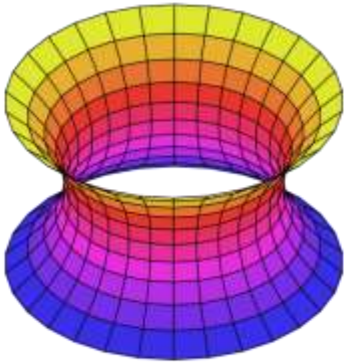
Calculus of Variations and Optimal Control



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Syllabus

1. Examples of problems addressed by the course.
2. Background on finite and infinite dimensional optimization.
3. Calculus of variations
4. Pontryagin’s Principle
5. The Hamilton-Jacobi-Bellman equation
6. The Linear Quadratic problem
7. Optimal Control Problems in Discrete time
8. Numerical Methods

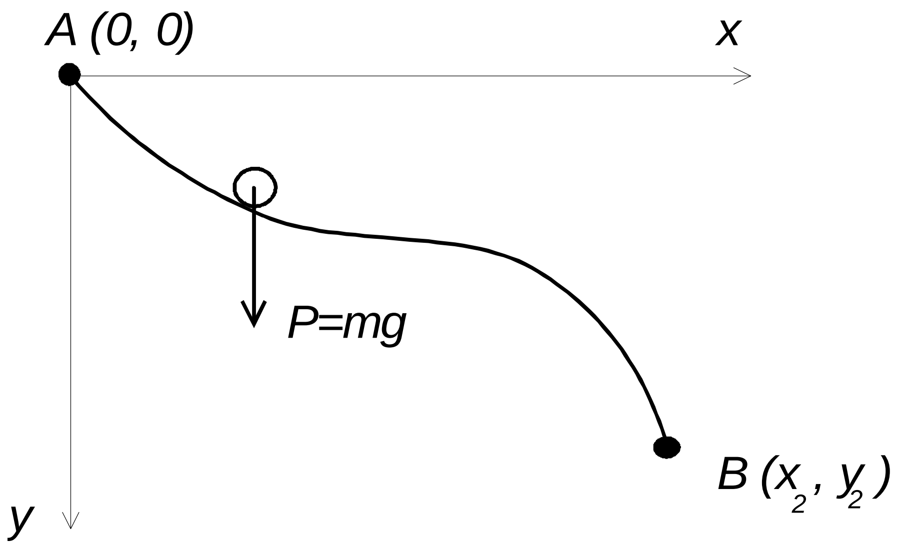
1.Examples of problems addressed by the course

Objective: *Introduce a novel class of optimization problems, that are solved with respect to infinite dimensional variables – Problems of Calculus of Variations and Optimal Control.*

Refs: [L2012], 26-31.

A classical problem: The brachistochrone curve

What is the shape of the curve that connects points A and B such data a point mass, under the force of gravity alone, slides (frictionless) from A to B in minimum time?



Which function *y(x)* minimizes the travel time between A and B?

Computing the travel time assuming $y(x) \quad 0 \le x \le x_2$ known

Without friction, the increase of kinetic energy is equal to the loss of potential energy, and $\frac{1}{2}mv^2 = mgy$ or

$$v(x) = \sqrt{2gy(x)}$$

Let s be the arclength. From Pythagoras theorem we get the kinematics relation

$$v(x) = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2}}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} . \frac{dx}{dt} = \sqrt{1 + \left(y'(x)\right)^2} . \frac{dx}{dt}$$

