbb{R} \rightarrow \mathbb{R}$



$$h'(t) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f(g(t)) \frac{d}{dt} g_i(t)$$ also written as $$h'(t) = \nabla f\big(g(t)\big) . g'(t)$$

END DETOUR

What is the implication of $g''(0) > 0$ on the minima of the function f?

[L2012] p. 6, 7

$$g(\alpha) \coloneqq f(x^* + \alpha d)$$

$$g'(\alpha) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x^* + \alpha d) d_i$$

$$g''(\alpha) = \sum_{i,j}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(x^* + \alpha d)\, d_i d_j$$

For $\alpha = 0$,

$$g''(0) = \sum_{i,j}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(x^*)\, d_i d_j$$

$$g''(0) = \sum_{i,j}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(x^*)\, d_i d_j$$

In matrix notation, this expression becomes

$$g''(0) = d^T \nabla^2 f(x^*) d$$

where the Hessian matrix of f is

$$\nabla^2 f(x^*) := \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \dfrac{\partial^2 f}{\partial x_n \partial n} \end{bmatrix}$$

Since

$$g''(0) = d^T \nabla^2 f(x^*) d \geq 0$$

And that this inequality holds for any d, if follows that the Hessian must be positive semidefinite at a minimum that is an interior point:

$$\nabla^2 f(x^*) \succcurlyeq 0$$

Second-order necessary condition for optimality [L2012] p. 6, 7

$f \in C^2$ (twice continuous differentiable)

x^* a local minimum in the interior of D

$$\nabla^2 f(x^*) \succcurlyeq 0$$

The Hessian must be positive semidefinite at a minimum that is an interior point.

2nd order sufficient conditions for optimality [L2012] p. 7

$$f \in C^2 \qquad f : D \to \mathbb{R}, \qquad x^* \in int(D)$$

$$\nabla f(x^*) = 0 \quad \text{and} \quad \nabla^2 f(x^*) \succ 0$$

Then, x^* is a strict local minimum.

If an interior point is such that the gradient vanishes and the hessian is positive definite, then the point is a strict local minimum.

Proof

Take an arbitrary d.

$$g(\alpha) \coloneqq f(x^* + \alpha d)$$

$$g(\alpha) = g(0) + g'(0)\alpha + \frac{1}{2} g''(0)\alpha^2 + o(\alpha^2)$$

Because of the assumption on the gradient

$$g'(0) = \nabla f(x^*) \cdot d = 0$$

$$g''(0) = d^T \nabla^2 f(x^*) d$$

Hence, the 2nd order Taylor expansion becomes

$$f(x^* + \alpha d) = f(x^*) + \frac{1}{2} d^T \nabla^2 f(x^*) d \alpha^2 + o(\alpha^2)$$

$$f(x^* + \alpha d) = f(x^*) + \frac{1}{2} d^T \nabla^2 f(x^*) d\alpha^2 + o(\alpha^2)$$

$$f(x^*) = f(x^* + \alpha d) - \frac{1}{2} d^T \nabla^2 f(x^*) d\alpha^2 + (-o(\alpha^2))$$

From Exercise 2, there is an interval of values of α such that

$$|o(\alpha^2)| < \frac{1}{2} d^T \nabla^2 f(x^*) d\alpha^2$$

Hence, for any d,

$$f(x^*) < f(x^* + \alpha d) - \frac{1}{2} d^T \nabla^2 f(x^*) d\alpha^2 + \frac{1}{2} d^T \nabla^2 f(x^*) d\alpha^2 = f(x^* + \alpha d)$$

$$f(x^*) < f(x^* + \alpha d),$$

and hence x^* is a strict local minimum. *Q.e.d.*

Exercise 4

The distance of a point A from a plane Π is defined as the minimum of the set of distances between the point A and all the points in the plane.

Find the distance from the point (-1, 4, 2) to the plane

$$\Pi = \{(x_1, x_2, x_3 : 2x_1 - 3x_2 + x_3 - 7 = 0\}$$

Hints:

i) Since distance is non-negative, it is equivalent to minimize the square of the distance, given by $F(x_1, x_2, x_3) = (x_1 + 1)^2 + (x_2 - 4)^2 + (x_3 - 2)^2$, where $(x_1, x_2, x_3) \in \Pi$.

ii) To minimize F while satisfying the constraint that $(x_1, x_2, x_3) \in \Pi$ you may simply express x_3 in terms of x_1 and x_2 using the plane equation to obtain a function $\varphi(x_1, x_2)$ that yields the distance from A to Π as a function of just x_1 and x_2.

iii) To show that the hessian matrix is positive definite, you may use Sylvester's criterium.

Sylvester's criterium

A symmetric matrix with real entries M is positive-definite if and only if all of the principal minors have a positive determinant. The principal minors are

- the upper left 1-by-1 corner of M,
- the upper left 2-by-2 corner of M,
- the upper left 3-by-3 corner of M,
- …
- M itself.

Solution of exercise 4

$$\varphi(x_1, x_2) = (x_1 + 1)^2 + (x_2 - 4)^2 + (5 - 2x_1 + 3x_2)^2$$

Compute the gradient and equate to zero

$$\frac{\partial \varphi}{\partial x_1} = 2(x_1 + 1) - 4(5 - 2x_1 + 3x_2) = 0$$

$$\frac{\partial \varphi}{\partial x_2} = 2(x_2 - 4) + 6(5 - 2x_1 + 3x_2) = 0$$

The solution of this system of equations is $\left(\frac{12}{7}, -\frac{1}{14}, \frac{47}{14}\right)$

The distance between A and Π is obtained by computing the square-root of either F or φ at this point, being $19/\sqrt{14}$.

Furthermore, the elements of the hessian matrix are

$$\frac{\partial^2\varphi}{\partial x_1^2} = 10 \qquad \frac{\partial^2\varphi}{\partial x_1 \partial x_2} = -12 \qquad \frac{\partial^2\varphi}{\partial x_2 1} = -12 \qquad \frac{\partial^2\varphi}{\partial x_2^2} = 20$$

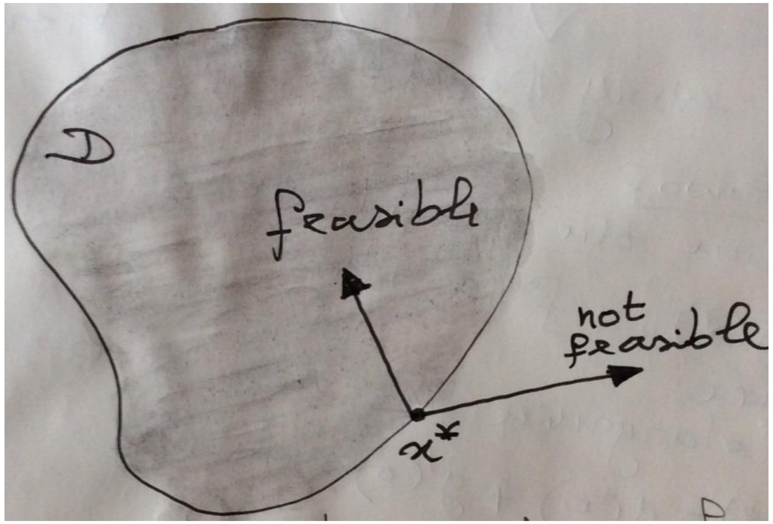
Since the determinant of the hessian is

$$\begin{vmatrix} 10 & -12 \\ -12 & 20 \end{vmatrix} = 200 - 144 = 56 > 0,$$

from the Sylvester criterion the hessian is positive definite, and the point that corresponds to the zero of the gradient is actually a minimum.

Feasible directions [L2012] p. 8

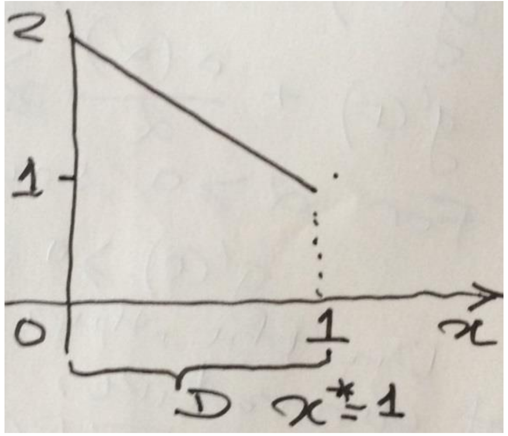
$d \in \mathbb{R}^n$ is a feasible direction at x^* if $x^* + \alpha d \in D$ for small enough $\alpha > 0$.



