P1 State model

$$A = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

FPE 0.458

$$G(s) = \frac{s+2}{(s+3)(s+4)}$$

7. L140

$$x' = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} x$$

$$e^{At} = \frac{1}{3}\begin{bmatrix} 2e^{t} + e^{4t} & e^{4t} - e^{t} \\ 2e^{4t} - 2e^{t} & e^{t} + 2e^{4t} \end{bmatrix}$$

P3 FPE, p. 466

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\omega_0^2 x_1 + u$$

Find the gains of a state-feedback control law that places both closed-loop poles at $-2\omega_0$

$$K = \begin{bmatrix} 3\omega_0^2 & 4\omega_0 \end{bmatrix}$$

P4 FPE p. 491

$y = x_1$

Design the estimator gains such that both estimation error poles are at $-10\omega_0$

$$L = \begin{bmatrix} 20\omega_0 \\ 99\omega_0^2 \end{bmatrix}$$

PI: Position control system

$$\dot{x}_1 = -x_2$$

$$\dot{x}_2 = -x_2 + g(x_1)$$

g verifies $\int_0^{x_1} g(z)\,dz > 0 \quad x_2 \neq 0$

$$g(z) = 0 \Rightarrow z = 0$$

candidate Lyapunov function

$$V = \frac{1}{2} x_2^2 + \int_0^{x_1} g(z)\,dz$$

$$\dot{V} = x_2 \dot{x}_2 + g(x_1)\dot{x}_1$$

$$= x_2\left[-x_2 + g(x_1)\right] - g(x_1)\,x_2$$

$$= -x_2^2 \leqslant 0$$

P6

$$\begin{cases} \dot{x}_1 = x_1 x_2 \\ \dot{x}_2 = 1 - \alpha x_2 - (1+\beta x_2) x_1 \end{cases}$$

$\alpha > 0 \quad \beta > 0$

Equilibria

$x_1 x_2 = 0 \Rightarrow x_1 = 0 \lor x_2 = 0$

$1 - \alpha x_2 - (1+\beta x_2) x_1 = 0$

$x_1 = 0$

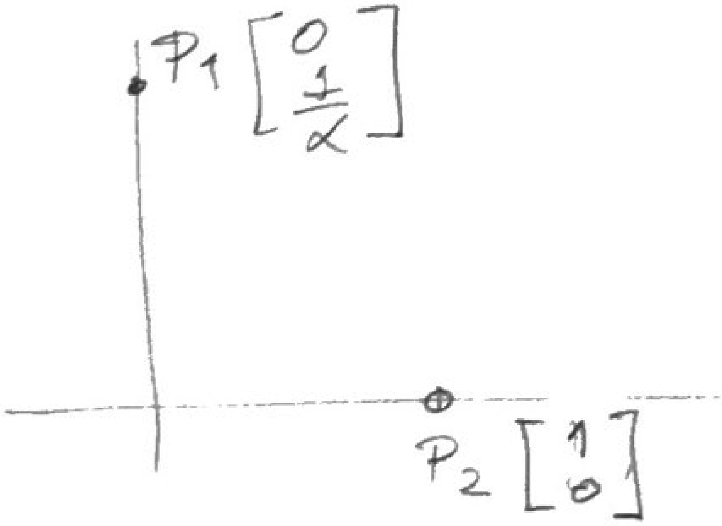
$1 - \alpha x_2 = 0 \rightarrow x_2 = \frac{1}{\alpha}$

$P_1 : \left(0, \frac{1}{\alpha}\right)$

$x_2 = 0$

$1 - x_1 = 0 \rightarrow x_1 = 1$

$P_2 = (1, 0)$



Jacobian

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x_2 & x_1 \\ -(1+\beta x_2) & -\alpha - \beta x_1 \end{bmatrix}$$

$\underline{\underline{\alpha = 1}}$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x_2 & x_1 \\ -(1+\beta x_2) & -(1+\beta x_1) \end{bmatrix}$$