Energy balance:

$$v(x) = \sqrt{2gy(x)}$$

Kynematics:

$$v(x) = \sqrt{1 + \left(y'(x)\right)^2} \cdot \frac{dx}{dt}$$

Eliminate v by equating the r.h.s.:

$$\sqrt{2gy(x)} = \sqrt{1 + \left(y'(x)\right)^2} \cdot \frac{dx}{dt}$$

$$\sqrt{2gy(x)} = \sqrt{1+\left(y'(x)\right)^2}\,.\frac{dx}{dt}$$

or

$$\frac{dt}{dx} = \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}$$

Traveling time is obtained by integration

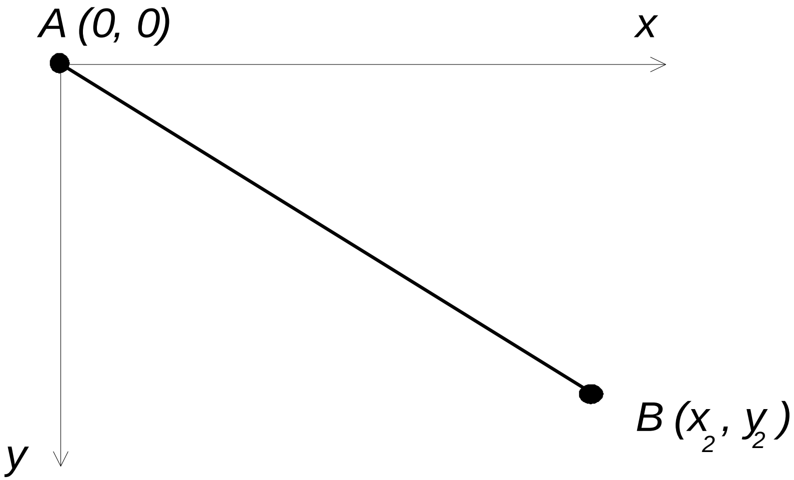
$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx$$

If we know the function $y(x)$, we can compute the travel time

$$T = \int_0^{x_2} \sqrt{\frac{1 + \left(y'(x)\right)^2}{2gy(x)}}\, dx$$

For instance, if the path to follow is a straight line between A and B,

$$y(x) = \alpha x \text{ with } \alpha = {}^{y_2}\!/_{x_2}$$



The travel time for the rectilinear path is

$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx = \int_0^{x_2} \sqrt{\frac{1+\alpha^2}{2g\alpha}}.x^{-1/2}dx = \sqrt{\frac{1+\alpha^2}{2g\alpha}}.2x_2^{1/2}$$

If we want to compare the travel time for the rectilinear path with the one of another curve (for instance an arc of circle), we can do it, and decide which one leads to the pastest path.

However, the point is that we don’t know the shape of the optimal curve.

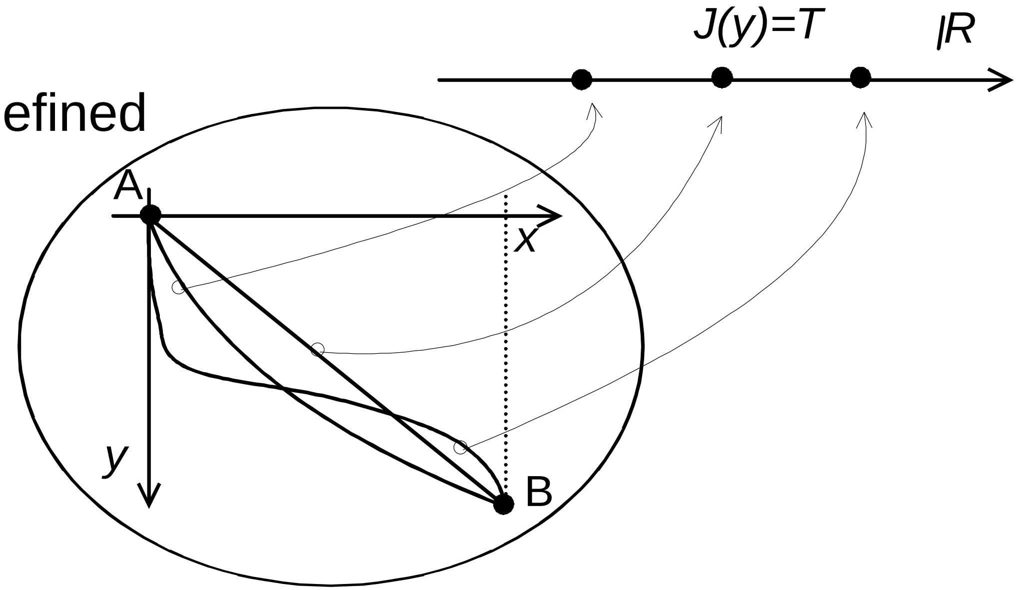
We want to optimize with respect to the curve and this is an infinite dimensional problem, because it depends on the position of the points on the curve (that are infinitely many).