If not all directions d at a minimum x^* are not feasible, then the condition that the gradient vanishes, $\nabla f(x^*) = 0$ is no longer necessary for optimality.

Example

$x \in [0,1] \subset \mathbb{R}$, $f(x) = 2 - x$



The minimum is $x^* = 1$, which is not an interior point to D.

However, at the minimum, $\nabla f(x^*) = -1 \neq 0$.

Exercise 7 1st order necessary condition for minimum

Prove that if x^* is a local minimum of f (not necessarily in the interior of D), then

$$\nabla f(x^*) \cdot d \geq 0$$

for every feasible direction d.

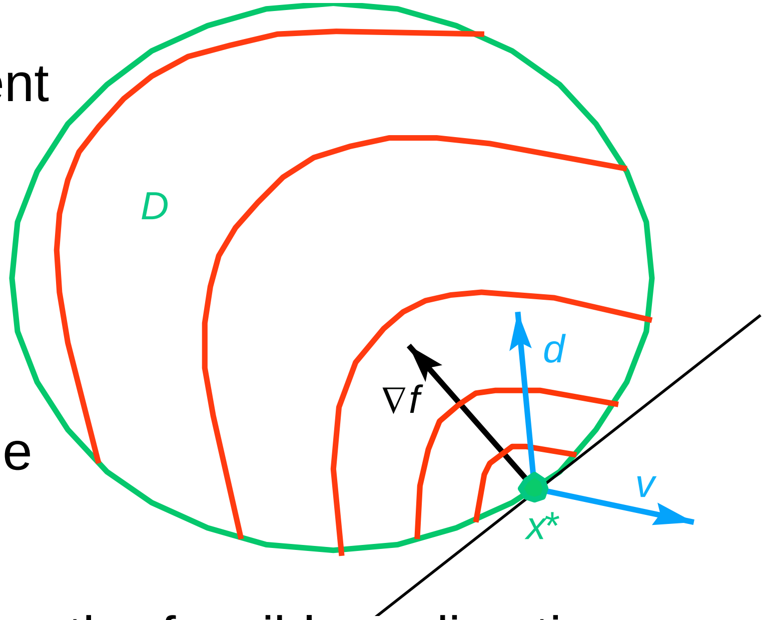
Hint: Modify the argument used in the 1st order necessary condition for minimum at an interior point.

Geometrical interpretation

For an interior point minimum, the gradient vanishes.

Consider a minimum at the boundary.

For the point to be a minimum, the function must grow when you move away from the minimum along any feasible direction



Since the gradient points to an ascent position, the feasible directions, such as d, must be "somehow aligned" with it, requiring that the internal product is positive. Non-feasible directions such as v, point in descent directions, and the internal product is negative.

Solution of Exercise 7

With d any feasible direction, define the function g by

$$g(\alpha) \coloneqq f(x^* + \alpha d)$$

Since

$$x^* = argmin\, f(x) \Rightarrow \alpha = 0 = argmin\, g(\alpha)$$

First order Taylor expansion

$$g(\alpha) - g(0) = g'(0)\alpha + o(\alpha)$$

Since $\alpha = 0$ is a minimum, $g(\alpha) - g(0) \geq 0$, and $g'(0)\alpha + o(\alpha) \geq 0$ or

$$g'(0) + \frac{o(\alpha)}{\alpha} \geq 0$$

For $\alpha > 0$ small enough, it is thus $g'(0) \geq 0$.

$$g(\alpha) \coloneqq f(x^* + \alpha d)$$

On the other way, by the chain rule of derivatives

$$g'(\alpha) = \nabla f(x^* + \alpha d) \cdot d$$

For $\alpha = 0$ this becomes

$$g'(0) = \nabla f(x^*) \cdot d$$

And since $g'(0) \geq 0$,

$$\nabla f(x^*) \cdot d \geq 0.$$

Q. e.d.

Exercise 8 – 2nd order necessary conditions for minimum

Prove that if x^* is a local minimum of f (not necessarily in the interior of D), then

$$d^T \nabla^2 f(x^*) \cdot d \geq 0$$

for every feasible direction d that satisfies

$$\nabla f(x^*) \cdot d = 0$$

Hint: Modify the argument used in the 2st order necessary condition for minimum at an interior point.

Proof: 2nd Series of homework problems.

Compact sub-sets of $\mathbb{R}^n$

$D \subset \mathbb{R}^n$ is compact if any of the following conditions apply:

1. D is closed and bounded
2. Every open cover of D has a finite subcover
3. D every sequence in D has a subsequence converging to some point in D

Of the above criteria, only 2) and 3) carry over to infinite dimensional sets.

A cover of a set X is a collection of sets whose union contains X.
A subcover of X is a subset of a cover of X that still covers .

Is $\{x: 1 \leq x \leq 10,\ \ and\ x\ rational\}$ a compact subset of $\mathbb{R}$?
Is $[1, 10]$ a compact subset of $\mathbb{R}$?
Is $]1, 10[$ a compact subset of $\mathbb{R}$?

Global minima: Weierstrass theorem [L2012] p. 9,10

If f is a continuous function and D is a compact set, then there exists a global minimum of f over D.

Procedure to find a global minimum

1. Find all the interior points of D that satisfy $\nabla f(x^*) = 0$ (stationary points).
2. If f is not differentiable, everywhere, include also points where ∇f does not exist (these points, together with the stationary points comprise the critical points).
3. Find all boundary points that satisfy $\nabla f(x^*) \cdot d \geq 0$ for all feasible d.
4. Compare the values of the function at all these candidate points and choose the smallest one.

Convex functions

A special case in which the minimum can be shown to be unique are convex functions.

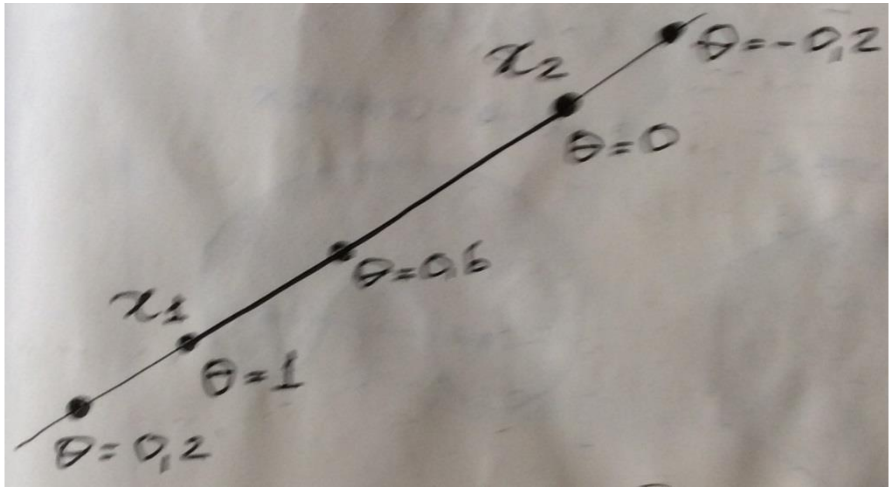
[BV2004], chs. 2, 3.

[BV2004] S. Boyd and L. Vanderberghe. *Convex Optimization*. Cambridge University Press, 2004.

This book is available in electronic form at http://stanford.edu/~boyd/cvxbook/

Parametrization of a line segment between two points

[BV2004], p.22



Line passing through x_1 and x_2 described parametrically by

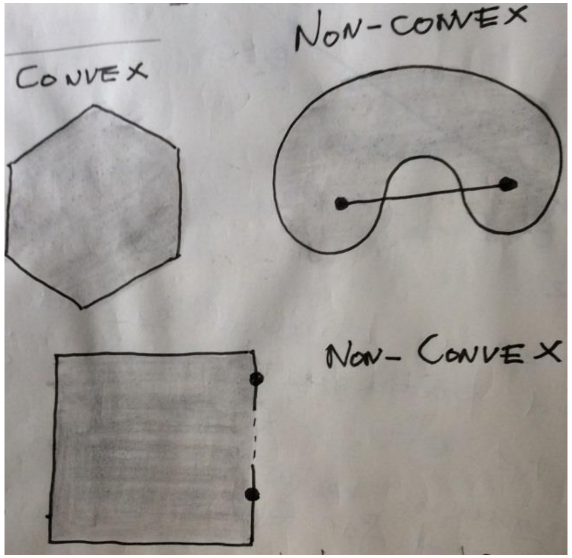
$$\theta x_1 + (1-\theta)x_2,\ \theta \in \mathbb{R}$$

The segment between x_1 and x_2 is obtained for $\theta \in [0,1]$.

