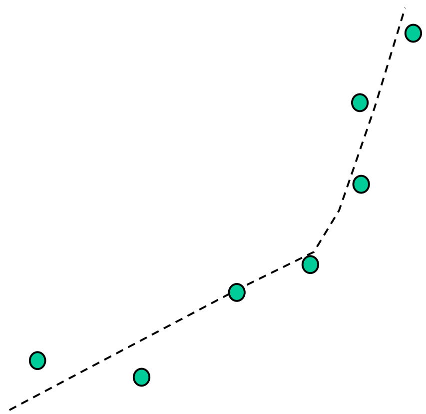
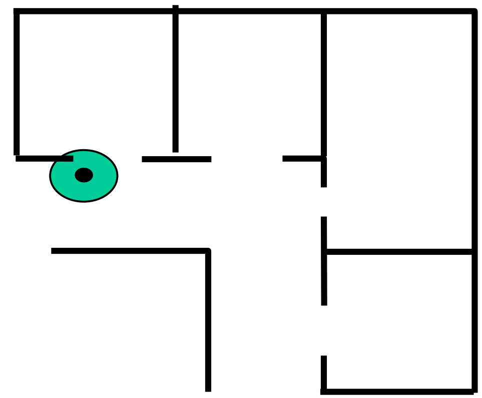
Non linear and Kalman Filtering

Challenges



moving target

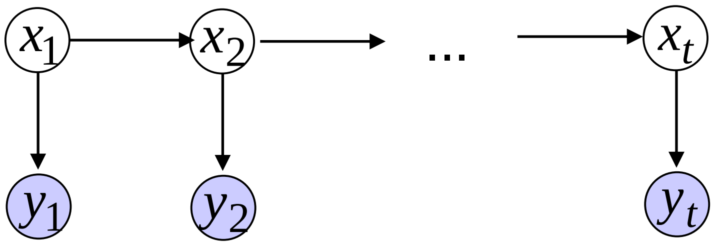


self-localization

Properties:

- observations are sequences
- dynamic problems

Sequence Estimation



Problem: given $Y^t = \{y_1, ..., y_t\}$ estimate x_t

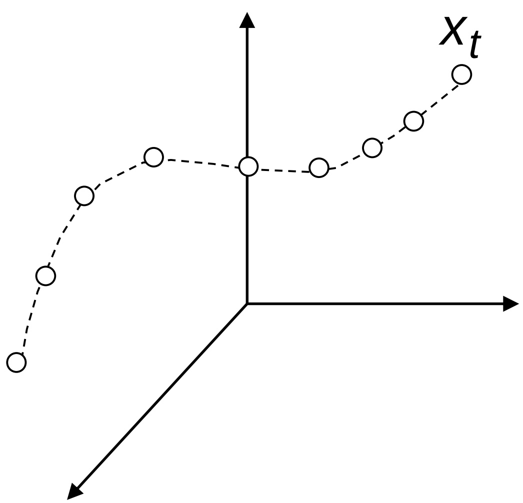
Examples:

$x_t \in R^n, y_t \in R^m$ NL, Wiener, and Kalman filters

$x_t \in \{1, ..., n\}, y_t \in \{1, ..., m\}$ or $y_t \in R^m$ HMM

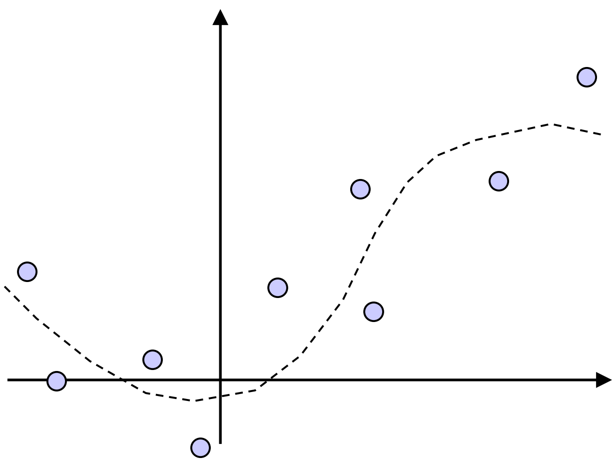
Geometric Interpretation

hidden sequence



(to be estimated)

observed sequence



Moving target (I)

Newton law

$m\ddot{p} = F$ $\quad$ *p=(p$_1$, p$_2$)* is the body position and F the applied force

Hipothesis: F is white Gaussian noise.