The expression

$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx$$

defines the functional to minimize.

To each differentiable function $y(x)$ defined on $[0,x_2]$ that satisfies the boundary conditions $y(0)=0$ and $y(x_2)=y_2$ if associates a real number (the travel time).



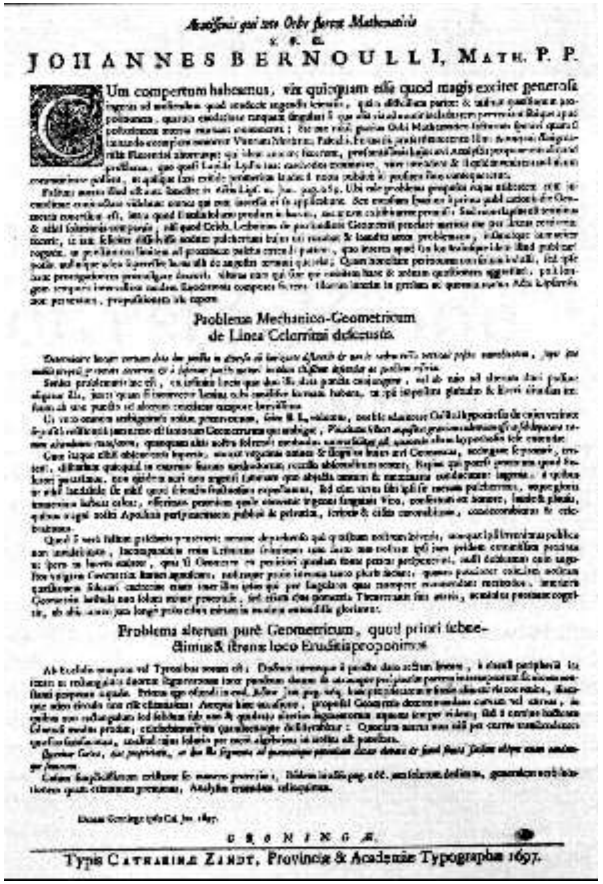
The Brachistochrone problem was published in 1 January 1667 by Johann Bernouilli, as a challenge to the scientific community: *Nothing is more attractive to intelligent persons than an honest problem that challenges them and which solution brings fame and stays as a lasting monument.*

60 years before, Galileo new already that the minimum time trajectory could not be a straight line, although he thought, erroneously, that it was a circumference arc.

This challenge was tackled by six of the most brilliant minds of the time: His elder broither Jacob, Leibniz, Tschirnhaus, l'Hopital and Newton (who published his solution anonymously).

An historical perspective (with technical content) of the Brachistochrone problem and of its relations with Optimal Control may be seen in

Sussmann, H. J. e J. C. Willems (1997). 300 Years of Optimal Control: From the Brachystochrone to the Maximum Principle. *IEEE Control Systems*, 17(3):32-44.