Linearization

$$P_1 : \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 \\ -(1+\beta) & -1 \end{bmatrix}$$

$$|\lambda I - A| = (\lambda - 1)(\lambda + 1)$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

$$\begin{bmatrix} \lambda - 1 & 0 \\ 1+\beta & \lambda + 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$(\lambda - 1) v_1 = 0$$

$$(1+\beta) v_1 + (\lambda + 1) v_2 = 0$$

$\lambda_1 = 1$

From the second equation

$$(1+\beta)\, v_1 + 2 v_2 = 0$$

$$v_1 = 1 \rightarrow v_2 = -\frac{1+\beta}{2}$$

$$v^1 = \begin{bmatrix} 1 \\ -\frac{1+\beta}{2} \end{bmatrix}$$

$\underline{\underline{\beta = 1}}$

$$v^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

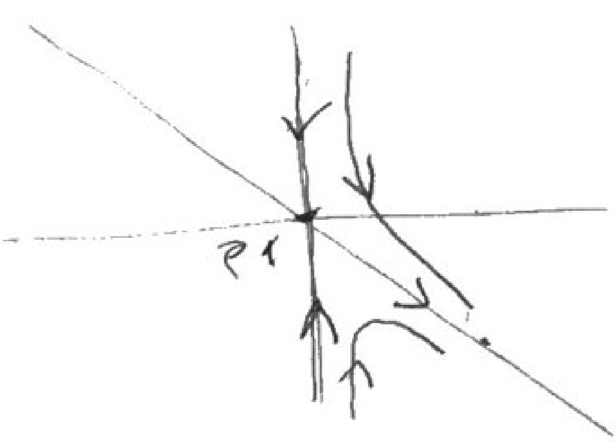
$\lambda_2 = -1$

From the 1^{st} eq.

$$-2 v_1 = 0 \Longrightarrow v_1 = 0$$

v_2 can be anything

$$v^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



$$P2 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -(1+\beta) \end{bmatrix}$$

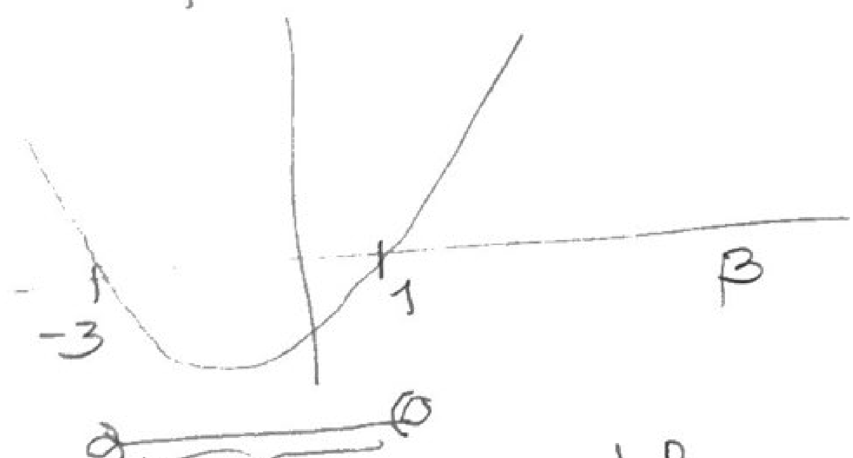
$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ 1 & \lambda + (1+\beta) \end{vmatrix} =$$

$$= \lambda^2 + (1+\beta)\lambda + 1$$

$$\lambda_{1,2} = \frac{-(1+\beta) \pm \sqrt{(1+\beta)^2 - 4}}{2}$$

$$(1+\beta)^2 - 4 = (1+\beta+2)(1+\beta-2) =$$

$$= (3+\beta)(\beta - 1)$$



Stable complex conjugate roots.

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = u - \beta x_2 \qquad \beta \geq 0$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$J(u) = \frac{1}{2} \int_0^1 u^2(t)\,dt \quad \text{minimize.}$$

$$f = \begin{bmatrix} x_2 \\ u - \beta x_2 \end{bmatrix} \qquad f_x = \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}$$

$$L = -\frac{1}{2} u^2 \qquad L_x = [0 \quad 0]$$

$$[-\dot{\lambda}_1 \quad -\dot{\lambda}_2] = [\lambda_1 \quad \lambda_2] \begin{bmatrix} 0 & 1 \\ 0 & -\beta \end{bmatrix}$$

$$\dot{\lambda}_1 = 0 \Rightarrow \lambda_1 = c_1$$

$$\dot{\lambda}_2 = -\lambda_1 + \beta \lambda_2$$

$$\dot{\lambda}_2 = -c_1 + \beta \lambda_2$$

$$\lambda_2(t) = c_2 e^{\beta t} - \frac{c_1}{\beta}\left(e^{\beta t} - 1\right)$$

$$H = \lambda_1 x_2 + \lambda_2 (u - \beta x_2) - \frac{1}{2} u^2$$

$$\frac{\partial H}{\partial u} = \lambda_2 - u \rightarrow u^* = \lambda_2$$

$$u = \underbrace{\left(c_2 - \frac{c_1}{\beta}\right)}_{:=k_1} e^{\beta t} + \underbrace{\frac{c_1}{\beta}}_{:=k_2}$$

