Advanced Plasma Physics

MEFT 2021/22

Problem Class 3

[SOLUTION]

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. The Kortweg-de Vries equation. Let us consider the propagation of nonlinear ion-acoustic waves in uniform, unmagnetized plasmas. For that, we should rely on a fluid description of the problem (consider one-dimensional electrostatic waves, for simplicity)

$$\frac{\partial n_\alpha}{\partial t} + \frac{\partial (n_\alpha u_\alpha)}{\partial x} = 0, \qquad \frac{\partial u_\alpha}{\partial t} + u_\alpha \frac{\partial u_\alpha}{\partial x} = \frac{q_\alpha}{m_e} E - \frac{1}{m_\alpha n_\alpha}\frac{\partial P_\alpha}{\partial x}.$$

a) At the scale of the ion motion, the electrons are not at rest. On the contrary, they move so fast that they follow the ions adiabatically, therefore being in thermal equilibrium. Show that the linearized Poisson equation yields

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\lambda_{De}^2}\right)\phi_1 = -\frac{e}{\epsilon_0} n_1.$$

The linearized Poisson equation for the electron-ion plasma is given by

$$\frac{\partial^2 \phi_1}{\partial x^2} = \frac{e}{\epsilon_0}(n_e - n_i) = \frac{e}{\epsilon_0}\left(n_0 e^{\frac{e\phi_1}{k_B T_e}} - n_0 - n_1\right) \simeq \frac{e}{\epsilon_0}\left[n_0\left(1 + \frac{e\phi_1}{k_B T_e}\right) - n_0 - n_1\right].$$

As $\lambda_{De} = \sqrt{k_B T_e \epsilon_0 / e^2 n_0}$, we obtain the result above.

b) Make use of the equations of motion for the ions to show that, in the limit $T_i \ll T_e$, we obtain

$$\omega = \frac{c_s k}{\sqrt{1 + k^2 \lambda_{De}^2}}. \tag{1}$$

From the continuity equation, we get

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial u_1}{\partial x} = 0 \quad \Longrightarrow \quad \tilde{n}_1 = \frac{n_0 k}{\omega} \tilde{u}_1.$$

Moreover, from the momentum conservation equation,

$$\frac{\partial u_1}{\partial t} = -\frac{e}{m_i} \frac{\partial \phi_1}{\partial x} \quad \Longrightarrow \quad \tilde{u}_1 = \frac{ek}{m_i \omega} \tilde{\phi}_1.$$

Upon Fourier transforming the Poisson equation in the previous point, we get

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{\lambda_{De}^2} \right) \phi_1 = -\frac{e}{\epsilon_0} n_1 \quad \Longrightarrow \quad \tilde{\phi}_1 = \frac{e}{\epsilon_0} \frac{\lambda_{De}^2}{1 + k^2 \lambda_{De}^2} \tilde{n}_1 = \frac{e n_0}{\epsilon_0} \frac{k \lambda_{De}^2}{\omega \left(1 + k^2 \lambda_{De}^2\right)} \tilde{u}_1.$$

Putting everything together, we have

$$\tilde{u}_1 = \underbrace{\frac{e^2 n_0}{\epsilon_0 m_i}}_{\omega_{pi}^2} \frac{k^2 \lambda_{De}^2}{\omega^2 \left(1 + k^2 \lambda_{De}^2\right)} \tilde{u}_1 \quad \Longrightarrow \quad \omega = \frac{c_s k}{\sqrt{1 + k^2 \lambda_{De}^2}}.$$

c) We now come back to the original equations, but keeping the nonlinearity appearing in the momentum conservation equation (the so-called convective term). Show that

$$\mathcal{F}\left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} \right) u_1 \right] = -i \frac{\omega_{pi}^2}{\omega} \frac{k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} \tilde{u}_1, \tag{2}$$

where $\mathcal{F}[A(x,t)] \equiv \tilde{A}(k,\omega)$ is the Fourier transform of a certain quantity $A(x,t)$.