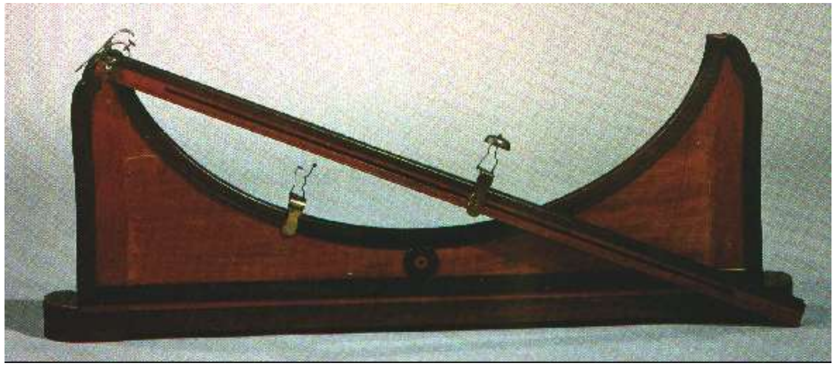

JOHANNES BERNOULLI, MATH. P. P.

Cum compertum habeamus, vix quicquam esse quod magis excitet generosa

Problema Mechanico-Geometricum
de Linea Celerrimi descensus.

Problema alterum purè Geometricum, quod priori substernimus & iterum loco Eruditis proponimus.

GRONINGÆ,
Typis CATHARINÆ ZANDT, Provinciæ & Academiæ Typographæ 1697.



A machine to exemplify the brachistochrone, Museu Pombalino de Física da Universida de Coimbra, Portugal.

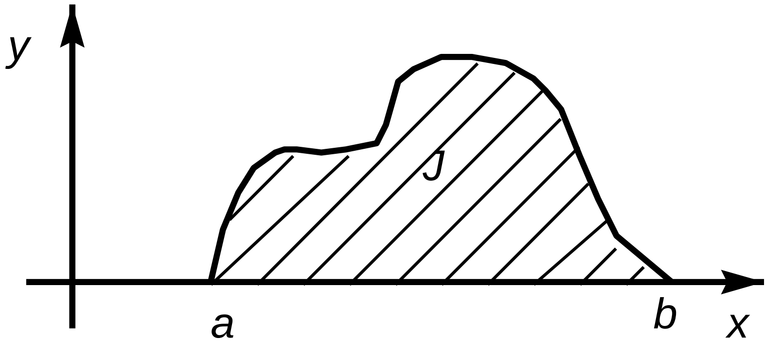
←The challenge of J. Bernouilli as published in *Acta Eroditorum*

What is the relation between ancient Rome poetry, the Phoenicians and Control and Estimation theory?

Queen Dido and the foundation of Charthage

According to an ancient roman poem, the Phoenician Dido was allowed to found the city of Carthage in the land she could embrace with a cow skin. She sliced the skin to form a long rope and disposed it in a shape that maximizes as much land as possible. This was probably the first variational problem ever considered (although her majesty the queen Dido was never enrolled in CVOC!).

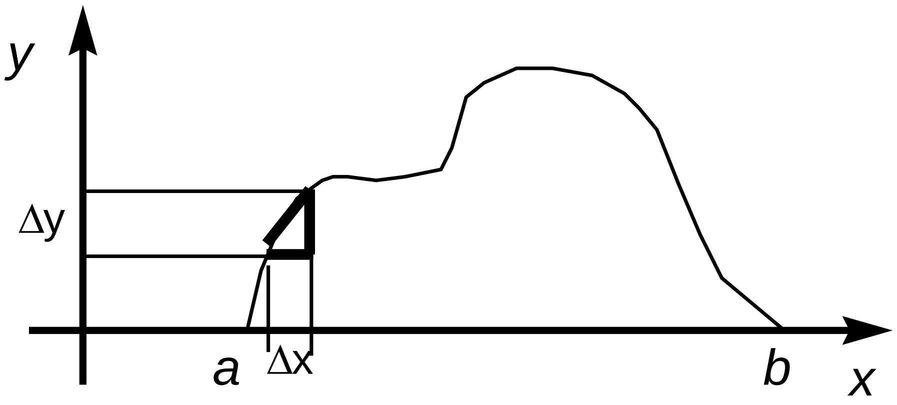
Dido's isoperimetric problem



What is the shape of y such that the area under it between a and b is maximum, under the constraint that the length of the graph of y is a given constant?