Convex sets [BV2004]

A set C is convex if the line segment between any two points in C lies in C, *i.e.*

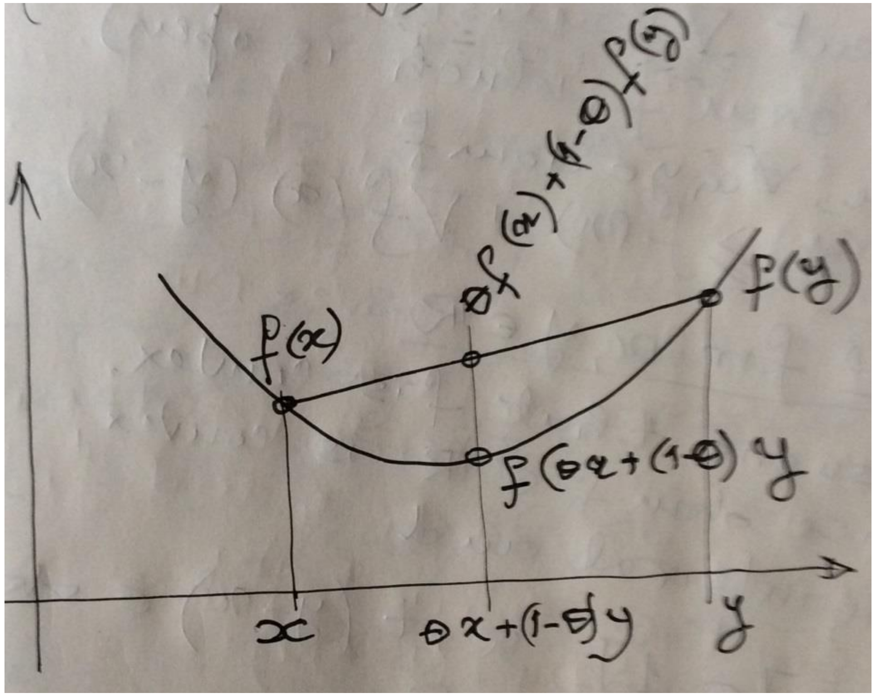
C is convex iff $\forall_{x_1,x_2 \in C}, \forall_{\theta \in [0,1]}$, then $\theta x_1 + (1-\theta) x_2 \in C$



Convex functions [BV2004], p. 67

A function $f: \mathbb{R}^n \rightarrow \mathbb{R}$ is a convex function iff its domain is a convex set and if for all $x, y \in dom\, f$, then

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)$$



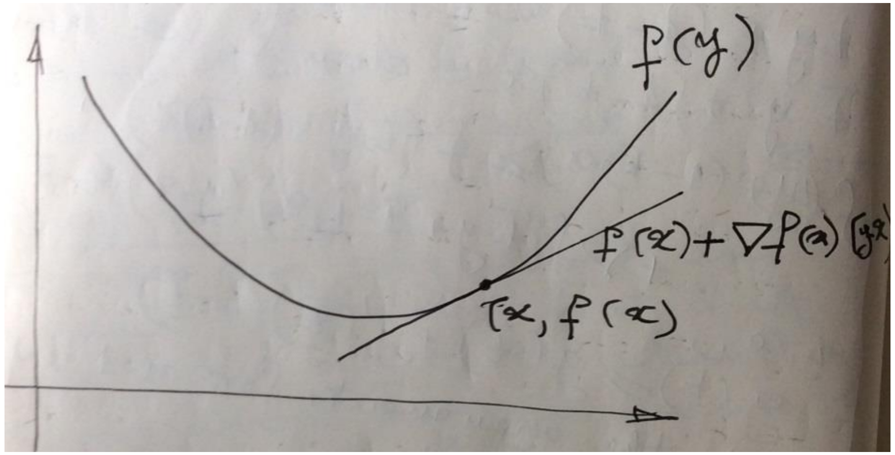
First order convexity condition [BV2004] p.69, 70

Assume that

1) $dom\ f$ is convex and open;

2) f is differentiable (∇f exists at each point in $dom\ f$, which is open;

Then, $\forall x, y \in dom\ f$, $f(y) \geq f(x) + \nabla f(x)(y - x)$



Proof: See companion document.

Uniqueness of the minimum of a C^1convex function

Let $f: D \subset \mathbb{R}^n \to \mathbb{R}$ be a C^1 convex function.

Consider a stationary point, i.e., a point x^* for which $\nabla f(x^*) = 0$.

By the first order convexity condition

$$f(y) \geq f(x^*) + \nabla f(x^*)(y - x^*) = f(x^*) + 0.(y - x^*) = f(x^*)$$

Hence

$$f(y) \geq f(x^*) \quad \forall y \in D,$$

meaning that x^* is the global minimum of f on D.

For a convex function f, the condition $\nabla f(x^*) = 0$ is both necessary and sufficient for minimum.

The minimum of f always exists and is either an interior point, or a boundary point if D is closed.

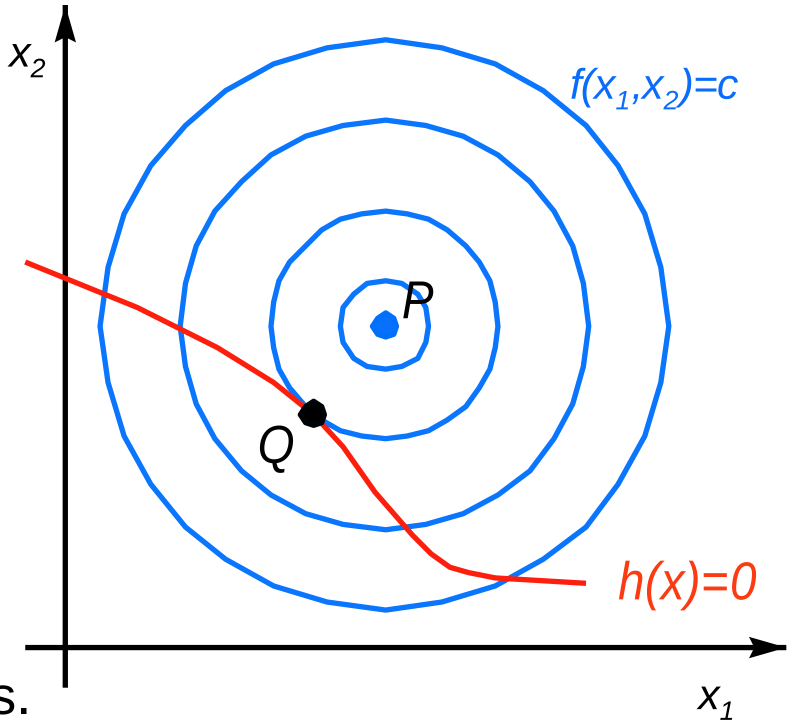
Constraints

Equality constraints: An example in $\mathbb{R}^2$

The unconstrained minimum of f is P.

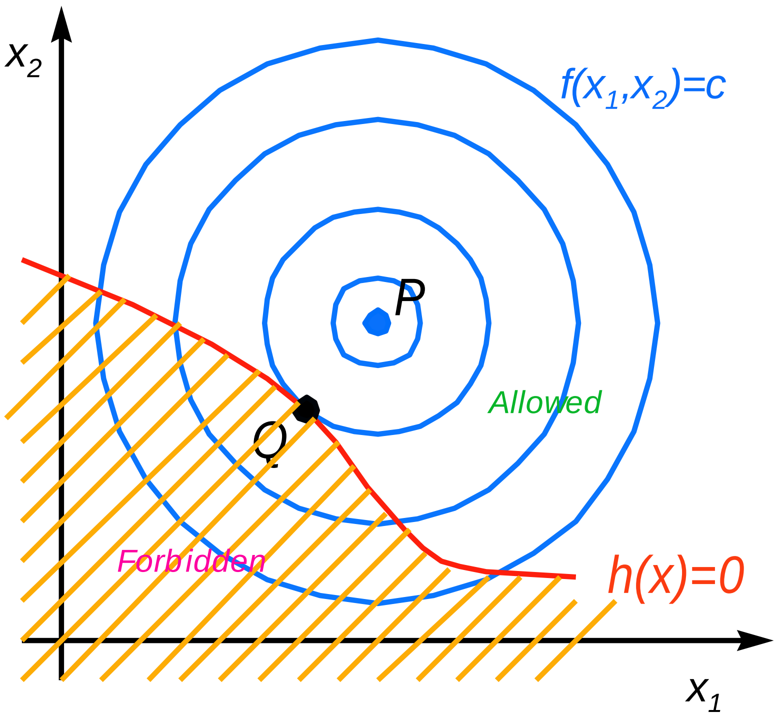
When x is constrained to be on the line defined by $h(x) = 0$, the constrained minimum is point Q.in

Point Q results from the intersection of the level curves of f with the line $h(x) = 0$, when this line is tangent to the level curves.