From the force equation, now accounting for the convective term, we have

$$\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} \right) u_1 = -\frac{e}{m_i} \frac{\partial \phi_1}{\partial x}.$$

By Fourier transforming both sides of the equation, we have

$$\mathcal{F}\left[\left(\frac{\partial}{\partial t} + u_1 \frac{\partial}{\partial x} \right) u_1 \right] = -i \frac{ek}{m_i} \tilde{\phi}_1 = -i \frac{\omega_{pi}^2}{\omega} \frac{k^2 \lambda_{De}^2}{1 + k^2 \lambda_{De}^2} \tilde{u}_1 \simeq -i \frac{\omega_{pi}^2}{\omega} k^2 \lambda_{De}^2 \left(1 - \frac{1}{2} k^2 \lambda_{De}^2 \right) \tilde{u}_1.$$

d) We are interested in the region of the ion spectrum where the dispersion starts loosing its acoustic character, $k\lambda_{De} \simeq 1$. For that, we replace ω in the denominator of Eq. (2). Then, we expand the denominator in the second factor of the RHS to first order. Upon replacing $k \to -i \dfrac{\partial}{\partial x}$ (momentum operator in quantum mechanics, right?), show that Eq. (2) reduces to the *Kortweg-de Vries equation*,

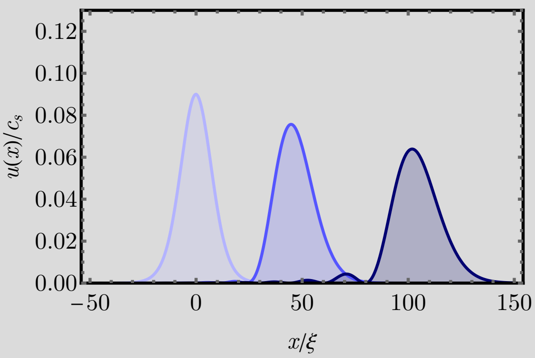
$$\frac{\partial u_i}{\partial t} + u_i \frac{\partial u_i}{\partial x} + c_s \frac{\partial u_i}{\partial x} + \frac{1}{2} c_s \lambda_{De}^2 \frac{\partial^3 u_i}{\partial x^3} = 0.$$

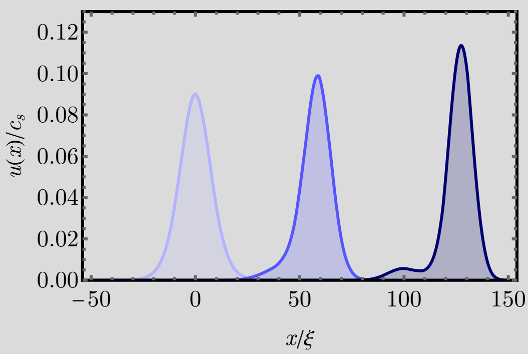
We now apply an inverse Fourier transform to the problem, but replace $\omega \simeq c_s k$ in the denominator of the RHS of the equation,

$$\mathcal{F}^{-1}\left\{\mathcal{F}\left[\left(\frac{\partial}{\partial t}+u_1\frac{\partial}{\partial x}\right)u_1\right]\right\} = \mathcal{F}^{-1}\left\{-i\frac{\omega_{pi}^2}{c_s k}k^2\lambda_{De}^2\left(1-\frac{1}{2}k^2\lambda_{De}^2\right)\tilde{u}_1\right\}$$
$$\frac{\partial u_1}{\partial t}+u_1\frac{\partial u_1}{\partial x} = \mathcal{F}^{-1}\left\{-ic_s k\left(1-\frac{1}{2}k^2\lambda_{De}^2\right)\tilde{u}_1\right\}$$
$$= -c_s\frac{\partial}{\partial x}\left(1+\frac{1}{2}\lambda_{De}^2\frac{\partial^2}{\partial x^2}\right)u_1.$$

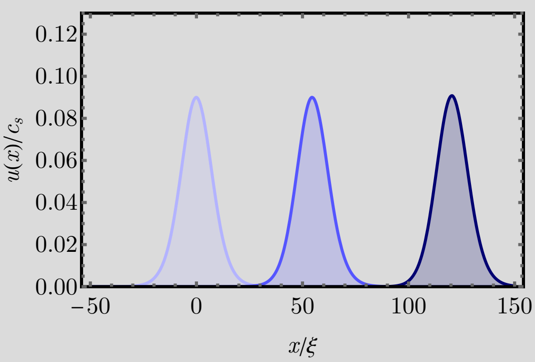
e) Make use of the Mathematica script available at our webpage to observe what happens in the following cases: i) neglecting the nonlinear term, ii) neglecting the dispersive term. Discuss with your colleagues the physics of both numerical solutions.