[L2012] p.26, 27.

Get a formula to the length of the graph:



Divide the line into strips

$$\Delta s = \sqrt{\Delta x^2 + \Delta y^2} \quad \text{or} \quad \Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Total length: $s_{tot} = \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$ (sum over all elements)

$$s_{tot} = \sum \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2} \Delta x$$

In the limit when $\Delta x \to 0$

$$S_{tot} = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

This expression for the arc-length assumes that y is $C^1([a, b] \to \mathbb{R})$.

Remember the above expression for arc-length.

A function $[a, b] \to \mathbb{R}$ is of class C^0 if it is continuous. It is of class $C^n, n \geq 1$ if it is continuous and its first n derivatives are continuous.

Formulation of DIDO's isoperimetric problem

Find $y \in C^1([a,b] \to \mathbb{R})$ that

Minimize $J = \int_a^b y(x)dx$

Subject to

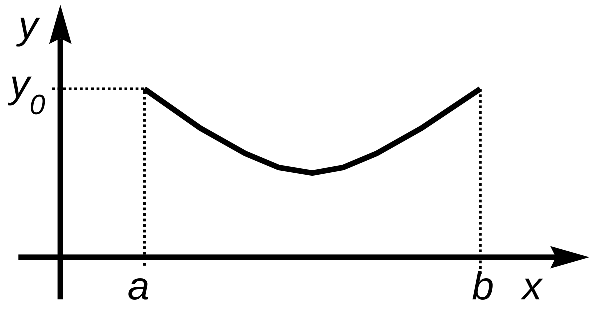
$$\int_a^b \sqrt{1 + (y'(x))^2} dx = C_0,$$

with $C_0 \in \mathbb{R}^+$ a constant.

The solution is an arc of circle.

The catenary [L2012] p.29

Find the shape of a rope of a given length C_0, with uniform mass density that is suspended freely between two fixed points $y(a) = y_0$ and $y(b) = y_1$.



It is assumed that the rope takes the shape that minimizes its potential energy.

Consider the case in which $y_0 = y_1$ and the rope does not touch the ground.

Rope potential energy

An element of rope between x and $x + dx$ has a mass $m = \gamma\sqrt{(\Delta x)^2 + (\Delta y)^2}$ where γ is the rope specific mass.

Since the elemnt of mass is at an eight $y(x)$, it has a potential energy

$$mgy = \gamma gy\sqrt{(\Delta x)^2 + (\Delta y)^2} = \gamma gy\sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$$

Potential energy of all the elements = $\gamma g \sum y(x)\sqrt{1\left(\frac{\Delta y}{\Delta x}\right)^2}\,\Delta x$

Making $\Delta x \to 0$, the potential energy is seen to be

$$J = \gamma g \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx$$

Formulation of the catenary problem

Find $y \in C^1([a,b] \to \mathbb{R})$ such that $y(a) = y(b) = y_0$ and that

Minimize

$$J = \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx$$

Subject to

$$\int_a^b \sqrt{1 + (y'(x))^2}dx = C_0,$$

with $C_0 \in \mathbb{R}^+$ a constant.

The solution is given by $y(x) = cosh\left(\frac{x}{c}\right)$, $c > 0$, modulo a translation along x.

