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Advanced Plasma Physics

MEFT 2021/22

Problem Class 5

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. Two-stream instability in 2D plasmas. Consider a two-dimensional plasma composed of electrons and ions. The electrons stream in the plasma with velocity $\mathbf{u}_0 = u_0\hat{\mathbf{x}}$. Remember that the two-dimensional Green's function is given by $G(\boldsymbol{\rho}) = \dfrac{e}{4\pi\epsilon_0|\boldsymbol{\rho}|}$, such that the electrostatic potential reads

$$\phi(\boldsymbol{\rho}) = \int \left[n_i(\boldsymbol{\rho}') - n_e(\boldsymbol{\rho}')\right] G(\boldsymbol{\rho} - \boldsymbol{\rho}')\, d\boldsymbol{\rho}'$$

a) Start from the fluid equations governing the motion of the electrons to show that the dielectric function reads

$$\epsilon(\omega, k) = 1 - \frac{g_e k}{(\omega - k u_0)^2} - \frac{g_i k}{\omega^2},$$

where $g_\alpha = \dfrac{e^2 n_0}{2\epsilon_0 m_\alpha}$ is the effective acceleration for the species $\alpha = \{e, i\}$.

b) Repeat the procedure you worked out in Week 2 to show that the dynamical instability happens provided the condition $k u_0 < \omega_c$, where

$$\omega_c = k u_0 \frac{(g_i/g_e)^{1/3}}{1 + (g_i/g_e)^{1/3}} \simeq k u_0 \left(\frac{m_e}{m_i}\right)^{1/3}.$$

c) Replace this condition to show that the dynamical instability takes place in the spectral range given by

$$k \leq \frac{g_e}{u_0^2}\left[\frac{1}{\left(1 - (g_i/g_e)^{1/3}\right)^2} + \left(\frac{g_i}{g_e}\right)^{1/3}\right] \simeq \frac{g_e}{u_0^2}\left[1 + 3\left(\frac{m_e}{m_i}\right)^{1/3}\right].$$

Compare with the result you obtained for the 3D case.

d) The most unstable mode is the one that resonates with the 2D electron plasma wave, $ku_0 \simeq \sqrt{g_e k}$, i.e. occurs for $k_{\max} \simeq g_e/u_0^2$. Assuming that $\omega \ll \sqrt{g_e k_{\max}} = g_e/u_0$, show that the maximum growth rate is expected to be

$$\gamma = \mathrm{Im}(\omega) \simeq \frac{\sqrt{3}}{2^{4/3}} \left(\frac{m_e}{m_i}\right)^{1/3} \frac{g_e}{u_0}.$$

Comparing with what you know for the 3D case, comment on the sensitivity of the growth rate with the mass ratio m_e/m_i and the relation between ω_{pe} and g_e/u_0.

e) With help of Mathematica, plot the dispersion relation numerically, observing the behaviour of both $\mathrm{Re}(\omega)$ and $\mathrm{Im}(\omega)$. Discuss with your colleagues, by putting some numbers on it (choose a ratio of $m_e/m_i \sim 10^{-1}$ and normalize the frequency as $\omega \to \omega u_0/g_e$ and the wavector as $k \to k u_0^2/g_e$, for the numerical evaluations) and conclude if the analytical estimates are better or worse when compared with the 3D case.

Problem 2. MHD waves. As discussed in the class, the magnetohydrodynamics (MHD) model allows for a remarkable simplification in the treatment of phenomena in magnetized plasmas, as the two-fluid model can be effectively reduced to a single fluid equation. The MHD model is valid is the magnetic field is sufficiently strong, such that electrostatic effects can be neglected (i.e. the electron-ion fluid moves as a whole, therefore meeting the quasi-neutrality condition identically, $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$). It is instructive to understand which waves can a plasma support within the MHD framework. Here, we will adopt a generic formulation of the such waves.

a) Consider the basic MHD equations,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0, \quad \frac{D\mathbf{u}}{Dt} = \frac{\mathbf{J} \times \mathbf{B}}{\rho} - \frac{\boldsymbol{\nabla} P}{\rho}, \quad \mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} - \frac{\boldsymbol{\nabla} P_e}{e n_0},$$

where $\eta = m_e \nu_{ei}/(n_0 e^2)$ is the plasma resistivity due to electron-ion collision at rate ν_{ei}, $D/Dt = \partial/\partial t + u \cdot \boldsymbol{\nabla} u$ stands for the material derivative, $\rho = m_i n_i + m_e n_e \simeq n_0(m_e + n_e)$ is the plasma (mass) density, $\mathbf{u} = (m_i \mathbf{u}_i + m_e \mathbf{u}_e)/(m_e + m_i)$ is the plasma velocity field and $\mathbf{J} = e(n_i \mathbf{u}_i - n_e \mathbf{u}_e) \simeq e n_0 (\mathbf{u}_i - \mathbf{u}_e)$ is the plasma current. Combine Faraday's and Ampère's laws to show that the magnetic field is governed by the following equation