Making use of the Mathematica notebook to solve the Kortweg-de Vries (K-dV) equation numerically, we obtain the following solutions



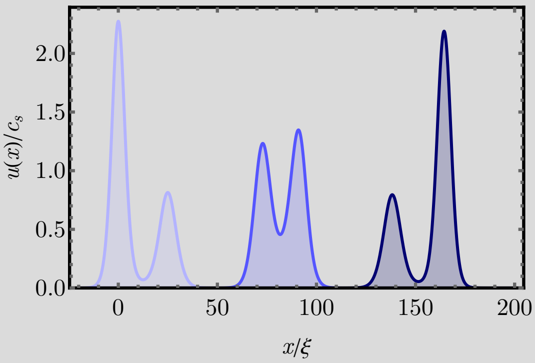


By neglecting the nonlinear ter, (case i)), we observe that dispersion starts to dominate the dynamics as the wave propagates (the wave packet spreads). Conversely, by neglecting the dispersive term (case ii)), the wave undergoes steeping as it propagates. In the situation where both terms are taken into account, numerical solutions produce solitons, i.e. the wave propagates without any deformation.



f) Simulate the case of two solitons colliding against each other. Observe the features of such collisions. Do the wavepackets break at anytime? What happens to the original form of the solitons after the collision? Maybe you are ready to explain to your colleagues why these nonlinear waves receive the name of *solitons*.

Let us consider two solitons, initially located at different positions, $x = 0$ and $x = 25\lambda_{De}$ (light blue line). The leftmost soliton is faster than the second one. At the collision point (blue line), their amplitudes equal. After this time, the faster soliton leaves the slower soliton behind (darker blue line). Interestingly, the shape of both solitons remain unchanged after the collision. This is why these nonlinear waves receive the name "soliton": they are solitary waves that behave like particles.



Problem 2. Trievelpiece-Gould waves. Consider a plasma produced at the interior of a cylindrical container of radius a. Let us assume, for definiteness, that such a container is metallic. In the following calculations, we make use of the cylindrical coordinates (r, θ, z), and consider waves propagating along the column axis, i.e. the $z-$ direction.

a) Start by showing that the Poisson equation can be written as

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right)\Phi = \frac{e}{\epsilon_0}\left(n_e - n_i\right),$$

where $\Phi = \Phi(r, \theta, z) = \phi(r, \theta)\varphi(z)$, and $n_i(x, y, z)$ and $n_e(x, y, z)$ are the 3D ionic and electronic densities, respectively.

This is obvious. This simply follow by splitting the Laplace operator into its terms,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla_\perp^2 + \frac{\partial^2}{\partial z^2}.$$

b) Consider the homogeneous Poisson (or Laplace) equation, resulting from the plasma approximation. Making use of the separation of variables above, show that the transverse component of the potential satisfies the Helmholtz equation

$$\nabla_\perp^2\phi + q^2\phi = 0, \tag{3}$$

where q is some arbitrary constant.

We can make use of the variable separation method to solve the homogeneous Poisson equation,

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right)\phi(x, y)\varphi(z) = \frac{e}{\epsilon_0}\left(n_e - n_i\right) \simeq 0,$$

where we assume quasi-neutrality. The latter equation can be recast in a more appealing form as

$$\frac{\nabla_\perp^2\phi}{\phi} = -\frac{1}{\varphi}\frac{\partial^2\varphi}{\partial z^2} = -q^2.$$

To get this last step, we observe that the RHS of the equation is a function of z alone, while the LHS is a function of $(x, y) = (r\theta)$. As such, the only nontrivial solutions is attainable if both members are equal to some arbitrary constant, that we call q^2 (the square comes for convenience - q then has the good physical units of wavector, and thus q^2 has the same units of the Laplace operator).

c) For symmetry reasons, we may expect $\phi(r, \theta)$ to display radial symmetry. As such, it can be decomposed as

$$\phi(r, \theta) = \sum_\ell R_\ell(r)e^{i\ell\theta},$$

where ℓ is an integer (why?). Show that the $R_\ell(r)$ satisfy the Bessel equation,

$$x^2 R_\ell'' + xR_\ell' + \left(x^2 - \ell^2\right)R_\ell = 0,$$

where $x = qr$.

The solution to the Helmholtz equation obtained in the last point is now straightforward. In polar coordinates, the transverse Laplacian reads

$$\nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Using the proposed decomposition, we have

$$\nabla_\perp^2 \phi + q^2\phi = 0 \quad \Leftrightarrow \quad R_\ell'' + \frac{1}{r}R_\ell' - \frac{\ell^2}{r^2}R_\ell + q^2 R_\ell = 0.$$

Multiplying the whole equation by r^2 and defining $x = qr$, the job is done.

d) Make use of the appropriate boundary conditions to show that the formal profile of the transverse potential is given by

$$\phi(r,\theta) = \sum_{n,\ell} \mathcal{A}_\ell J_\ell(k_{n,\ell} r) e^{i\ell\theta},$$

where $k_{n,\ell} = \alpha_{n,\ell}/a$ and $\alpha_{n,\ell}$ is the nth zero of the ℓth Bessel function of the first kind, $J_\ell(x)$.

