Formulae — Probability Theory — 1st. Semester 2021/22 — IST

Discrete and continuous univariate distributions

X (r.v.)	Values	$P(X=x)$ or $f_X(x)$	$E(X)$	$V(X)$	$M_X(t)$ or $E(X^k)$	$P_X(s)$
Binomial(n,p)	$\{0,\ldots,n\}$	$\binom{n}{x}p^x(1-p)^{n-x}$	np	$np(1-p)$	$[pe^t+(1-p)]^n$	$(1-p+ps)^n$
HyperG(N,M,n)	$\{\max\{0,n-N+M\}, \ldots, \min\{n,M\}\}$	$\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$	$n\frac{M}{N}$	$n\frac{M}{N}\frac{N-M}{N}\frac{N-n}{N-1}$	not interesting	not interesting
Geometric(p)	$\mathbb{N}$	$(1-p)^{x-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{ps}{1-(1-p)s}$
Geometric$^\star(p)$	$\mathbb{N}_0$	$(1-p)^x p$	$\frac{1-p}{p}$	$\frac{1-p}{p^2}$	$\frac{p}{1-(1-p)e^t}$	$\frac{p}{1-(1-p)s}$
NegativeBin(r,p)	$\{r,r+1,\ldots\}$	$\binom{x-1}{r-1}p^r(1-p)^{x-r}$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$	$\left[\frac{ps}{1-(1-p)s}\right]^r$
NegativeBin$^\star(r,p)$	$\{0,1,\ldots\}$	$\binom{y+r-1}{r-1}p^r(1-p)^y$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{p}{1-(1-p)e^t}\right]^r$	$\left[\frac{p}{1-(1-p)s}\right]^r$
Poisson(λ)	$\mathbb{N}_0$	$e^{-\lambda}\frac{\lambda^x}{x!}$	λ	λ	$e^{\lambda(e^t-1)}$	$e^{-\lambda(1-s)}$
Uniform$(\{1,\ldots,n\})$	$\{1,\ldots,n\}$	$\frac{1}{n}$	$\frac{n+1}{2}$	$\frac{n^2-1}{12}$	$\frac{e^t(1-e^{tn})}{n(1-e^t)}$	$\frac{s(1-s^n)}{n(1-s)}$
Beta(α,β)	$[0,1]$	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{+\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$	—
Cauchy(μ,σ)	$\mathbb{R}$	$\frac{1}{\pi\sigma}\frac{1}{1+\left(\frac{x-\mu}{\sigma}\right)^2}$	nonexistent	nonexistent	nonexistent	—
$\chi^2_{(n)}$	$\mathbb{R}_0^+$	$\frac{(\frac{1}{2})^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}x^{\frac{n}{2}-1}e^{-\frac{x}{2}}$	n	$2n$	$\left(\frac{\frac{1}{2}}{\frac{1}{2}-t}\right)^{\frac{n}{2}}, \quad t<\frac{1}{2}$	—
Exponential(λ)	$\mathbb{R}_0^+$	$\lambda e^{-\lambda x}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$	$\frac{\lambda}{\lambda-t}, \quad t<\lambda$	—
Gamma(α,λ)	$\mathbb{R}_0^+$	$\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\lambda x}$	$\frac{\alpha}{\lambda}$	$\frac{\alpha}{\lambda^2}$	$\left(\frac{\lambda}{\lambda-t}\right)^\alpha, \quad t<\lambda$	—
LogNormal(μ,σ^2)	$\mathbb{R}^+$	$\frac{1}{\sqrt{2\pi}\sigma x}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}$	$e^{\mu+\frac{\sigma^2}{2}}$	$\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}$	$E(X^k)=e^{k\mu+\frac{k^2\sigma^2}{2}}$	—
Normal(μ,σ^2)	$\mathbb{R}$	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ^2	$e^{\mu t+\frac{(t\sigma)^2}{2}}$	—
Rayleigh(σ)	$\mathbb{R}_0^+$	$\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}$	$\sigma\sqrt{\frac{\pi}{2}}$	$\frac{4-\pi}{2}\sigma^2$	$E(X^k)=\left(\sqrt{2}\sigma\right)^k\Gamma\left(1+\frac{k}{2}\right)$	—
Uniform(a,b)	$[a,b]$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}, \ t\neq 0$	—
Weibull(α,β)	$\mathbb{R}_0^+$	$\frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}e^{-\left(\frac{x}{\alpha}\right)^\beta}$	$\alpha\Gamma\left(1+\frac{1}{\beta}\right)$	$\alpha^2\left[\Gamma\left(1+\frac{2}{\beta}\right)-\Gamma\left(1+\frac{1}{\beta}\right)^2\right]$	$E(X^k)=\alpha^k\Gamma\left(1+\frac{k}{\beta}\right)$	—

$\Gamma(\alpha)=\int_0^{+\infty}x^{\alpha-1}e^{-x}dx, \ \alpha>0$; $\quad\Gamma(n)=(n-1)!, \ n\in\mathbb{N}$; $\quad\Gamma(\alpha+1)=\alpha\Gamma(\alpha), \ \alpha>0$; $\quad\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Relating c.d.f.