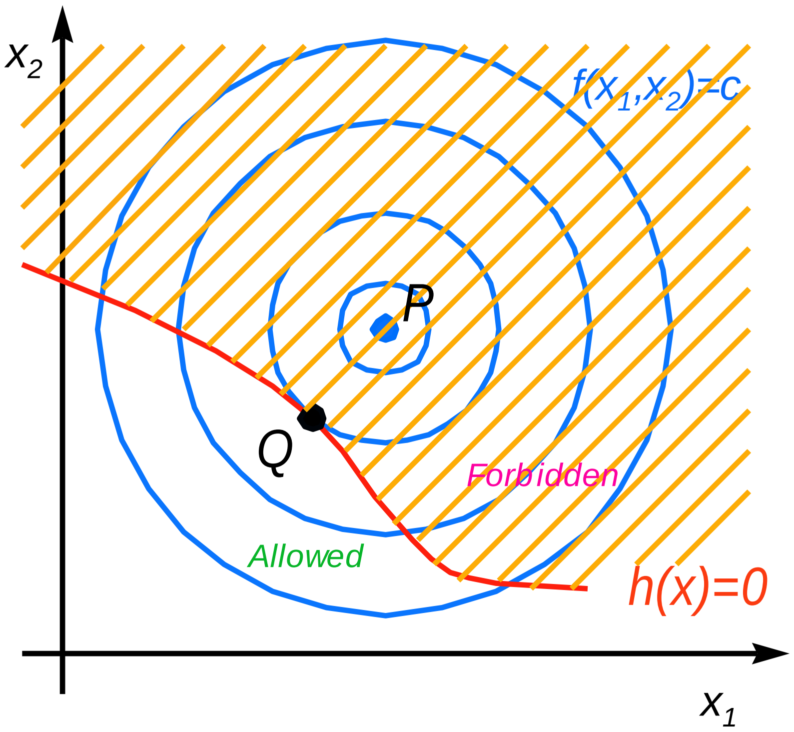


The MATLAB function *contour* plots the level curves of functions defined in $\mathbb{R}^2$

Inequality constrains



(A)



(B)

For the moment, in the sequel we will study only equality constraints

In situation *A* the unconstrained minimum P is in the allowed region and the constrained minimum is equal to it. The constraint is not active.

In situation B the unconstrained minimum P is in the forbidden region and the constrained minimum Q is at the boundary of the allowed region. The constraint is active.

Constrained optimization [L2012] p. 11-16

$D \subset \mathbb{R}^n$ a “surface” defined by the equality constraints

$$h_1(x) = h_2(x) = \cdots = h_m(x) = 0$$

$h_i: \mathbb{R}^n \rightarrow \mathbb{R}, \quad i = 1, \dots, m \quad C^1$ functions

f a C^1 function

Objective: Study the minima of f over D (constrained minima)

Regular points

$x^* \in D$ a local minimum

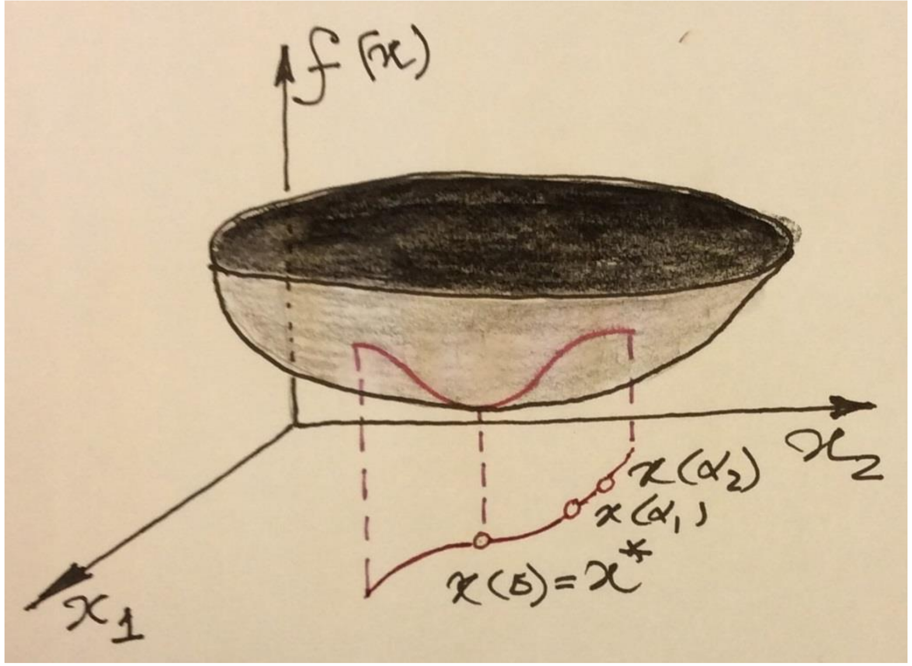
x^* is be assumed to a regular point. This means that $\nabla h_i, i = 1, \ldots, m$ are linearly independent at x^*.

This assumption rules out degenerate situations in which the necessary condition for minima to be presented may not hold.

Generalization of directional derivatives

Instead of straight line segments, use curves in D that pass through x^*.

A family of points $x(\alpha) \in D$ parameterized by $\alpha \in \mathbb{R}$, with $x(0) = x^*$ and $x(\cdot)$ a C^1 function for α near 0.



$$g(\alpha) \coloneqq f(x(\alpha))$$

Exercise 11

Prove that $\alpha = 0$ is a minimum of g and conclude that $g'(0) = 0$.

Hints: Use the definition of $g(\alpha)$ and the fact that $x(\alpha) = x^*$ for $\alpha = 0$.

Proof

Let $\alpha \neq 0$.

$$g(\alpha) = f\big(x(\alpha)\big) \geq f(x^*) = g(0)$$

And hence $g(\alpha) \geq g(0)$ for $\alpha \neq 0$, meaning that $\alpha = 0$ is a minimum of g.

Since $\alpha = 0$ is a minimum of g, it follows that $g'(0) = 0$ and since

$$g'(\alpha) = \nabla f(x(\alpha)) \cdot x'(\alpha),$$

for $\alpha = 0,\ g'(0) = \nabla f(x(0)) \cdot x'(0)$ and hence

$$\nabla f(x^*)x'(0) = 0$$

This expression will be used later.

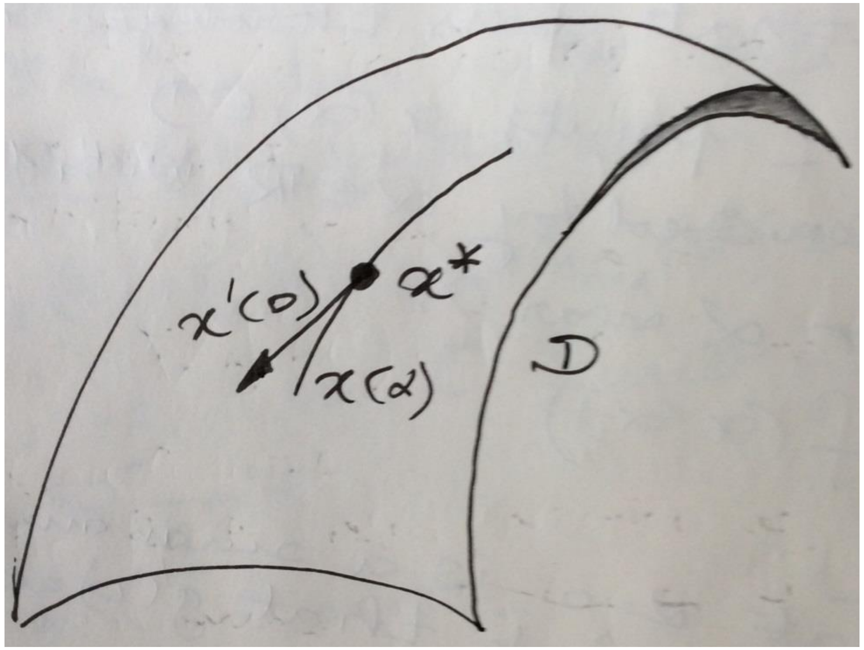
Tangent space

$x'(0)$ defines a linear approximation of $x(\cdot)$ at x^*:

$$x(\alpha) = x^* + x'(0)\alpha + o(\alpha)$$

$x'(0)$ is a tangent vector of D at x^*.

