DM
DEPARTAMENTO DE MATEMÁTICA
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Probability Theory
LMAC, MMA

1st. Semester – 2021/2022
2021/12/14 — 12:10
MAP30 #3

Duration: **30** minutes

- Write your number and name below.
- Add your answers on this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and THREE QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. Let two fair dice be tossed (independently) and $\Omega = \{(i,j) : i,j = 1,\dots,6\}$ be the associated sample space. (**1.0**)

Consider the events:

- $A = \{$points of the first die $= 1,2$ or $3\}$,
- $B = \{$points of the first die $= 3,4$ or $5\}$,
- $C = \{$the sum of points of the two dies $= 9\}$.

Show that $P(A\cap B\cap C) = P(A)\times P(B)\times P(C)$ and yet these three events are not mutually independent.

- **Events and probabilities**

  Since the 36 elementary events of the sample space are equiprobable, we get

$$
\begin{aligned}
A &= \{(i,j) : i = 1,2,3,\ j = 1,2,3,4,5,6\}\\
P(A) &= \#A \times \frac{1}{36} = \frac{3\times 6}{36} = \frac{1}{2}\\
B &= \{(i,j) : i = 3,4,5,\ j = 1,2,3,4,5,6\}\\
P(B) &= \#B \times \frac{1}{36} = \frac{3\times 6}{36} = \frac{1}{2}\\
C &= \{(i,j) : i,j = 1,2,3,4,5,6,\ i+j = 9\} = \{(3,6),(4,5),(5,4),(6,3)\}\\
P(C) &= \#C \times \frac{1}{36} = \frac{4}{36} = \frac{1}{9}
\end{aligned}
$$

- **Requested proof**

$$
\begin{aligned}
A\cap B\cap C &= (3,6)\\
P(A\cap B\cap C) &= P((3,6)) = \frac{1}{36} \quad = \quad P(A)\times P(B)\times P(C) = \frac{1}{2}\times\frac{1}{2}\times\frac{1}{9} = \frac{1}{36} \qquad \checkmark\\
A\cap B &= \{(3,j) : j = 1,2,3,4,5,6\}\\
P(A\cap B) &= \#(A\cap B)\times\frac{1}{36} = \frac{6}{36} = \frac{1}{6} \quad \neq \quad P(A)\times P(B) = \frac{1}{2}\times\frac{1}{2} = \frac{1}{4},
\end{aligned}
$$

  thus, the 3 events are not pairwise independent and therefore not mutually independent. ✓

2. Let X and Y two i.i.d. r.v. with common standard normal distribution. (**1.5**)

Derive the p.d.f. of $W = \frac{X}{Y}$.

- **Random vector and range**

  (X,Y), $\quad X \overset{indep.}{\sim} Y \sim \text{normal}(0,1)$

  $\mathbb{R}_{X,Y} = (\mathbb{R})^2$

- **Transformation of (X,Y) and its range**

  $W = g(X,Y) = \frac{X}{Y}$

$\mathbb{R}_W = g(\mathbb{R}_{X,Y}) = \mathbb{R}$

- **P.d.f. of W**

$$\begin{aligned}
f_W(w) &= f_{X/Y}(w) \\
&\overset{X \perp\!\!\!\perp Y}{=} \int_{-\infty}^{+\infty} f_X(wy) \times f_Y(y) \times |y| \, dy \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{w^2 y^2}{2}\right) \times \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right) \times |y| \, dy \\
&= -\int_{-\infty}^{0} \frac{1}{2\pi} \exp\left[-\frac{y^2}{2} \times (w^2+1)\right] \times y \, dy + \int_{-\infty}^{0} \frac{1}{2\pi} \exp\left[-\frac{y^2}{2} \times (w^2+1)\right] \times y \, dy \\
&= \frac{1}{2\pi} \frac{1}{w^2+1} \int_{-\infty}^{0} (-y)(w^2+1) \times \exp\left[-\frac{y^2}{2} \times (w^2+1)\right] dy \\
&\quad -\frac{1}{2\pi} \frac{1}{w^2+1} \int_{0}^{\infty} (-y)(w^2+1) \times \exp\left[-\frac{y^2}{2} \times (w^2+1)\right] dy \\
&= \frac{1}{2\pi} \frac{1}{w^2+1} \times \exp\left[-\frac{y^2}{2} \times (w^2+1)\right]\Bigg|_{-\infty}^{0} - \frac{1}{2\pi} \frac{1}{w^2+1} \times \exp\left[-\frac{y^2}{2} \times (w^2+1)\right]\Bigg|_{0}^{\infty} \\
&= \frac{1}{\pi \times (1+w^2)}, \quad w \in \mathbb{R}.
\end{aligned}$$

3. Let $\{X_i : i \in \mathbb{N}\}$ be a Bernoulli process with parameter p and $S_n = \sum_{i=1}^{n} X_i$. **(1.5)**

Derive the p.f. of $(S_m \mid S_n = k)$, for $n \in \mathbb{N}$, $m = 1, \ldots, n$, and $k = 0, 1, \ldots, n$.

- **Stochastic process**

$\{X_i : i \in \mathbb{N}\} \sim BP(p)$

$X_i \overset{i.i.d.}{\sim} \text{Bernoulli}(p), \quad i \in \mathbb{N}$

$p = P(\text{success}), \quad p \in (0,1)$

- **R.v.**

$S_n = \sum_{i=1}^{n} X_i$ = number of successes in the first n Bernoulli trials $\sim$ binomial(n,p), $\quad n \in \mathbb{N}$

- **Requested p.f.**

$P(S_m = x \mid S_n = k) = \frac{P(S_m = x, S_n = k)}{P(S_n = k)}$, where:

$$\begin{aligned}
P(S_n = k) &= \binom{n}{k} p^k (1-p)^{n-k}; \\
P(S_m = x, S_n = k) &= P\left(\sum_{i=1}^{m} X_i = x, \sum_{i=m+1}^{n} X_i = k - x\right) \\
&\overset{disj.\,block\,theo.}{=} P\left(\sum_{i=1}^{m} X_i = x\right) \times P\left(\sum_{i=m+1}^{n} X_i = k - x\right) \\
&= P_{binomial(m,p)}(x) \times P_{binomial(n-(m+1)+1,p)}(k-x) \\
&= \binom{m}{x} p^x (1-p)^{m-x} \times \binom{n-m}{k-x} p^{k-x} (1-p)^{(n-m)-(k-x)} \\
&= \binom{m}{x} \binom{n-m}{k-x} p^k (1-p)^{n-k}.
\end{aligned}$$

Thus,

$$P(S_m = x \mid S_n = k) = \frac{\binom{m}{x}\binom{n-m}{k-x} p^k (1-p)^{n-k}}{\binom{n}{k} p^k (1-p)^{n-k}} = \frac{\binom{m}{x}\binom{n-m}{k-x}}{\binom{n}{k}},$$

[for $x \in \{\max\{0, k-(n-m)\}, \ldots, \min\{k, n\}\}$]. ✓