Duration: **30** minutes

- Write your number and name below.
- Add your answers on this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and THREE QUESTIONS. The total of points is 4.0.

Number: **Name**:

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1. Let X be a r.v. with p.f. $P(X=x)=\frac{3}{\pi^2 x^2}$, for $x \in \mathbb{Z}\backslash\{0\}$. **(1.0)**

After having derived the p.f. of X^+ and of X^-, show that $E(X^+)=E(X^-)=+\infty$ and therefore: (i) $X \not\in L^1$; (ii) $E(X)$ does not exist.

- **R.v.**

  X

  $P(X=x)=P(X=-x)=\frac{3}{\pi^2 x^2}, \quad x \in \mathbb{R}_X=\mathbb{Z}\backslash\{0\}$ (symmetric around zero, ($\star$))

- **Positive part of** X

$$\begin{aligned}
X^+ &= \max\{X,0\} \\
P(X^+=x) &= P(\max\{X,0\}=x) \\
&= \begin{cases} P(X\le 0)=P(X\le -1)=P(X\ge 1)\overset{(\star)}{=}\frac{1}{2}, & x=0 \\ P(X=x)=\frac{3}{\pi^2x^2}, & x\in\mathbb{N} \end{cases} \\
E(X^+) &\overset{X^+\ge 0,\, Cor.\ 4.78}{=} \sum_{x=0}^{+\infty} x\times P(X=x)=\sum_{x=1}^{+\infty} x\times\frac{3}{\pi^2x^2}=\frac{3}{\pi^2}\sum_{x=1}^{+\infty}\frac{1}{x}=+\infty
\end{aligned}$$

- **Negative part of** X

$$\begin{aligned}
X^- &= -\min\{X,0\}=\max\{-X,0\} \\
P(X^-=x) &= P(\max\{-X,0\}=x)=\begin{cases} P(X\ge 0)=P(X\ge 1)\overset{(\star)}{=}P(X\le -1)=\frac{1}{2}, & x=0 \\ P(X=-x)\overset{(\star)}{=}P(X=x)=\frac{3}{\pi^2x^2}, & x\in\mathbb{N}\end{cases} \\
X^- &\sim X^+ \\
E(X^-) &= E(X^+)=+\infty
\end{aligned}$$

- **Conclusion**

  Since $E(X^+)=E(X^-)=+\infty$, we can add that:

  (i) $E(|X|)=E(X^+)+E(X^-)\not< +\infty$ and thus $X\not\in L^1$, that is, X is not an integrable r.v.;

  (ii) $E(X)$ does not exist, according to the conventions when X is non-integrable. □

2. State Minkowski's moment inequality. **(1.5)**

Illustrate this inequality, when $p=3$ and the random vector (X,Y) has independent components that are uniformly distributed in the interval $[0,1]$.

- **Statement of the Minkowski's moment inequality**

$$X,Y\in L^p,\ p\in[1,+\infty) \quad\Rightarrow\quad E^{\frac{1}{p}}(|X+Y|^p)\le E^{\frac{1}{p}}(|X|^p)+E^{\frac{1}{p}}(|Y|^p)$$

- **Random vector**

$$(X,Y), \quad X \overset{i.i.d.}{\sim} Y \sim \text{uniform}(0,1), \quad f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Transformation of** (X,Y)

$$Z = X+Y, \quad f_Z(z) \overset{Prop.\ 3.98}{=} \begin{cases} z, & 0 \le z \le 1 \\ 2-z, & 1 < z \le 2 \\ 0, & \text{otherwise} \end{cases}$$

- **Requested illustration** $(p=3)$ — Since $X \perp\!\!\!\perp Y$ and $X \sim Y$ we get:

$$E^{\frac{1}{p}}(|X+Y|^p) = \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x+y|^p \times f_{X,Y}(x,y)\,dy\,dx\right]^{\frac{1}{p}} \le E^{\frac{1}{p}}(|X|^p) + E^{\frac{1}{p}}(|Y|^p)$$

$$\left[\int_{-\infty}^{+\infty} |z|^p \times f_Z(z)\,dz\right]^{\frac{1}{p}} = \left[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x+y|^p \times f_X(x) \times f_Y(y)\,dy\,dx\right]^{\frac{1}{p}} \le 2 \times \left[\int_{-\infty}^{+\infty} |x|^p \times f_X(x)\,dx\right]^{\frac{1}{p}}$$

$$\left[\int_0^1 z^3 \times z\,dz + \int_1^2 z^3 \times (2-z)\,dz\right]^{\frac{1}{3}} = \left[\int_0^1\int_0^1 (x+y)^3\,dy\,dx\right]^{\frac{1}{3}} \le 2 \times \left[\int_0^1 x^3\,dx\right]^{\frac{1}{3}}$$

$$\left(\left.\frac{z^5}{5}\right|_0^1 + \left.\frac{z^4}{2}\right|_1^2 - \left.\frac{z^5}{5}\right|_1^2\right)^{\frac{1}{3}} = [\ldots] \le 2 \times \left(\left.\frac{x^4}{4}\right|_0^1\right)^{\frac{1}{3}}$$

$$\left(\frac{1}{5} + 8 - \frac{1}{2} - \frac{32}{5} + \frac{1}{5}\right)^{\frac{1}{3}} \le 2 \times \left(\frac{1}{4}\right)^{\frac{1}{3}}$$

$$\left(\frac{3}{2}\right)^{\frac{1}{3}} \simeq 1.144714 \le 2 \times \left(\frac{1}{4}\right)^{\frac{1}{3}} \simeq 1.259921. \quad \checkmark$$

3. Admit that $(X,Y,Z) \sim \text{normal}_3(\underline{\mu}, \Sigma)$, where $\underline{\mu}$ and Σ are such that: $\mu_X = \mu_Y = \mu_Z = 0$; $\sigma_X^2 = \sigma_Y^2 = \sigma_Z^2 = 1$, $corr(X,Y) = corr(X,Z) = 0$, and $corr(Y,Z) = \rho = 0.5$. **(1.5)**
What is the probability that $\frac{X+Y}{2}$ does not exceed $(Z+1)$?

- **Random vector** (X,Y,Z)

$$(X,Y,Z) \sim \text{normal}_3(\underline{\mu}, \Sigma), \quad \text{where:} \quad \underline{\mu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix}.$$

- **Requested probability and auxiliary r.v.**

$$P[(X+Y)/2 \le Z+1] = P(W = X+Y-2Z \le 2)$$

$$W = X+Y-2Z = \mathbf{C} \times \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \underline{b}, \quad \text{where:} \quad \mathbf{C} = [1 \quad 1 \quad -2]; \quad \underline{b} = [0].$$

$$W \overset{Th.\ 4.216}{\sim} \text{normal}(E(W), V(W)), \quad \text{where:}$$

$$E(W) = \mathbf{C}\underline{\mu} + \underline{b} = 0;$$

$$V(W) = \mathbf{C}\Sigma\mathbf{C}^\top = [1 \quad 1 \quad -2] \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \rho \\ 0 & \rho & 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 & 1-2\rho & \rho-2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = 6 - 4\rho.$$

Hence,

$$\begin{aligned} P(W = X+Y-2Z \le 2) &= \Phi\left\{[2 - E(W)]/\sqrt{V(W)}\right\} \\ &= \Phi\left[(2-0)/\sqrt{6-4\rho}\right] \\ &\overset{\rho=0.5}{=} \Phi(1) = 0.8413. \end{aligned}$$