State model (continuous)

$$\begin{bmatrix} \dot{p}_1 \\ \dot{p}_2 \\ \ddot{p}_1 \\ \ddot{p}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ w_3 \\ w_4 \end{bmatrix} \qquad w_3, w_4 \sim N(0,q)$$

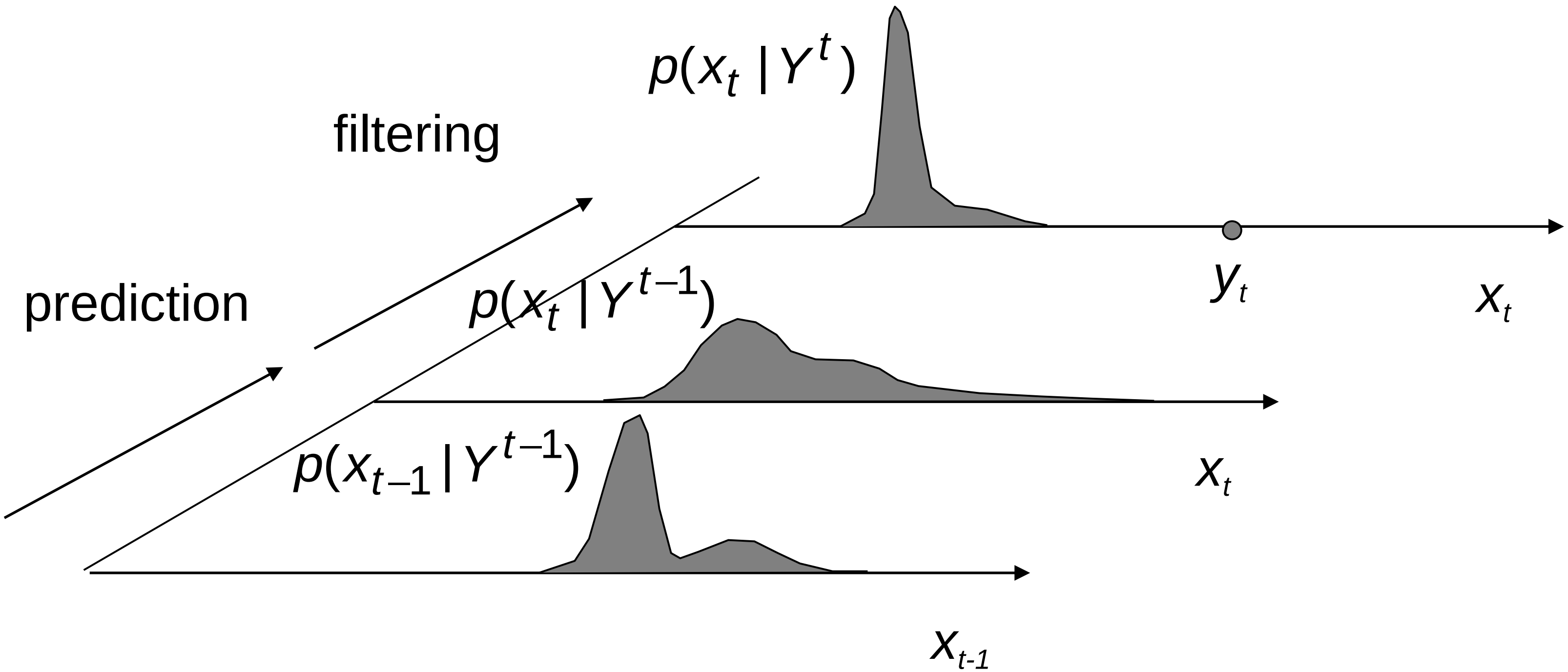
Moving target(II)

After a first order discretization

$$x_t = \begin{bmatrix} 0 & \Delta I \\ 0 & 0 \end{bmatrix} x_{t-1} - w_t \qquad w_t \sim N(0,Q) \qquad Q = \begin{bmatrix} 0 & 0 \\ 0 & qI \end{bmatrix}$$