It leaves in $T_{x^*}D$, the tangent space to D at x^*



Characterization of the tangent space

Since D is defined as the set of points that satisfy the constraints, and since $x(\alpha)$ satisfies the constraints,

$$h_i\big(x(\alpha)\big) = 0 \qquad \forall_\alpha, \quad i = 1, \dots, m$$

Differentiating

$$0 = \frac{d}{d\alpha} h_i\big(x(\alpha)\big) = \nabla h_i(x(\alpha)) \cdot x'(\alpha), \qquad i = 1, \dots, m$$

for all α close to 0.

For $\alpha = 0$ (remember that $x(0) = x^*$):

$$0 = \frac{d}{d\alpha}|_{\alpha=0} h_i\big(x(\alpha)\big) = \nabla h_i(x^*) \cdot x'(0), \quad i = 1, \dots, m$$

$$0 = \frac{d}{d\alpha}|_{\alpha=0} h_i\big(x(\alpha)\big) = \nabla h_i(x^*) \cdot x'(0), \quad i = 1, \dots, m$$

This shows that for an arbitrary curve $x(\cdot)$ in D with $x(0) = x^*$, its tangent vector must satisfy

$$\nabla h_i(x^*) \cdot x'(0) = 0, \text{ for each } i = 1, \dots, m$$

Actually, it is possible to prove that the converse is also true:

Every vector $d \in \mathbb{R}^n$ such that

$$\nabla h_i(x^*) \cdot d = 0, \text{ for each } i = 1, \dots, m$$

Is a tangent vector to D at x^* corresponding to some curve.

Characterization of the tangent space

The tangent space $T_{x^*}D$ is the space generated by the linear combination of the vectors d that satisfy

$$\nabla h_i(x^*) \cdot d = 0, \text{ for each } i = 1, \ldots, m$$

First order necessary conditions of minimum under equality constraints

We have seen that

$$\nabla f(x^*)x'(0) = 0$$

Since $x(\cdot)$ can be chosen in an arbitrary way, it follows that

$$\text{For all } d \in T_{x^*}D, \qquad \nabla f(x^*) \cdot d = 0$$

The first order necessary conditions of minimum under equality constraints my be written

$$\nabla f(x^*) \cdot d = 0 \qquad \forall d\!:\ \nabla h_i(x^*) \cdot d = 0$$

First order necessary conditions of minimum under equality constraints:

$$\nabla f(x^*) \cdot d = 0 \qquad \forall d \colon\ \nabla h_i(x^*) \cdot d = 0$$

We will now prove that these conditions are equivalent to

At a regular point, the gradient of f at x^* is a linear combination of the gradients of the constraint functions $h_1, \dots, h_m$ at x^*:

$$\nabla f(x^*) \in span\{\nabla h_i(x^*), i = 1, \dots, m\}$$

Proof

If the claim is not true, there is a $d \neq 0$ satisfying $\nabla h_i(x^*) \cdot d = 0$, $i = 1, \dots, m$, such that

$$\nabla f(x^*) = d - \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*)$$

for some $\lambda_1^*, \dots, \lambda_m^*$. But then

$$\nabla f(x^*) \cdot d = \left(d - \sum_{i=1}^{m} \lambda_i^* \nabla h_i(x^*) \right) \cdot d = d \cdot d \neq 0$$

That contradicts the assumption that

$$\nabla f(x^*) \cdot d = 0.$$

Q.e.d.

First order necessary condition for constrained optimality

The condition

$$\nabla f(x^*) \in span\{\nabla h_i(x^*), i = 1, \dots, m\}$$

means that, at the regular points, there exist real numbers $\lambda_1^*, \dots, \lambda_m^*$ (known as Lagrange multipliers) such that the 1st order necessary condition for optimality holds:

$$\nabla f(x^*) + \lambda_1^* \nabla h_1(x^*) + \cdots + \lambda_m^* \nabla h_m(x^*) = 0$$

Geometrical interpretation

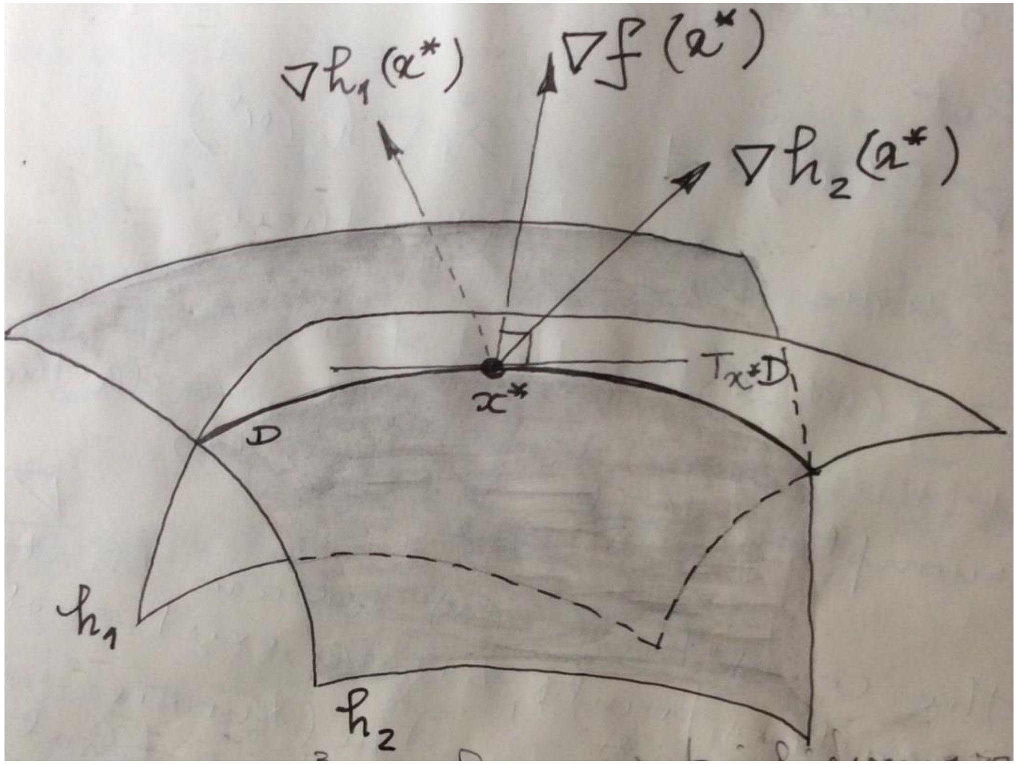
$n = 3$, $m = 2$

$h_1(x) = 0$, $x \in \mathbb{R}^3$ defines a bidimensional surface.

Idem for $h_2(x) = 0$.

The intersection of these two surfaces defines D.

$\nabla f(x^*)$ is a linear combination of $\nabla h_1(x^*)$ and $\nabla h_2(x^*)$ and is orthogonal to the tangent to D at x^*.



The Lagrange dual function

[BV2004] ch. 5

Special case for equality constraints

Primal problem

Minimize $f(x)$ $\qquad f: \mathbb{R}^n \to \mathbb{R}$, $D = dom\, f$

Subject to $h_i(x), i = 1, \dots, m$

Lagrangian function: $\quad L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \quad dom\, L = D \times \mathbb{R}^n$

$$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$$

$\lambda = [\lambda_1 \quad \dots \quad \lambda_m]$ Lagrange multiplier vector or dual variable

Warning: *Don't get confused with the integrand function of the cost in optimal control problems that is also called "Lagrangian function". This is another function, although in some problems they are the same.*

Dual function

$$g(\lambda) = \inf_{x \in D} L(x, \lambda)$$

The dual function yields lower bounds on the optimal value of the primal problem.