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left(\mathbf{u} \times \mathbf{B} + \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times B - \frac{\boldsymbol{\nabla} P_e}{e n_0} \right).$$

b) Let us now focus on the special class of *ideal MHD waves*, by setting $\eta \to 0$, and recast the velocity field in terms of the *displacement vector* $\boldsymbol{\xi}$,

$$\mathbf{u} = \frac{\partial \boldsymbol{\xi}}{\partial t}.$$

Linearize the ideal MHD equations (i.e. make $X = X_0 + \delta X$, where X is any relevant physical quantity) to obtain

$$\frac{\delta \rho}{\rho_0} = -\boldsymbol{\nabla} \cdot \boldsymbol{\xi}, \quad \frac{\delta P}{P_0} = -\gamma \boldsymbol{\nabla} \cdot \boldsymbol{\xi}, \quad \frac{\partial}{\partial t} \delta \mathbf{B} = (\mathbf{B}_0 \cdot \boldsymbol{\nabla}) \frac{\partial \boldsymbol{\xi}}{\partial t} - \mathbf{B}_0 \left(\boldsymbol{\nabla} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} \right),$$

where γ is the adiabatic index. [<u>Hint</u>: no need to linearize the momentum equation at this point...]

c) Decompose the previous equations into their parallel and perpendicular components, (i.e., define $\mathbf{X}_{\parallel} = \left(\hat{\mathbf{b}} \cdot \mathbf{X} \right) \hat{\mathbf{b}}$, with $\hat{\mathbf{b}} = \mathbf{B}_0 / B_0$, and $\mathbf{X}_{\perp} = \mathbf{X} - \mathbf{X}_{\parallel}$) to show that

$$\frac{\delta \mathbf{B}_{\parallel}}{B_0} = \nabla_{\parallel} \boldsymbol{\xi}_{\perp}, \quad \frac{\delta \mathbf{B}_{\perp}}{B_0} = -\boldsymbol{\nabla}_{\perp} \cdot \boldsymbol{\xi}_{\perp},$$

where $\nabla_{\parallel} = \hat{\mathbf{b}} \cdot \boldsymbol{\nabla}_{\parallel}$.

d) Now proceed to the linearization of the momentum equation, and combine with the previous results, to obtain

$$\rho_0 \frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = \gamma P_0 \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) + \frac{1}{\mu_0} \left[B_0^2 \boldsymbol{\nabla}_{\perp} \left(\boldsymbol{\nabla}_{\perp} \cdot \boldsymbol{\xi} \right) + B_0^2 \nabla_{\parallel}^2 \boldsymbol{\xi}_{\perp} \right].$$

By diving everything by ρ_0, show that the wave equation reads

$$\frac{\partial^2 \boldsymbol{\xi}}{\partial t^2} = C_s^2 \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{\xi} \right) + V_{\mathrm{A}}^2 \left[\boldsymbol{\nabla}_\perp \left(\boldsymbol{\nabla}_\perp \cdot \boldsymbol{\xi} \right) + \nabla_\parallel^2 \boldsymbol{\xi}_\perp \right],$$

where $C_s = \sqrt{\gamma P_0/\rho_0}$ is the sound speed and $V_{\mathrm{A}} = B_0/(\mu_0 \rho_0)$ is the *Alfvén speed.*

e) Consider propagation along the magnetic field lines, $\mathbf{k} \parallel \mathbf{B}_0 = B_0 \hat{\mathbf{z}}$. For definiteness, assume then that the displacement field is defined in the $\hat{\mathbf{y}}$ direction (you could consider the generic case, but this simplifies a bit the calculations). Obtain the dispersion relation for the shear Alfvén waves,

$$\omega = V_{\mathrm{A}} k.$$

Show that these waves correspond to incompressible electromagnetic modes.

f) Similarly, consider now the compressional sector, $\boldsymbol{\xi} = (\xi_x, 0, \xi_z)$. Obtain the dispersion relation for the magnetosonic waves,

$$\omega_\pm = \frac{1}{2} k^2 \left[C_s^2 + V_{\mathrm{A}}^2 \pm \sqrt{(C_s^2 + V_{\mathrm{A}}^2)^2 - 4 C_s^2 V_{\mathrm{A}}^2 \cos\theta^2} \right], \quad \cos\theta = \frac{k_\parallel}{\sqrt{k_\parallel^2 + k_\perp^2}},$$

with the $\pm$ signs standing for the *fast* and *slow* magnetosonic modes. For the case of $\xi_x = 0$, we simply get $\omega = C_s k$, which clearly means that the wave is compressive (why)?.