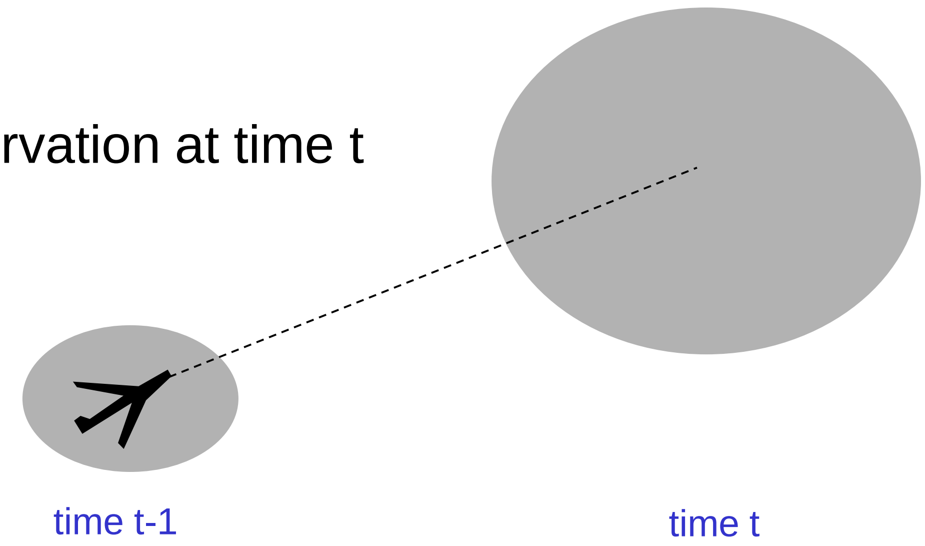
$$y_t = \begin{bmatrix} 0 & I \end{bmatrix} x_t + v_t \qquad v_t \sim N(0,R) \qquad R = rI$$

Uncertainty propagation

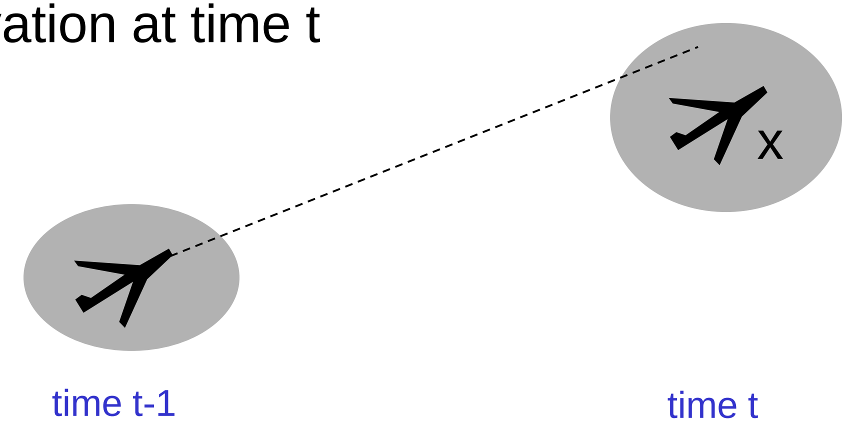


Example

before the observation at time t



after the observation at time t



Uncertainty propagation

Prediction

$$p(x_t \mid Y^{t-1}) = \int p(x_t \mid x_{t-1})\, p(x_{t-1} \mid Y^{t-1})\; dx_{t-1}$$

Filtering

$$p(x_t \mid Y^{t-1}) = k\; p(y_t \mid x_t)\, p(x_t \mid Y^{t-1}) \qquad \text{(Bayes law)}$$

Difficulty: the analytic solution can only be computed in special cases.

Point Mass Filter

state vector discretization: $x_t, \in \{x_1, ..., x_n\}$

Prediction: $\pi_t^- = A^T \pi_{t-1}$

Filtering: $\pi_t = k D_{y_t} \pi_t^-$

A,D are $n \times n$ matrices

$$A_{kl} = p(x_l \mid x_k)$$

$$D_{y_t} = diag(p(y_t \mid x_1), ..., p(y_t \mid x_n))$$

Particle Filter

The distribution is represented by a set of realizations of the random variable (particles) $x^1,\ldots,x^N$ updated according to the following algorithm.

Initialization: generate N particles $x^1,\ldots,x^N$ with initial distribution $p(x_1)$
and assign a probability $P(x^i) = 1/N$

Ciclo (increment t)

Prediction: randomly select N particles, according to the distribution P.
Replace each of them by a new particle generated according to the distribution $p(x_t \mid x_{t-1} = x^i)$

