Test preparation

Algebraic and Geometric Methods for Engineering and Physics

1. Which of the following sets of matrices form groups under matrix multiplication? Provide a short proof or a counterexample.

   (a) $M_{2\times 2}(\mathbb{Z})$;

   (b) $\{A \in M_{2\times 2}(\mathbb{Z}) \,:\, \det A \neq 0\}$;

   (c) $\{A \in M_{2\times 2}(\mathbb{Z}) : \det A = 1\}$;

   (d) $\{A \in M_{2\times 2}(\mathbb{Z}_2) : \det A \neq 0\}$.

2. Find a multiplicative inverse for $[3] \in \mathbb{Z}_{16}$.

3. Mark as true or false. providing a short proof or counterexample:

   (a) $\mathbb{Z}_n$ is cyclic for any $n \in \mathbb{N}$.

   (b) If $p = |G|$ is prime, then G is isomorphic to $\mathbb{Z}_p$.

   (c) If all proper subgroups of G are abelian then G is abelian.

   (d) Every subgroup of a normal subgroup is normal.

   (e) If H is a normal subgroup of G, then G/H is abelian.

   (f) Every group has a conjugacy class with a single element.

   (g) Every free action is effective.

   (h) If an action is effective, then no isotropy subgroup is trivial.

   (i) An action is free if and only if it has no fixed points.

4. If all months were 30 days long, what would the periodicity of the calendar be? (That is, how long would it take for a Friday the 13th, say, to repeat itself?)

5. Show that the subset
$$K = \{e,\, (12)(34),\, (13)(24),\, (14)(23)\}$$
is a normal subgroup of S_4.

6. Let G be a group and $H \subseteq G$ be a nonempty, finite subset. Show that H is a subgroup if and only if $xy \in H$ for all $x, y \in H$.

7. Let G be a finite group and $C \subseteq G$ be a conjugacy class. Using the fact that G acts transitively on C by conjugation, prove that $|C|$ divides $|G|$.

8. Determine whether each of the following group actions $G \overset{\varphi}{\curvearrowright} \mathbb{C}$ are effective, free or transitive, and compute the set of orbits $\mathbb{C}/G$, the set of fixed points $\mathbb{C}^G$, and isotropy subgroups G_z for all points $z \in \mathbb{C}$:

   (a) $G = \mathbb{R}^*$, $\varphi_t(z) = tz$;

   (b) $G = \mathbb{R}$, $\quad \varphi_t(z) = e^{it} z$;

   (c) $G = S^1$, $\quad \varphi_t(z) = tz$;

   (d) $G = \mathbb{C}^*$, $\quad \varphi_t(z) = tz$;

   (e) $G = \mathbb{C}$, $\quad \varphi_t(z) = z + t$.

9. Which of the group actions in Problem 7 are representations?

10. Show that the standard representation $S_n \overset{\varphi^{\mathrm{st}}}{\curvearrowright} \mathbb{C}^n$ is reducible.