$F_{NegativeBin(r,p)}(x) = 1 - F_{Binomial(x,p)}(r-1)$

$F_{Erlang(n,\lambda)}(x) = 1 - F_{Poisson(\lambda x)}(n-1)$

$F_{Gamma(\alpha,\beta)}(x) = F_{\chi^2_{(2\alpha)}}(2\beta x)$

$F_{Beta(\alpha,\beta)}(x) = 1 - F_{Binomial(\alpha+\beta-1,x)}(\alpha-1)$

Moment/probability generating function; moments

$M_X(t) = E\left(e^{tX}\right)$

$E(X^k) = \left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0}$

$P_X(s) = E(s^X); \quad P(X=k) = \frac{1}{k!} \times \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=0}$

$E[X(X-1)\cdots(X-k+1)] = \left.\frac{d^k P_X(s)}{ds^k}\right|_{s=1}, \, k \in \mathbb{N}$

$E(X) = \int_0^{+\infty}[1-F_X(x)]\,dx, \ \text{for } X \geq 0$

$E(X^k) = \int_0^{+\infty} kx^{k-1}[1-F_X(x)]\,dx, \ \text{for } X \geq 0$

$SC(X) = \frac{E\{[X-E(X)]^3\}}{[SD(X)]^3}$

$KC(X) = \frac{E\{[X-E(X)]^4\}}{[SD(X)]^4} - 3$

Multinomial distribution

$P(N_1 = n_1, \ldots, N_d = n_d) = \frac{n!}{\prod_{i=1}^d n_i!} \times \prod_{i=1}^d p_i^{n_i}$

$\{(n_1,\ldots,n_d) \in I\!N_0^d : \sum_{i=1}^d n_i = n\}$

$M_{N_1,\ldots,N_{d-1}}(t_1,\ldots,t_{d-1}) = \left[\left(\sum_{i=1}^{d-1} p_i\, e^{t_i}\right) + p_d\right]^n$

$N_i \sim \text{Binomial}(n, p_i); \quad Cov(N_i, N_j) = -n\, p_i\, p_j, \, i \neq j$

$M_{\underline{X}}(\underline{t}) = E\left[\exp\left(\sum_{i=1}^n t_i\, X_i\right)\right]$

$E\left(\prod_{i=1}^n X_i^{k_i}\right) = \left.\frac{\partial^{\sum_{i=1}^n k_i} M_{\underline{X}}(\underline{t})}{\partial t_1^{k_1} \ldots \partial t_n^{k_n}}\right|_{\underline{t}=\underline{0}}$

Functions of r.v.

$F_{Y=g(X)}(y) = P[X \in g^{-1}((-\infty, y])]$

$f_{Y=g(X)}(y) = f_X[g^{-1}(y)] \times \left|\frac{dg^{-1}(y)}{dy}\right|$

Hierarchical models resulting from mixtures

$P(X=x) = \sum_y P(X=x|Y=y) \times P(Y=y)$

$P(X=x) = \int_{I\!R_Y} P(X=x|Y=y) \times f_Y(y)\,dy$

$E[g(X)] = E\{E[g(X)|Y)]\}$

$V[g(X)] = V\{E[g(X)|Y]\} + E\{V[g(X)|Y]\}$

Functions of random vectors

$F_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = P\left[\underline{X} \in \underline{g}^{-1}\left(\prod_{i=1}^m (-\infty, y_i]\right)\right]$

$f_{\underline{Y}=\underline{g}(\underline{X})}(\underline{y}) = f_{\underline{X}}\left[\underline{g}^{-1}(\underline{y})\right] \times |J(\underline{y})|$

$J(\underline{y}) = \begin{vmatrix} \frac{\partial g_1^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_1^{-1}(\underline{y})}{\partial y_n} \\ \vdots & \ldots & \vdots \\ \frac{\partial g_n^{-1}(\underline{y})}{\partial y_1} & \ldots & \frac{\partial g_n^{-1}(\underline{y})}{\partial y_n} \end{vmatrix}$

$f_{X+Y}(z) = \int_{-\infty}^{+\infty} f_{X,Y}(z-y, y)\,dy$

$f_{X-Y}(u) = \int_{-\infty}^{+\infty} f_{X,Y}(u+y, y)\,dy$

$f_{XY}(v) = \int_{-\infty}^{+\infty} f_{X,Y}(v/y, y) \times \frac{1}{|y|}\,dy$

$f_{X/Y}(w) = \int_{-\infty}^{+\infty} f_{X,Y}(wy, y) \times |y|\,dy$

Order statistics

$P[X_{(n-k+1)} > x] = 1 - F_{Binomial(n, 1-F_X(x))}(k-1)$

$f_{X_{(1)},\ldots,X_{(n)}}(x_{(1)},\ldots,x_{(n)}) = n! \times \prod_{i=1}^n f_X(x_{(i)})$

$F_{X_{(i)}}(x) = 1 - F_{Binomial(n, F_X(x))}(i-1)$

$f_{X_{(i)}}(x) = \frac{n!}{(i-1)!(n-i)!}\,[F_X(x)]^{i-1}\,[1-F_X(x)]^{n-i}\,f_X(x)$

$f_{\left(X_{(i)}, X_{(j)}\right)}(x,y) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\,[F_X(x)]^{i-1}\,[F_X(y)-F_X(x)]^{j-i-1}\,[1-F_X(y)]^{n-j}\,f_X(x) f_X(y), \, x < y$