Filtering: update the particle probabilities multiplying each of them by the likelihood function

$$P(x^i) \leftarrow p(y_t / x_t = x^i) P(x^i)$$

end of cycle

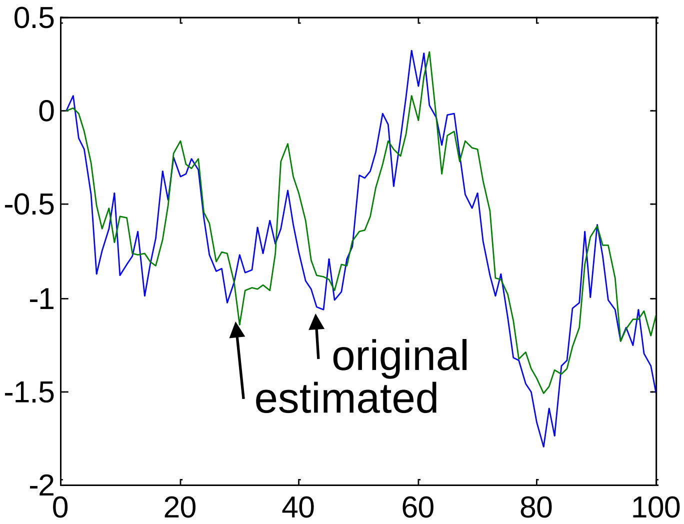
Example – Particle Filter

System:

$$x_t = x_{t-1} + w_t \qquad w_t \sim N(0,Q)$$

$$y_t = \begin{bmatrix} \cos x_t \\ \sin x_t \end{bmatrix} + v_t \qquad v_t \sim N(0,R)$$

Problem: estimate phase from the quadrature components.



Q=0.04, R=0.1

Matlab Program

```
T=100; Q=.04; x=[0 cumsum(sqrt(Q)*randn(1,T-1))];
R=.1; y=[cos(x);sin(x)]+sqrt(R)*randn(2,T);

% particle filter
N=100;
xp=x(:,1)*ones(1,N); P=ones(N,1)/N;
for t=2:T,
   k=sum(ones(N,1)*rand(1,N)>cumsum(P)*ones(1,N))+1;
   xp=xp(k)+sqrt(Q)*randn(1,N);  yp=[cos(xp);sin(xp)];
   P=P(k).*exp(-0.5*sum((y(:,t)*ones(1,N)-yp).^2)/R)';
   P=P/sum(P);
   xav(t)=sum(P'.*xp);
end
plot(1:T,x,1:T,xav)
```

Linear Dynamic Systems

How to characterize $p(x_t)$, $p(x_t|x_{t-1})$, $p(y_t|x_t)$?

One approach is based on stochastic linear systems:

$$x_t = Ax_{t-1} + Bu_t + w_t \qquad x_1 \sim N(\bar{x}, \bar{P})$$
$$y_t = Cx_t + v_t$$

where:

$x_t \in R^{n_x}$ state vector

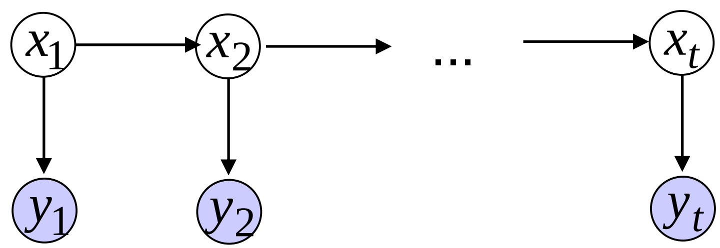
$u_t \in R^{n_u}$ input vector

$y_t \in R^{n_u y}$ output vector

$w_t \in R^{n_x}, v_t \in R^{n_y}, w_t \sim N(0, Q_t), v_t \sim N(0, R_t)$ white noise

(w_t, v_t independent r.v. and independent with respect to x_t)

Filtro de Kalman



If x_t, y_t are generated by a LDS, the ***conditional distribution of the hidden variables, given the observations, is normal !***

$$p(x_t \mid y^t) = N(\hat{x}_t, P_t)$$

$\hat{x}_t, P_t$ are updated by the Kalman filter.

