Probability Theory

LMAC, MMA

1st. Semester – 2021/2022
2021/10/19 — 12:10
MAP30 #1

Duration: **30** minutes

- Write your number and name below.
- Add your answers on this and the following page.
- Please justify all your answers.
- This test has ONE PAGE and THREE QUESTIONS. The total of points is 4.0.

Number: **Name**:

1. $\mathscr{P}$ is a $\pi-$system on Ω if it is non-empty family of subsets of Ω such that: $A, B \in \mathscr{P} \Rightarrow A \cap B \in \mathscr{P}$. **(1.5)**

$\mathscr{D}$ is a $D-$system on Ω if it is a non-empty family of subsets of Ω and the following three conditions hold: (i) $\Omega \in \mathscr{D}$; (ii) $A \in \mathscr{D} \Rightarrow A^c \in \mathscr{D}$; (iii) $A_1, A_2, \dots \in \mathscr{D}$ and $A_i \cap A_j = \emptyset$, for $i \neq j \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathscr{D}$.[1]

After identifying the sole difference between a $D-$system and a $\sigma-$algebra, prove that a $D-$system which is also a $\pi-$system is a $\sigma-$algebra on Ω.

Hint: disjointification technique.

- **Sole difference**

  A $D-$system on Ω is closed under countable unions of pairwise disjoint events (Resnick, 1999, p. 36), whereas a $\sigma-$algebra on Ω is closed under countable unions of ANY EVENTS.

- **Proof**

  Since conditions (i) and (ii) hold for both $D-$systems and $\sigma-$algebras on Ω, we have to prove that condition (iii) combined with the closure under finite intersections of $\pi-$systems lead to the closure under countable unions of ANY EVENTS.

  Let $A_1, A_2, \dots \in \mathscr{D}$ be ANY EVENTS from the $D-$system which is also a $\pi-$system on Ω. Then:

  - $A_1^c, A_2^c, \dots \in \mathscr{D}$ because a $D-$system is closed under complementation (condition (ii));
  - $B_1 = A_1, B_2 = A_2 \backslash A_1, B_3 = A_3 \backslash (A_1 \cup A_2), \dots, B_n = A_n \backslash (\cup_{i=1}^{n-1} A_i), \dots$, are disjoint events obtained by using the disjointification technique;
  - $B_2 = A_1 \cap A_2^c$, $B_3 \cap (A_1^c \cap A_2^c)$, $\dots$, $B_n = A_n \cap (\cap_{i=1}^{n-1} A_i^c)$, $\dots$, result from finite intersections of events of the $D-$system which is also a $\pi-$system, therefore also belong to $\mathscr{D}$;
  - $\cup_{n=1}^{\infty} A_n = \cup_{n=1}^{\infty} B_n$ belongs to $\mathscr{D}$ as well, because a $D-$system is closed under countable unions of pairwise disjoint events.

2. Let $N_{k,n}$ be the event that EXACTLY k of n events $(A_1, A_2, A_3, \dots, A_n)$ occur. Obtain $P(N_{1,3})$.[2] **(1.5)**

- **Requested probability**

$$\begin{aligned} P(N_{1,3}) &= P(\text{EXACTLY 1 of 3 events occur}) \\ &= P\left([A_1 \backslash (A_2 \cup A_3)] \cup [A_2 \backslash (A_1 \cup A_3)] \cup [A_3 \backslash (A_1 \cup A_2)]\right) \\ &\overset{disj.\ events}{=} P[A_1 \backslash (A_2 \cup A_3)] + P[A_2 \backslash (A_1 \cup A_3)] + P[A_3 \backslash (A_1 \cup A_2)] \end{aligned}$$

[1] This definition was taken from (Resnick, 1999, p. 36). $D-$systems or Dynkin systems are named after the Soviet and American mathematician *Eugene Borisovich Dynkin* (1924–2014). *Dynkin systems are sometimes referred to as $\lambda-$systems (Dynkin himself used this term) or d-systems.*

[2] The result for $n \in \mathbb{N}$ and $k = 1, \dots, n$ is usually referred to as WARING'S THEOREM (Grimmett and Stirzaker, 2001b, p. 5) and can be stated as follows: $P(N_{k,n}) = \sum_{i=0}^{n-k} (-1)^i \binom{k+i}{k} S_{k+i,n}$, where $S_{0,n} = 1$, $S_{1,n} = \sum_{i=1}^{n} P(A_i)$, $S_{2,n} = \sum_{i<j} P(A_i \cap A_j)$, $S_{3,n} = \sum_{i<j<k} P(A_i \cap A_j \cap A_k)$ and so on. *This and similar results have applications in the valuation of assurances and annuities contingent upon the death or survival of a large number of lives.*

$$
\begin{aligned}
P(N_{1,3}) &= P(A_1) - P[A_1 \cap (A_2 \cup A_3)] \\
&\quad + P(A_2) - P[A_2 \cap (A_1 \cup A_3)] \\
&\quad\quad + P(A_3) - P[A_3 \cap (A_1 \cup A_2)] \\
&= P(A_1) - P[(A_1 \cap A_2) \cup (A_1 \cap A_3)] \\
&\quad + P(A_2) - P[(A_1 \cap A_2) \cup (A_2 \cap A_3)] \\
&\quad\quad + P(A_3) - P[(A_1 \cap A_3) \cup (A_2 \cap A_3)] \\
&= P(A_1) - P(A_1 \cap A_2) - P(A_1 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&\quad + P(A_2) - P(A_1 \cap A_2) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&\quad\quad + P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3) \\
&= P(A_1) + P(A_2) + P(A_3) \\
&\quad -2 \times [P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)] \\
&\quad\quad +3 \times P(A_1 \cap A_2 \cap A_3).
\end{aligned}
$$

3. Consider $P(A) = \frac{1}{2} \times \epsilon_{\{0\}}(A) + \frac{1}{2} \int_{A \cap \mathbb{R}^+} 2\,\phi(x)\,dx$, for $A \in \mathscr{B}(\mathbb{R})$, where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$, for $x \in \mathbb{R}$.[3] **(1.0)**

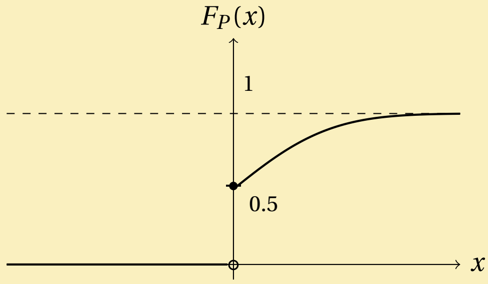
Derive the distribution function associated with P, $F_P(x)$, for $x \in \mathbb{R}$, and plot its graph.

- **D.f. associated with P**

$$
\begin{aligned}
F_P(x) &= P((-\infty, x]) \\
&= \frac{1}{2} \times \epsilon_{\{0\}}((-\infty, x]) + \frac{1}{2} \int_{(-\infty,x] \cap \mathbb{R}^+} 2\,\phi(t)\,dt \\
&= \begin{cases} 0, & x < 0 \\ \frac{1}{2}, & x = 0 \\ \frac{1}{2} + \int_0^x \phi(t)\,dt = \frac{1}{2} + [\Phi(x) - \Phi(0)] = \Phi(x), & x > 0 \end{cases}
\end{aligned}
$$

where $\Phi(x) = \int_{-\infty}^x \phi(t)\,dt$ [represents the c.d.f. of the standard normal distribution].

- **Plot of F_P**



[3] This is a particular case of the rectified Gaussian distribution *used in many applications, namely in biological neural network and signal processing.*