The best lower bound is given by the solution of the dual problem

$$\arg\max_{\lambda} \; g(\lambda).$$

KKT (Karush-Kuhn-Tucker) conditions

Lagrangian function $L(x,\lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$

Special case for equality constraints [L2012] p.15

At (x^*, λ^*),

$$\begin{cases} \frac{\partial}{\partial x} L(x,\lambda) = 0 \\ \frac{\partial}{\partial \lambda} L(x,\lambda) = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x) \\ h(x^*) \end{bmatrix} = 0$$

Loosely speaking, adding Lagrange multipliers converts a constrained problem into an unconstrained one.

Warning: There are cases in which the stationary point of the Lagrangian function is not the constrained minimum.

How to solve the KKT conditions

1) In $\frac{\partial L}{\partial x} = 0$ express x in terms of λ;

2) Insert the expression of x in terms of λ in $\frac{\partial L}{\partial \lambda} = h(x) = 0$;

3) Solve the equation on λ that results from step 2) to get the stationary points for λ (λ^*);

4) Go back to the expressions of x in terms of λ obtained in step 1), and use the results of step 3) co cancel λ and obtain x^*.

Exercise 13 (KKT conditions)

Consider the problem

$$\underset{x_1,x_2}{minimize}\ f(x_1,x_2) = x_1 + 1$$

Subject to $h(x_1,x_2) = x_1^2 + x_2^2 - 1 = 0$

a) Write the Lagrangian function

b) Write the KKT conditions

c) Find the stationary points of the Lagrangian function

d) Make a sketch to provide a geometrical interpretation

Solution

a) $L(x_1, x_2, \lambda) = x_1 + 1 + \lambda(x_1^2 + x_2^2 - 1)$

b) $\frac{\partial L}{\partial x_1} = 1 + 2\lambda x_1 = 0$ $\quad \frac{\partial L}{\partial x_2} = 2\lambda x_2 = 0$ $\quad \frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 - 1 = 0$

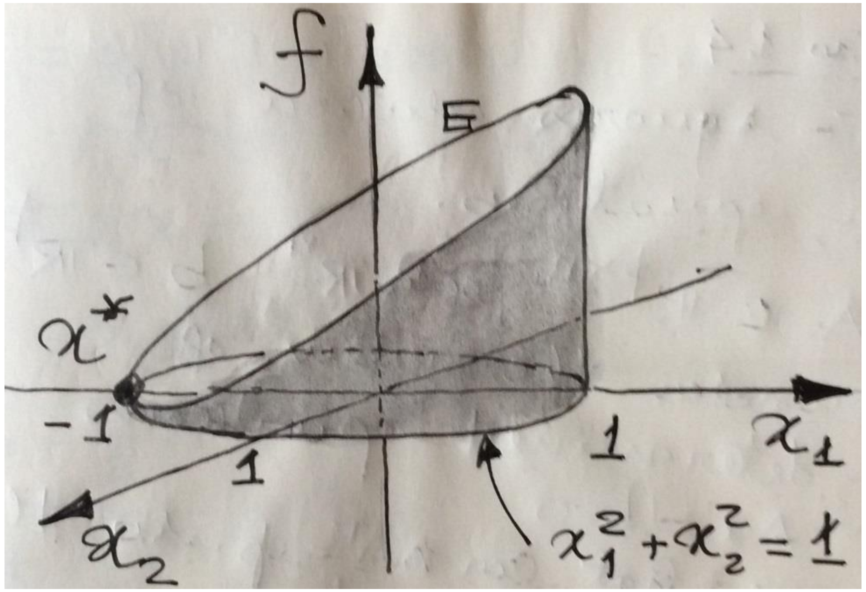
c) From the first two equations: $x_1 = -\frac{1}{2\lambda}$ $\quad x_2 = 0$

Insert these values for x_1 and x_2 in the 3rd equation and solve with respect to λ

$$\frac{1}{4\lambda^2} - 1 = 0 \quad \rightarrow \quad \lambda = \frac{1}{2}$$

Hence, $x_1 = -1$

e)The constraint $x_1^2 + x_2^2 - 1 = 0$ defines a circumference in the (x_1, x_2) plane. The function $f(x_1, x_2) = x_1 + 1$ defines a plane in the space (x_1, x_2, f). The intersection of this plane with the cylinder having as a basis that circumference yields the elipse E, made by points with ordinate given by the value of f at the feasible points (the points of the circumference that satisfy the constraint). The minimum value for the constrained problem is attained at $(-1,0)$, in accordance with the stationary point of the KKT conditions.



Exercise 14 – Least-squares solution of linear equations

Let $A \subset \mathbb{R}^{p\times n}$ a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^p$ vectors.

Consider the system of under-determined equations

$$Ax = b$$

Assume that $rank\ (AA^T) = p$ so that the inverse of AA^T exists.

Since there are infinite values of x that satisfy the equation, one possibility is to look for the minimum norm solution. As such, use the KKT conditions to solve the following minimization problem with equality constraints

$$\text{Minimize } x^T x$$

$$\text{Subject to } Ax - b = 0$$

Solution

$$L(x,\lambda) = x^T x + \lambda^T (Ax - b)$$

$$\nabla_x L = 2x^T + \lambda^T A = 0 \quad \text{or, transposing} \quad 2x + A^T\lambda = 0$$

From which $x = -\frac{1}{2}A^T\lambda$

$$\nabla_\lambda L = x^T A^T - b^T = 0 \rightarrow Ax - b = 0$$

Insert now $x = -\frac{1}{2}A^T\lambda$ in this expression:

$$-\frac{1}{2}AA^T\lambda - b = 0 \quad \rightarrow \quad \lambda = -2(AA^T)^{-1}b$$

$$x = -\frac{1}{2}A^T(-2(AA^T)^{-1}b) \quad \rightarrow \quad x = A^T(AA^T)^{-1}b$$

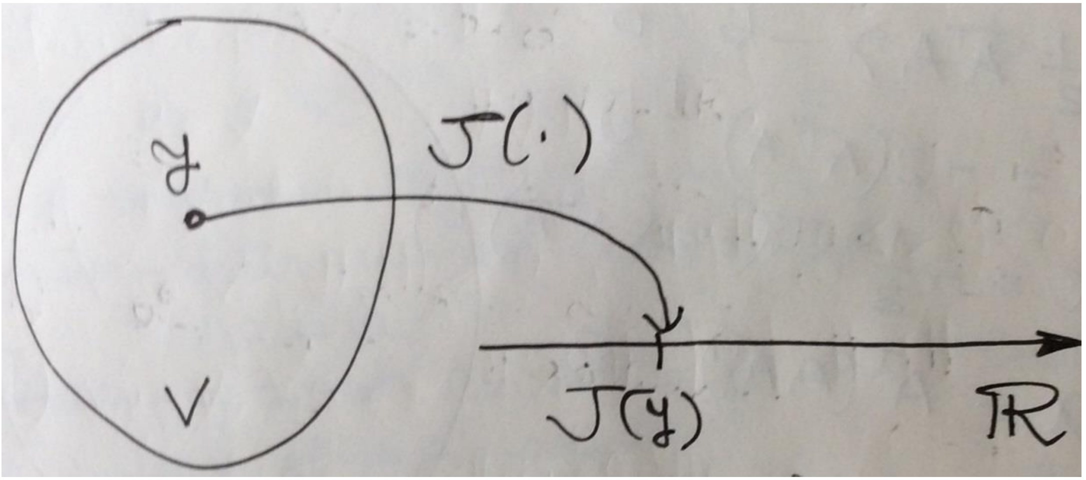
Preview of infinite dimensional optimization problems

- [L2012] p. 17-24.
- [L1969]. This is not the road followed in this course, but is certainly a very interesting (and classical) reference for those who want to start going deeply in infinite dimensional optimization. In its crystalline style that makes difficult issues to look understandable and attractive, Luenberger guides the reader in a functional analysis travel along optimization in infinite dimensional spaces, up to optimal control.

[L1969] D. G. Luenberger. *Optimization by vector space methods*. *John Wiley & Sons, 1969.* Modern reprint by Wiley Interscience.

Functionals

Functionals are maps of a vector space of functions V on $\mathbb{R}_0^+$



V is infinite dimensional.

Many different choices possible for V

Local minima imply the notion of closeness (norm)

In function spaces different choices of norm lead to drastically different notions of closeness.