Kalman Filter

Inicialization: $\hat{x}_1^- = \bar{x} \quad P_1^- = \bar{P}, t = 1$

Cicle: *t=1,2,...,n*

Filtering

$$\hat{x}_t = \hat{x}_t^- + K(y_t - C\hat{x}_t^-)$$

$$K_t = P_t^- C' S^{-1} \qquad S = CP_t^- C' + R$$

$$P_t = (I - KC)P_t^- \qquad \rho_t = N(y_t - C\hat{x}_t^-; 0, S)$$

Prediction

$$\hat{x}_t^- = A\hat{x}_{t-1} + Bu_{t-1}$$

$$P_t^- = AP_{t-1}A^T + Q$$

end cycle $(t \leftarrow t+1)$

likelihood function: $l = \log p(y^t) = \sum_{\tau=1}^{t} \log \rho_t$

Demonstração

Prediction

Since the density $p(x_{t+1}|y^t)$ is Gaussian, only the mean and covariance have to be specified

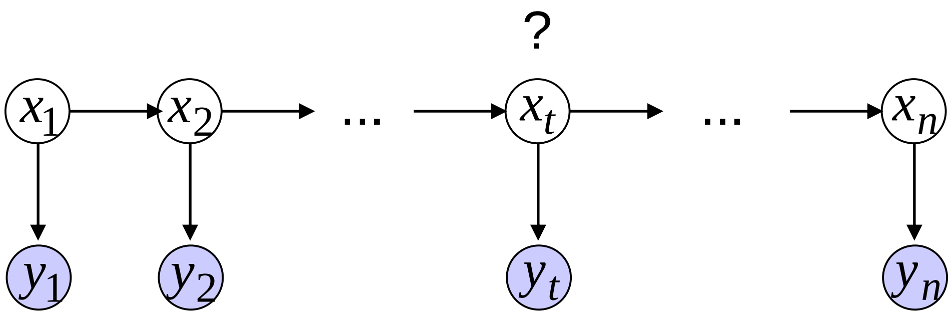
$$\hat{x}_t^- = E\{x_t \mid y^{t-1}\} = AE\{\hat{x}_{t-1} \mid y^{t-1}\} + Bu_{t-1} = A\hat{x}_{t-1}$$

$$P_t^- = Cov\{Ax_{t-1} + Bu_t + w_{t-1}\} = AP_{t-1}A' + Q_t$$

Filtering

Filtering equations were proved before.

Kalman Smoother



Let $x_1,\ldots, x_n$, e $y_1,\ldots, y_n$, be sequences generated by a LDS. The output sequence is known.

We wish to estimate y_t (t<n). This is known as the smoothing problem.

In this case $p(x_t \mid y^n) = N(\hat{x}, \hat{P})$ with mean and covariance updated by the Kalman smoother.

Kalman Smoother

Step 1: apply the Kalman filter to the data sequence.Denote the estimate obtained by x_t, P_t.

Passo 2: (backward step)

initialization $\hat{x}_n = x_n \quad \hat{P}_n = P_n$

cycle t=n-1, ..., 1

$$P^- = AP_t A' + Q$$

$$J = P_t A' (P^-)^{-1}$$

$$\hat{x}_t = x_t + \mathrm{J}(\hat{\mathrm{x}}_{\mathrm{t+1}} - \mathrm{Ax}_{\mathrm{t}}) \qquad \hat{P}_t = P_t + J(\hat{P}_{t+1} - P^-)J'$$

end of cycle

Computer Work

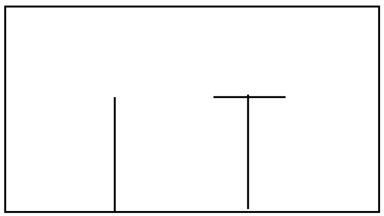
We wish to develop a self-localization system for a mobile robot. Suppose the initial position of the robot is unknown and the robot has sonar sensors which detect obstacles closer than 40cm in the directions NESW.

Simulate the robot and the sensors in a computer.

Develop a self-localization algorithm and compute the uncertainty ofeach admissible position.

Test the algorithm by showing the correct position as well as the uncertainty associate to all admissible positions.

environment (known)



Exercícios

Let x,y be sequences of discrete random variables generated by the model:

$$x_t = x_{t-1} + w_t \qquad w_t = \pm 1, \quad P\{w_t = 1\} = P\{w_t = -1\} = 0.5$$

$$y_t = x_t + v_t \qquad v_t = \pm 2, \quad P\{v_t = 2\} = P\{v_t = -2\} = 0.5$$

$$P(x_1) = \begin{cases} 0.2 & \forall x \in \{-2,-1,0,1,2\} \\ 0 & \text{c.c.} \end{cases}$$

w, v are independent processes.

Given a the output sequence y**=**(0 1 -2 -1 -2) propagate the distribution of the state variable x_t for $t \in \{1,2,3,4,5\}$ and determine the MAP estimate of x_t.

Solve the previous problem assuming that processes x, y, w, v are continuous and $w_t \sim N(0,1)$, $v_t \sim N(0,4)$, $x_1 \sim N(0,1)$.