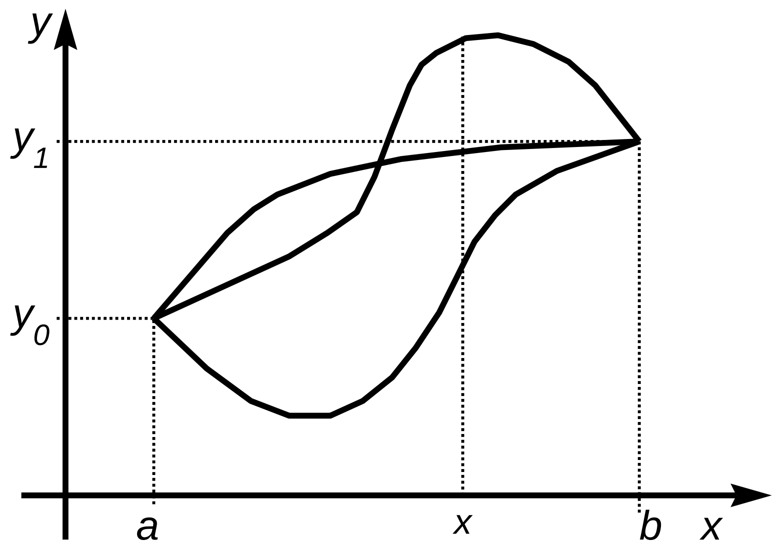
Basic Calculus of Variations Problem

Among all C^1 curves $y\colon [a,b] \to \mathbb{R}$ that the given boundary conditions

$$y(a) = y_0, \; y(b) = y_1,$$

Find the (local) minima of the cost function

$$J(y) \coloneqq \int_a^b L(x, y(x), y'(x))\, dx$$



$L\colon \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is the "lagrangian" or "running cost".

Can be extended to $y\colon [a,b] \to \mathbb{R}^n$

Solution of the basic CV problem

A necessary condition for a $C^1([a,b] \to \mathbb{R})$ function y to be an extremum (maximum or minimum) of the CV basic problem

Minimize $J(y) \coloneqq \int_a^b L(x, y(x), y'(x)\, dx$

s. t. $y(a) = y_0,\ \ y(b) = y_1$

is that it satisfies the Euler-Lagrange equation

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

We need to develop new tools to solve these infinite dimensional problems, that generalize the method of "equating the derivative to zero".

Exercise 1 – A simple CV problem

Use the EL equation

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right)$$

to find the extremal for the following fixed end-points problem:

minimize $J(y) = \int_0^{\pi} ((y')^2 + 2y\, sin\, x)dx$

s. t. $y(0) = y(\pi) = 0$

Solution:

$$L = (y')^2 + 2y\sin x$$

$$\frac{\partial L}{\partial y} = 2\sin x \quad \frac{d}{dx}\left(\frac{\partial L}{\partial y\prime}\right) = \frac{d}{dx}2y' = 2\ddot{y}$$

Thus, the EL eq. is in this case

$$y'' = \sin x$$

$$y' = -\cos x + \alpha, \quad y = -\sin x + \ \alpha x + \beta$$

Apply the boundary conditions:

$$y(0) = -\sin(0) + \alpha 0 + \beta = \beta \ \Rightarrow \ \beta = 0, \ \ y(\pi) = -\sin(\pi) + \alpha\pi = \alpha\pi \ \Rightarrow \ \alpha = 0$$

The solution is thus

$$y = -\sin x$$

Extensions of the basic CV problem

- The isoperimetric problem and integral constraints
  - Lagrange multipliers
- Free terminal “time”
  - Requires extra conditions
- Final value constrained to lay on a given line (or surface)
  - Requires extra conditions (transversality conditions)

CV problems with integral constraints

Among all C^1 curves $y\colon [a,b] \to \mathbb{R}$ that the given boundary conditions

$$y(a) = y_0,\ \ y(b) = y_1,$$

Find the (local) minima of the cost function

$$J(y) \coloneqq \int_a^b L(x, y(x), y'(x))\, dx$$

Subject to the integral constraint

$$\int_a^b M(x, y(x), y'(x))\, dx = C_0$$

Approach: Solve the EL equation with the extended lagrangian $L + \lambda^* M$, where λ^* is a Lagrange multiplier.