$$u = k_1 e^{\beta t} + k_2$$

Impose the terminal constraints

$$\dot{x}_2 = k_1 e^{\beta t} + k_2 - \beta x_2$$

$$s X_2 - x_2(0) = k_1 \frac{1}{s-\beta} + \frac{k_2}{s} - \beta X_2$$

$$X_2 = \frac{1}{s+\beta} x_2(0) + k_1 \frac{1}{(s-\beta)(s+\beta)} + \frac{k_2}{s(s+\beta)}$$

$$\frac{1}{(s-\beta)(s+\beta)} = \frac{A}{s-\beta} + \frac{B}{s+\beta}$$

$$A = \frac{1}{2\beta} \quad B = -\frac{1}{2\beta}$$

$$\frac{1}{(s-\beta)(s+\beta)} = \frac{1}{2\beta}\frac{1}{s-\beta} - \frac{1}{2\beta}\frac{1}{s+\beta}$$

$$\frac{1}{s(s+\beta)} = \frac{A}{s} + \frac{B}{s+\beta}$$

$$A = \frac{1}{\beta} \qquad B = -\frac{1}{\beta}$$

$$\frac{1}{s(s+\beta)} = \frac{1}{\beta}\cdot\frac{1}{s} - \frac{1}{\beta}\frac{1}{s+\beta}$$

$$X_2 = \frac{1}{s+\beta}x_2(0) + \frac{K_1}{2\beta}\left(\frac{1}{s-\beta} - \frac{1}{s+\beta}\right) +$$

$$+ \frac{K_2}{\beta}\left(\frac{1}{s} - \frac{1}{s+\beta}\right)$$

$$x_2(0) = 1$$

$$x_2(t) = e^{-\beta t} + \frac{K_1}{2\beta}\left(e^{\beta t} - e^{-\beta t}\right) +$$

$$+ \frac{K_2}{\beta}\left(1 - e^{-\beta t}\right)$$

$$x_2(t) = \frac{K_2}{\beta} + \frac{K_1}{2\beta}e^{\beta t} +$$

$$+ \left(1 - \frac{K_1}{2\beta} - \frac{K_2}{\beta}\right)e^{-\beta t}$$

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$\beta = 1$

$$x_2(t) = K_2 + \frac{K_1}{2} e^{t} + \left(1 - \frac{K_1}{2} - K_2\right) e^{-t}$$

$$x_1(t) = \underset{\overset{\shortparallel}{0}}{x_1(0)} + \int_0^t x_2(\tau)\, d\tau$$

$$x_1(t) = K_2 t + \frac{K_1}{2} e^{t} - \left(1 - \frac{K_1}{2} - K_2\right) e^{-t}$$