The general solution to the Bessel equation is given in terms of the Bessel function,

$$R_\ell(x) = \mathcal{A}_\ell J_\ell(x) = \mathcal{A}_\ell J_\ell(qr),$$

where $\mathcal{A}_\ell$ is some arbitrary coefficient. The presence of a metallic boundary at $r = a$ forces the potencial to vanish at that point. As such,

$$\phi(a) = \sum_\ell \mathcal{A}_\ell J_\ell(qa) e^{i\ell\theta} = 0 \quad \Longrightarrow \quad qa = \alpha_{n,\ell},$$

where $\alpha_{n,\ell}$ is the n-th zero of the Bessel function $J_\ell(x)$. As such, we have that $q = \alpha_{n,\ell}/a \equiv k_{n,\ell}$.

e) We now restrict the discussion to the first harmonic, i.e. $\ell = 0$, corresponding to the lowest excitation along the transverse direction (i.e. the potential vanishes only at the border of the container). In what follows, we show that the longitudinal electron waves inherit the structure of the transverse potential. First, convince yourself that the resulting potential along the $z-$direction reads

$$\left(\frac{\partial^2}{\partial z^2} - k_n^2\right)\Phi = \frac{e}{\epsilon_0}\left(n_e - n_i\right),$$

where $k_n \equiv \alpha_{n,0}/a$. Then, work out the fluid equations to obtain the dispersion relation of the Trivelpiece-Gould waves

$$\omega^2 = \omega_{pe}^2 \frac{k^2}{k^2 + k_n^2} + \gamma_e v_e^2 k^2, \tag{4}$$

where $v_e = \sqrt{k_B T_e / m_e}$. Plot the dispersion relation for the first and second harmonics ($n = 0$ and $n = 1$) and explain what is happening physically. What is apparently strange with these waves? Does it remind you of something?

If we assume quasi-neutrality along the transverse direction, the only fluctuations possible are along the $z-$direction. From the Poisson equation, we have

$$\left(\nabla_\perp^2 + \frac{\partial^2}{\partial z^2}\right)\phi(r,\theta)\varphi(z) = \frac{e}{\epsilon_0}(n_e - n_i) = f(z)$$

$$\left(-k_{n,\ell}^2 + \frac{\partial^2}{\partial z^2}\right)\phi(r,\theta)\varphi(z) = \frac{e}{\epsilon_0}(n_e - n_i) = f(z)$$

$$\left(-k_{n,\ell}^2 + \frac{\partial^2}{\partial z^2}\right)\Phi = \frac{e}{\epsilon_0}(n_e - n_i) = f(z).$$

We now repeat the process that allows us to derive the dispersion relation from fluid equations, by taking $n_e = n_0 + n_1$, $n_i = n_0$. The difference is, now, the inclusion of the electron pressure explicitly in the momentum conservation equation,

$$\frac{\partial P_1}{\partial x} = \gamma_e k_B T_e \frac{\partial n_1}{\partial x}.$$

Considering the lowest mode, $\ell = 0$, we have

$$\omega^2 = \omega_{pe}^2 \frac{k^2}{k^2 + k_n^2} + \gamma_e v_e^2 k^2.$$

In the cold plasma limit, we have

$$\omega = \omega_{pe} \frac{k}{\sqrt{k^2 + k_n^2}} \neq \omega_{pe} \;\; (!)$$

This is very similar to the case of ion-acoustic waves, if we replace k_n by $1/\lambda_{De}$. The reason why electrons now display acoustic behaviour comes from the shielding imposed by the metallic cylinder containing the plasma. In other words, the metallic cylinder acts as a capacitor, which establishes a <u>local</u> relation between the charge and the potential,

$$C = \frac{\Delta Q}{\Delta V} = \frac{-e n_1 \mathcal{V}}{-e\Phi_1} \quad \Rightarrow \quad \Phi_1(z) \propto n_1(z),$$

where $\mathcal{V} = \pi a^2 L$ is the volume. In unbounded plasmas, the relation between the potential and the density is <u>nonlocal</u> (governed by the Poisson equation with $k_n = 0$)

$$\Phi_1(z) \propto \int \frac{n_1(z')}{|z - z'|} dz'.$$