Information and Communication Theory

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Problem Set 0 (Probability Refresher)

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1. Consider the random draw of a card from a shuffled deck (with the standard set of 52 cards), that is, $\{A\clubsuit, K\clubsuit, ..., 2\clubsuit, A\heartsuit, ..., 2\heartsuit, A\diamondsuit, ..., 2\diamondsuit, A\spadesuit, ..., 2\spadesuit\}$.

   a) What is the probability of drawing any particular card, say $7\clubsuit$?

   b) What is the probability of drawing a king?

   c) What is the probability of drawing a card of spades?

   d) Consider drawing a second card, assuming that the first one is returned to the deck. Repeat the three previous questions, now for the second card.

   e) Repeat the previous question, now considering that the first card is **not** returned to the deck.

   f) If you draw 5 cards at random from the shuffled deck (without replacement), what is the probability that 4 of them are equal (a poker)?

2. Let $X \in \mathcal{X} = \{A\clubsuit, K\clubsuit, ..., 2\clubsuit, A\heartsuit, ..., 2\heartsuit, A\diamondsuit, ..., 2\diamondsuit, A\spadesuit, ..., 2\spadesuit\}$ be the random variable that represents the outcome of the random draw of a card from a shuffled deck, and consider the score function $f : \mathcal{X} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 11 & \text{if } x \text{ is an ace} \\ 10 & \text{if } x \text{ is a 7} \\ 3 & \text{if } x \text{ is a king} \\ 2 & \text{if } x \text{ is a queen} \\ 1 & \text{if } x \text{ is a jack} \\ 0 & \text{if } x \text{ is any other card} \end{cases}$$

   a) Let $Y = f(X) \in \{0, 1, 2, 3, 10, 11\}$ be random variable that represents the random score obtained by drawing a card; compute the expected value of Y, that is, $\mathbb{E}[Y] = \mathbb{E}[f(X)]$ and its variance $\text{var}[Y] = \mathbb{E}\left[\left(Y - \mathbb{E}[Y]\right)^2\right]$.

   b) Compute the probability that the score Y is larger than 7, that is, $\mathbb{P}[Y > 7]$.

   c) Now consider two random variables Y_1 and Y_2 corresponding to the scores of the first and second drawn cards (with replacement); determine the joint probability function $\mathbb{P}[Y_1 = y, Y_2 = z]$, for all possible outcomes $(x, z) \in \{0, 1, 2, 3, 10, 11\}^2$.

d) Use the result of the previous question to compute the expected value of the score obtained by drawing two cards (with replacement).

e) Repeat the two previous questions, now considering that (as is more natural) the cards are drawn without replacement. Hint, in this case it may be easier to first obtain the conditional probability function $\mathbb{P}[Y_2 = z|Y_1 = y]$, and then obtain the joint probability function via $\mathbb{P}[Y_1 = y, Y_2 = z] = \mathbb{P}[Y_2 = z|Y_1 = y]\,\mathbb{P}[Y_1 = y]$.

3. Let the random variable $X \in \{2, 3, ..., 12\}$ correspond to the sum of the outcomes of a pair of fair dice.

   a) Find the probability distribution of the random variable X and compute its expected value.

   b) Now let $Z \in \mathbb{N}$ denote the random variable that corresponds to the number of tosses necessary to obtain the first 12. Find the probability distribution of the random variable Z and compute its expected value.

4. Consider a pair of binary random variables $(X, Y) \in \{(0, 0), (0, 1), (1, 0), (1, 1)\}$, with the following joint probability function: $P(X = 0, Y = 0) = 1/4$, $P(X = 0, Y = 1) = 1/3$, $P(X = 1, Y = 0) = 1/12$, $P(X = 1, Y = 1) = 1/3$.

   a) Find the (marginal) probability functions of X and Y.

   b) Obtain the conditional probability functions of X given Y and of Y given X.

   c) Compute the expected values $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X + Y]$, and $\mathbb{E}[XY]$.

Solutions

1.a) 1/52
1.b) 1/13
1.c) 1/4
1.d) nothing changes
1.e) nothing changes
1.f) 0.00024

2.a) $\mathbb{E}[Y] = 27/13$; $\text{var}[Y] = 2326/169$
2.b) 2/13
2.c) $\mathbb{P}[Y_1 = y, Y_2 = z] = \mathbb{P}[Y_1 = y]\mathbb{P}[Y_2 = z] = ...$
2.d) 54/13

3.a) $\mathbb{E}[X] = 7$
3.b) $\mathbb{P}[Z = z] = (35/36)^{z-1}(1/36)$; $\mathbb{E}[Z] = 36$

4.a) $\mathbb{P}[X = 0] = 7/12; \mathbb{P}[X = 1] = 5/12; \mathbb{P}[Y = 0] = 1/3; \mathbb{P}[Y = 1] = 2/3$
4.c) $\mathbb{E}[X] = 5/12; \mathbb{E}[Y] = 2/3; \mathbb{E}[X + Y] = 13/12; \mathbb{E}[XY] = 1/3$