The Principle of Maximum Entropy

General problem: Find an unknown function given an incomplete set of facts that concern its properties.

PME: Look for a function that complies with the known data, while maximizing the entropy.

Entropy and Information

- If Jorge Jesus says: "*I lost, but I am the best*"
  - He says this all the time. The information content is very low
- If Jorge Jesus says: "*I lost because the other coach was better*"
  - This is quite rare. The information content is very high.

Source: Getty Images

The information carried by the observation of an event is inversely proportional to the probability of the event.

Entropy and information: Discrete processes

An event with n possible outcomes with probabilities p_k, $k = 1, \cdots, n$

The information gained by observing one particular outcome is

$$I_k = \log\left(\frac{1}{p_k}\right)$$

The smaller p_k, (the rarer the event), the larger the information gained.

Logarithm makes the information gained by two independent events additive.

Entropy = the total uncertainty represented by an ensemble of random events:

$$H = \sum_k p_k \log\left(\frac{1}{p_k}\right) = -\sum_k p_k \log(p_k)$$

Weighted sum of individual uncertainties.

Entropy for continuous distributions

$$H(p) = -\int_{\Omega} p(x) \log p(x) \, dx$$

p a pdf associated to the r. v. x

Example: Boltzmann's kinetic theory of gas

A gas fills a "rectangular" (one dimensional) box of length L

A) The probability of finding a molecule between x and $x + dx$ is $\cong p(x)dx$

Find p that maximizes $H(p)$ under the constraints

(C1) $\int_0^L p(x)dx = 1$

(C2) $p(x) = 0$ for $x < 0$ or $x > L$

This is a variational problem with an isoperimetric constraint.

Using the EL equation it may be shown that the solution is a uniform pdf.

(problem to tackle in detail later in the course).

B) Find the pdf p of the energy of the molecules of the gas that maximizes $H(p)$ subject to

(C1) $\int_0^\infty p(E)dE = 1$

(C2) $\int_0^\infty Ep(E)dE = \bar{E}$ Given average energy of the gas

(C3) $p(E) = 0$ for $E < 0$

From the EL equation:

$$\frac{\partial}{\partial p}(-p\,log\,p - \lambda Ep - \gamma p) = 0 \;\rightarrow\; p = exp(-1 - \gamma - \lambda E)$$

The Lagrange multipliers γ and λ are found by imposing the constraints.

Result (Boltzmann distribution):

$$p(E) = \frac{1}{\bar{E}} e^{-\frac{E}{\bar{E}}}$$

C) Find the pdf of the velocities v of the molecules that form the gas, by maximizing $H(p)$ subject to the constraints

(C1) $\int_{-\infty}^{\infty} p(v)dv = 1$

(C2) $\int_{-\infty}^{\infty} vp(v)dv = 0$ Zero mean

(C3) $\int_{-\infty}^{\infty} v^2 p(v)dv = \sigma^2$ given variance

Solution: Gaussian distribution.

A major application of the PME: High resolution spectral estimation (Burg)

More general CV problems

Not addressed, but use the ideas explained in this course.

See e.g. p. 81 [C2013] K. W. Cassel. Variational Methods, Cambridge Univ. Press, 2013.

Example: For the lagrangian

$$L = L(x, y, y', y'')$$

The EL equation is

$$\frac{\partial L}{\partial y} = \frac{d}{dx}\left(\frac{\partial L}{\partial y'}\right) - \frac{d^2}{dx^2}\left(\frac{\partial L}{\partial y''}\right)$$

With suitable boundary conditions.

Other types of conditions for CV problems

The EL equation provides a first order necessary condition for C^1 functions (weak extrema).

- Second order necessary conditions (Legendre condition)
- Sufficient conditions
- Piecewise C^1 functions (strong extrema).
  - Crucial in Optimal Control (e. g. bang-bang control).
  - In this course, addressed in the context of Optimal Control only.