A few notions on metric and normed spaces

For further examples see

[KREYSIG1978]

E. Kreysig (1978). Introductory Functional Analysis with Applications. Wiley.

Cartesian product of two sets

Given two sets A and B the cartesian product of A by B, is denoted by $A \times B$ and is given by the set of all ordered pairs in which the first element belongs to A and the second to B:

$$A \times B = \{(x, y) \mid x \in A, y \in B\}$$

Metric space

Let M be a set and d a function of $M \times M$ on $\mathbb{R}_0^+$. The couple (M, d) is said to be a metric space if the function d (called metrics or distance) satisfies the following properties:

For all $x, y, z \in M$:

1) $d(x, y) \geq 0$, being $d(x, y) = 0$ iff $x = y$

2) $d(x, y) = d(y, x)$

3) The triangular inequality holds

$$d(x, z) \leq d(x, y) + d(y, z)$$

Example of a metric space

Let M be the set of all the continuous functions in a closed set $[t_0, t_1]$. Given $f, g \in M$ define the distance as

$$d(f,g) = \max_{t \in [t_0, t_1]} |f(t) - g(t)|$$

Remark that the maximum exists since $|f(t) - g(t)|$ is continuous.

Show that $d(f,g)$ *satisfies the properties of a distance*.

Hint: Use the properties of the modulo function.

Proof

1) Since the modulus is non-negative, $d(f,g) \geq 0$. Furthermore, if $f = g$ then $d(f,g) = \max_{t_0 \leq t \leq t_1} |f(t) - g(t)| = \max_{t_0 \leq t \leq t_1} 0 = 0$.

On the other way, if $d(f,g) = 0$ then $\max_{t_0 \leq t \leq t_1} |f(t) - g(t)| = 0$ and $f(t) = g(t)\ \forall t \in [t_0, t_1]$

2) $d(f,g) = \max_{t_0 \leq t \leq t_1} |f(t) - g(t)| = \max_{t_0 \leq t \leq t_1} |g(t) - f(t)| = d(g,f)$

3) $d(x,z)=\max\limits_{t_0\le\tau\le t_1}\left|x(\tau)-z(\tau)\right|$ Let $\tau*$ be the value of the independent variable where the maximum is attained. Then, using the triangle inequality for the modulus of real numbers:

$$
\begin{aligned}
&d(x,z)=\max_{t_0\le\tau\le t_1}\left|x(\tau)-z(\tau)\right|=\left|x(\tau*)-z(\tau*)\right|\le\\
&\le\left|x(\tau*)-y(\tau*)\right|+\left|y(\tau*)-z(\tau*)\right|\le\\
&\le\max_{t_0\le\tau\le t_1}\left|x(\tau)-y(\tau)\right|+\max_{t_0\le\tau\le t_1}\left|y(\tau)-z(\tau)\right|=d(x,y)+d(y,z)
\end{aligned}
$$

Q.e.d.

Function spaces to consider

Typical function spaces to consider are spaces of functions that map

$$[a, b] \subset \mathbb{R} \to \mathbb{R}^n$$

Different spaces result from placing different requirements on these functions.

Examples:

- $C^k([a, b], \mathbb{R}^n)$, $k \geq 0$

  Function elements are assumed to be k times differentiable.

- Piecewise continuous functions
- Measurable functions (this case is not addressed in this Course)

Convergence

Given a sequence $\{x_k\}_{k=0}^{\infty}$ of elements in a metric space, we say that it converges to the limit $x^* \in M$ if $d(x_k, x^*) \to 0$ when $k \to \infty$.

This definition reduces the notion of "convergence" in a metric space to convergence to 0 in $\mathbb{R}$.

Uniqueness of the limit

When it exists, the limit is unique.

Assume that there are two limits x^* and x^{**}. Then,

$$d(x^*, x^{**}) \leq d(x^*, x_k) + d(x_k, x^{**}) \to 0$$

Therefore, $d(x^*, x^{**}) = 0$ and hence it must be (by the properties of distance)

$$x^* = x^{**}$$

Cauchy sequences

A sequence $\{x_k\}_{k=0}^{\infty}$ of elements of M is a Cauchy sequence if $d(x_n, x_m) \to 0$ when $n, m \to \infty$.

Proposition: If $\{x_k\}_{k=0}^{\infty}$ converges, then it is a Cauchy sequence.

Proof Let the limit be $x*$. From the triangular inequality

$$d(x_n, x_m) \le d(x_n, x^*) + d(x^*, x_m)$$

When $n, m \to \infty$ both terms on the right approach zero, and hence $d(x_n, x_m) \to 0$, *q. e. d.*

Complete metric spaces

Although all convergence sequences are Cauchy sequences, the converse is not true: Depending on the space, there can be Cauchy sequences that do not converge to an element in the space.

Definition A metric space is complete if all Cauchy sequences converge.

Examples

$\mathbb{R}$ is complete.

Is $\mathbb{Q}$ complete?

Is the set of C^1 functions defined in a interval complete?

Topological concepts

Consider a metric space (V, d).

Given a point $x_0 \in V$ and a real number $r > 0$, we define

- The open ball: $B(x_0, r) = \{x \in V | d(x, x_0) < r\}$
- The closed ball: $\bar{B}(x_0, r) = \{x \in V | d(x, x_0) \leq r\}$
- The sphere $S(x_0, r) = \{x \in V | d(x, x_0) = r\}$

The interior of a subsect $K \subset V$ is the set of all the points of K such that there is a ball around it, of sufficiently small radius, that only contains points of K.

The boundary of K is the set of points of V such that any ball around it always contains points of K and points that do not belong to K, no matter how small is the radius.

The set K is said to be closed if it contains its boundary, being open otherwise.

The interior of a set is an open set.

Recollection of a few algebraic concepts

Indeed very, very, few. But if you want to know more you may look at

F. M. Goodman (1998). *Algebra*. Prentice Hall.

Algebra might look somewhat static, but is the key to understand dynamical systems. It is worth studying it. The young Evariste Galois and Niels Abel set the basis of Group theory and the existence of formulas that express the solution of polynomial equations in a finite number of root extraction operations. Inspired by their work, Sophus Lie developed an algebraic theory for differential equations.

In both cases symmetry played a major role, the same concept that is a corner stone in Variational Mechanics, where every quantity that is conserved is associated to a symmetry (that is to say, an invariant) of the Euler-Lagrange equation, a fact established by Emmy Noether in the early XX century.

Both Galois and Lie have Romanesque stories associated to them. Galois wrote some notes about his theory on the eve of a duel in the sequel of which he died. "Je n'ait pas du temps" – I have not enough time – , he wrote in anguish.

The story of Lie is much less dramatic, and even fun: During the Franco-Prussian war in 1970 he was leaving is hotel every day in Alsace and going to the forest to think about his mathematical problems. People started saying that he was a spy and when they arrested him they found the undisputable evidence: His notebook was covered with symbols that no one could understand, for sure code messages to be sent to the obscure powers that he was serving. When the story came to the hears of the President of the French Republic, Carnot, himself a scientist, he was released from prison. As a comment, Lie said the time spent in prison was very good for mathematical research. They were treating him well, feeding him and nothing deviated himself from work.

Groups

A group G is a set of elements equipped with a map $V \times V \rightarrow V$ that for any two elements $x, y \in G$ associates an element denoted $x \otimes y$, such that the "operation" $\otimes$ has the following properties

1. The operation is "associative": $\forall_{a,b,c \in G}, (a \otimes b) \otimes c = a \otimes (b \otimes c)$
2. There is an element $e \in G$ called "identity" : $\forall_{a \in G}$, $a \otimes e = e \otimes a = a$
3. For each $a \in G$ there is an element $a^{-1} \in G$ such that

$$a \otimes a^{-1} = a^{-1} \otimes a = e$$

Which ones are groups: $(\mathbb{N}, +)$, $(\mathbb{N},\times)$, $(\mathbb{Z}, +)$, $(\mathbb{Z},\times)$, : $(\mathbb{Q}, +)$, $(\mathbb{Q},\times)$, $(\mathbb{R}, +)$, $(R,\times)$, Vectors on $\mathbb{R}^2$ with the parallelogram addition rule.

Fields

A field K is a set equipped with two operations called “addition” $+$ and “multiplication” $\times$, with identify elements called “0” and “1”, such tat

1. $(K,+)$ is a commutative group, $\forall_{a,b\in K}\ a+b=b+a$
2. $(K\backslash\{0\},\times)$ is a commutative group, $\forall_{a,b\in K}\ a\times b=b\times a$
3. $\forall_{a\in K}\ a\times 0=0,\ \ \forall_{a\in K}\ a\times 1=a$
4. $1\neq 0$
5. Distributivity: $\forall_{a,b,c\in K}, a\times(b+c)=a\times b+a\times c$ and $(b+c)\times a=b\times a+c\times a$

Examples of fields: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, with the “usual” addition and multiplication.