terminal conditions

$x_1(1) = 1$, $x_2(1) = 0$

$$K_2 + \frac{K_1 e}{2} - \left(1 - \frac{K_1}{2} - K_2\right)\frac{1}{e} = 1$$

$$K_2 + \frac{K_1 e}{2} + \left(1 - \frac{K_1}{2} - K_2\right)\frac{1}{e} = 0$$

(+) $2K_2 + K_1 e = 1$ //

(−) $-2\left(1 - \frac{K_1}{2} - K_2\right)\frac{1}{e} = 1$

$$K_1 + 2K_2 = e + 2$$ //

$$k_1 e + 2k_2 = 1$$
$$k_1 + 2k_2 = e + 2$$

---

(1) $$k_1(e-1) = -e-1$$

$$\boxed{k_1 = -\frac{e+1}{e-1}}$$

$$k_2 = \frac{1}{2}(1 - k_1 e)$$

$$k_2 = \frac{1}{2}\left(1 + \frac{e+1}{e-1}e\right)$$

$$k_2 = \frac{1}{2}\left(\frac{e-1+e^2+e}{e-1}\right)$$

$$\boxed{k_2 = \frac{1}{2}\,\frac{e^2+2e-1}{e-1}}$$

$$u^*(t) = -\frac{e+1}{e-1}e^{t} + \frac{1}{2}\,\frac{e^2+2e-1}{e-1}$$

P8/ $\dot{x} = 0,5\,u\,x \qquad x(0) > 0$ 1'1/

$$0 \le u(t) \le 1$$

$$\max J = x(T) + \int_0^T (1-u)\,x\,dt$$

$f = 0,5\,u\,x \qquad f_x = 0,5\,u$

$L = (1-u)\,x \qquad L_x = 1-u$

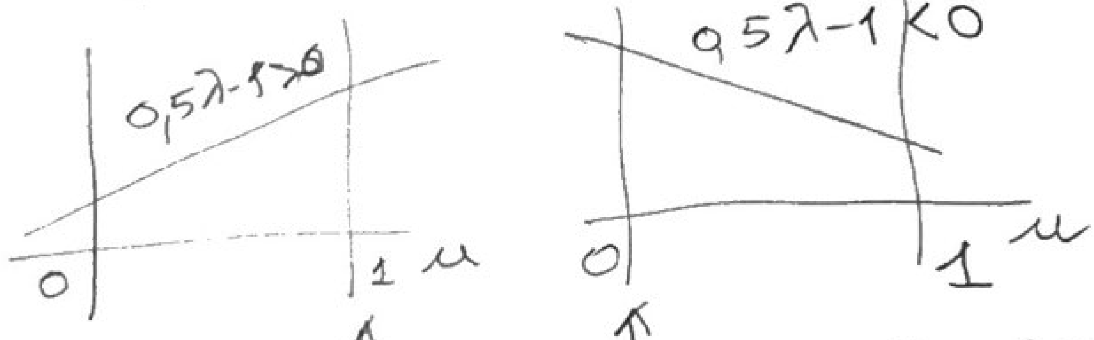
$\varphi = x(T) \qquad \varphi_x = 1 \qquad \lambda(T) = 1$