Vector spaces

A vector space (or linear space) V over a field K is a nonempty set of elements $x, y, z, ...$, called vectors, with the operations

- Vector addition → commutative group
- Multiplication of vectors by scalars ($\alpha, \beta, ... \in K$), such that

1. $\alpha(\beta x) = (\alpha\beta)x$
2. $1x = x$
3. $\alpha(x + y) = \alpha x + \beta y$
4. $(\alpha + \beta)x = \alpha x + \beta x$

Hereafter we will take $K = \mathbb{R}$.

Exercise

Prove that

1. $0x = \bar{0}$
2. $\alpha\bar{0} = \bar{0}$
3. $(-1)x = -x$

Norms

V a vector space.

V is a normed space if there is a function $V \to \mathbb{R}_0^+$

$$y \in V \to \|y\| \in \mathbb{R}_0^+$$

such that

1. $\forall_{y \in V}, \ \|y\| \geq 0, \quad \|y\| = 0 \Rightarrow y = 0$
2. $\forall_{y \in V}, \forall_{\lambda \in \mathbb{R}} \ \|\lambda y\| = |\lambda| \cdot \|y\|$
3. Triangle inequality: $\forall_{x,y,z \in V}, \|y + z\| \leq \|y\| + \|z\|$

Distance induced by a norm

V is a normed space.

The norm induces a distance $d: V \times V \to \mathbb{R}_0^+$ by

$$d(y,z) = \|y - z\|$$

<u>Exercise</u>

Prove that d as defined above satisfies the conditions to be a distance.

Examples of normed spaces [L2012] p.18

Space $C^0([a,b],\mathbb{R}^n)$

$$\|y\|_0 \coloneqq \max_{a\le x\le b}\|y(x)\|$$

Where ||·|| without any index denotes the standard Euclidean norm.

The above norm is also known as the L_∞ norm.

Space $C^1([a,b],\mathbb{R}^n)$

$$\|y\|_1 \coloneqq \max_{a\le x\le b}\|y(x)\| + \max_{a\le x\le b}\|y'(x)\|$$

L_p norm:

$$\|y\|_{L_p} \coloneqq \left(\int_a^b \|y(x)\|^p dx\right)^{1/p}$$

Local minima of a functional [L2012] p. 19

V is a vector space of functions equipped with a norm $\|\cdot\|$.

$$A \subset V$$

J a real valued function defined on A (a functional)

Definition: A function $y^* \in A$ is a local minimum of J over A if

$$\exists_{\varepsilon>0} : \forall_{y \in A} : \|y - y^*\| < \varepsilon \text{ , then } J(y^*) \leq J(y)$$

The notions of strict local minima, global minimum, and the corresponding notions for maxima are defined in a similar way.

First variation

V a function space, $J \to \mathbb{R}$.

Definition: A linear functional $\delta J|_y : V \to \mathbb{R}$ is called the first variation of J at y if

$$\forall_{\eta \in V}, \forall_{\alpha \in \mathbb{R}}, \qquad J(y + \alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + o(\alpha)$$

where $o(\alpha)$ satisfies: $\displaystyle \lim_{\alpha \to 0} \frac{o(\alpha)}{\alpha} = 0$

Requirement that $\delta J|_y$ is linear:

$$\delta J|_y(\alpha_1\eta_1 + \alpha_2\eta_2) = \alpha_1 \delta J|_y(\eta_1) + \alpha_2 \delta J|_y(\eta_2)$$

$$\forall_{\eta_1, \eta_2 \in V}, \forall_{\alpha_1, \alpha_2 \in \mathbb{R}}$$

Gateaux derivative

The Gateaux derivative is defined as

$$G(y,\eta) \coloneqq \lim_{\alpha\to 0}\frac{J(y+\alpha\eta)-J(j)}{\alpha}$$

The Gateaux derivative is equal to the first variation

$$\delta J|_y(\eta) = G(y,\eta)$$

Defining $$g(\alpha) \coloneqq J(y+\alpha\eta)$$

Then $$\delta J|_y(\eta) = g'(0)$$

A related concept is the Fréchet derivative, that will not be used here.

René Gateaux (1889-1914), A young and promising French mathematician that died prematurely at the beginning of World War I.

Maurice Fréchet (1878–1973) was a French mathematician who made major contributions to the topology of point sets and introduced the concept of metric spaces. He also made important pioneering contributions to functional analysis in relation to compactness and the representation theorem.

Exercise 18 (Example 5-6 of [ATHANS2007], p. 239)

[ATHANS2007] M. Athans and P. Falb. *Optimal Control*. MacGraw-Hill, 1966. Dover reprint, 2007.

Consider the space $V = C^0([0,1], \mathbb{R})$

Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a C^1 function

Let $J: V \to \mathbb{R}$, $\quad J(y) \coloneqq \int_0^1 \varphi\big(y(x)\big)dx$

Compute the first variation of J, $\delta J|_y(\eta)$.

Hints: Consider the function $\varphi(y(x) + \alpha\eta)$ and expand it in a Taylor series in α.

Use the definition of first variation $J(y + \alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + o(\alpha)$

Assume that you can interchange the limit and the integral.

Solution

$$J(y+\alpha\eta)=\int_0^1 \varphi\big(y(x)+\alpha\eta(x)\big)dx=\int_0^1 \varphi\big(y(x)\big)+\alpha\varphi'(y(x))\eta(x)+o(\alpha))dx=$$

$$=J(y)+\alpha\int_0^1 \varphi'\big(y(x)\big)\,\eta(x)dx+o(\alpha)$$

Remark that $\lim\limits_{\alpha\to 0}\frac{\int_0^1 o(\alpha)d\alpha}{\alpha}=\int_0^1 \lim\limits_{\alpha\to 0}\frac{o(\alpha)}{\alpha}dx=\int_0^1 0dx=0$

From the definition of first variation it follows that

$$\delta J|_y(\eta)=\int_0^1 \varphi'\big(y(x)\big)\,\eta(x)dx.$$

First order necessary condition for optimality

Let y^* be a local minimum of J over a subset $A \subset V$.

A perturbation $\eta \in V$ is admissible if $y^* + \alpha\eta \in A$ for all α sufficiently close to zero.

As a function of α, $J(y^* + \alpha\eta)$ has a local minimum at $\alpha = 0$ for each admissible perturbation η.

Necessary condition for optimality

$$\delta J|_y(\eta) = 0 \quad \text{for all admissible } \eta$$

Bilinear functionals and quadratic functionals

A real valued functional B on $V \times V$ is called bilinear if it is linear in each argument (when the other one is fixed).

Setting $Q(y) \coloneqq B(y, y)$ we then obtain a quadratic functional or quadratic form on V.

Second variation

A quadratic form $\delta^2 J|_y(\eta): V \to \mathbb{R}$ is called the second variation of J at y if

$$\forall_{\eta \in V}, \forall_{\alpha \in \mathbb{R}}, \qquad J(y+\alpha\eta) = J(y) + \delta J|_y(\eta)\alpha + \delta^2 J|_y(\eta)\alpha^2 + o(\alpha^2)$$

<u>Example</u> (continuation of exercise 18)

Let $J: V \to \mathbb{R}$, $\quad J(y) \coloneqq \int_0^1 \varphi\big(y(x)\big)dx$ but assume now that φ is C^2.

Since

$$J(y+\alpha\eta) = J(y) + \alpha \int_0^1 \varphi'\big(y(x)\big)\,\eta(x)dx + \tfrac{1}{2}\alpha^2 \int_0^1 \varphi''\big(y(x)\big)\eta(x)dx + o(\alpha^2),$$

the second variation is

$$\delta^2 J|_y(\eta) = \tfrac{1}{2}\int_0^1 \varphi''\big(y(x)\big)\eta(x)dx.$$

Second-order necessary condition for optimality

If y^* is a local minimum of J over $A \subset V$, then, for all admissible perturbations η we have

$$\delta^2 J|_y(\eta) \geq 0$$

In other words, the second variation of J at y^* must be positive semidefinite on the space of admissible perturbations.

The proof follows the same arguments of the one given in the finite dimensional case.

Sufficient condition for optimality

Based on a superficial comparison with the finite dimensional case, one might be tempted to state that a sufficient for optimality is $\delta J|_y(\eta) = 0$ and

$$\delta^2 J|_y(\eta) > 0$$

Actually, this is not true. Further discussion will be made afterwards.