$\dot{\lambda} = -0,5\,u\,\lambda + u - 1$

$\dot{\lambda} = (1 - 0,5\,\lambda)\,u - 1$

$H = 0,5\,u\,x\,\lambda + (1-u)\,x$

$H = (0,5\,\lambda - 1)\,x\,u + x$



$\lambda > 2 \qquad u^* = 1 \qquad\qquad u^* = 0 \qquad \lambda < 2$

At the end of the optimization interval

$\lambda(T) = 1 < 2 \Rightarrow u^* = 0$

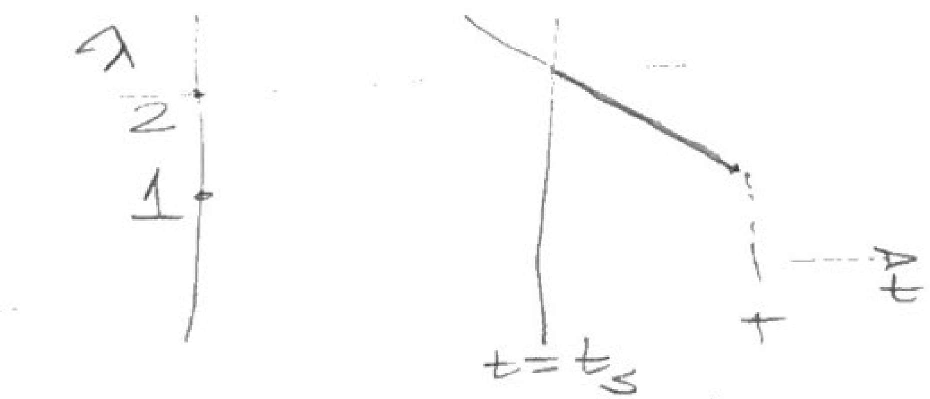
For t close to T, $u(t) = 0$

$\dot{\lambda} = -1 \qquad \lambda(T) = 1$

$\lambda(t) = c - t$

$c - T = 1 \Rightarrow c = 1 + T$

$\lambda(t) = 1 + T - t$



switching condition

$\frac{\lambda}{2} - 1 = 0 \rightarrow \lambda = 2$

$1 + T - t_s = 2 \rightarrow t_s = T - 1$

For $t < t_s$, $\lambda > 2 \Rightarrow u(t) = 1$

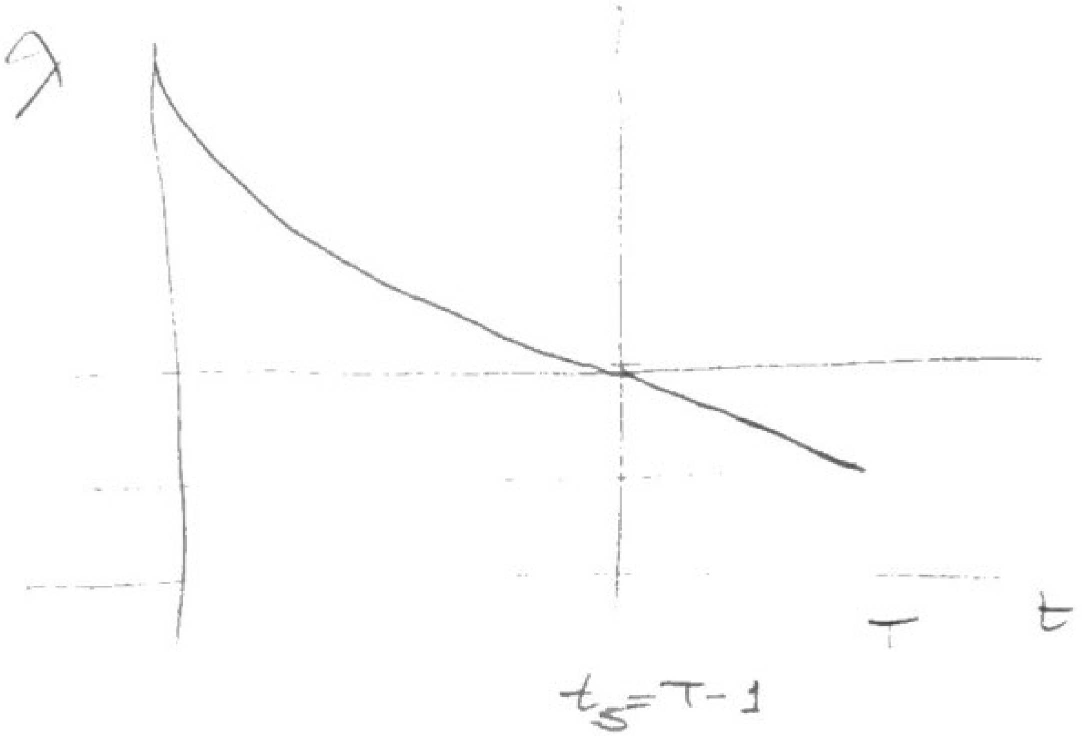
$\dot{\lambda} = -0{,}5\,\lambda$

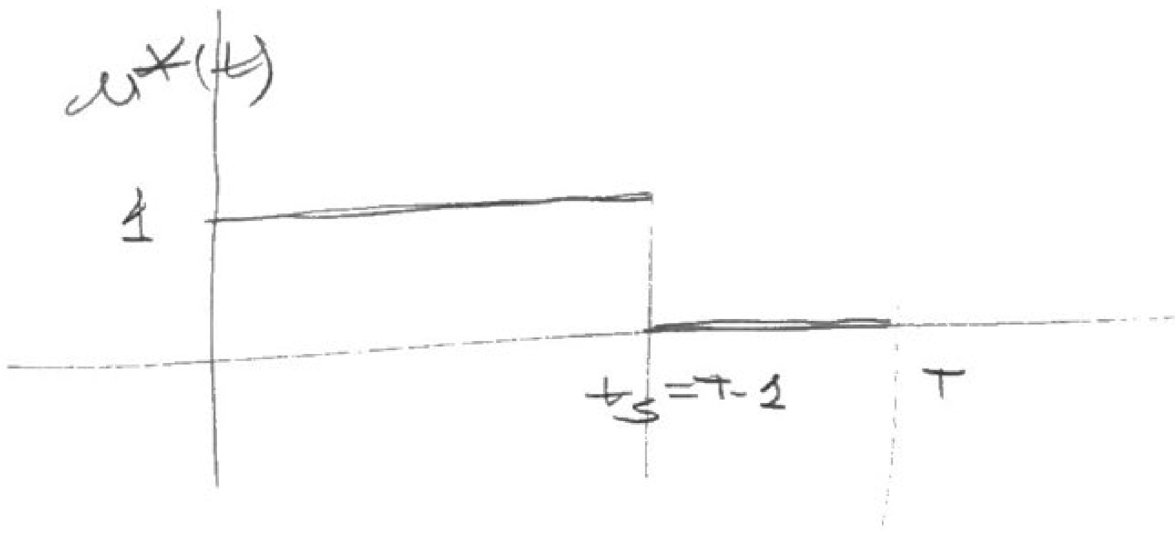
$\lambda(t) = c_2\, e^{-0{,}5\,t} \qquad \lambda(T-1) = 2$

$2 = c_2 e^{-0{,}5(T-1)}$

$c_2 = 2\, e^{0{,}5(T-1)}$

$$\lambda(t) = 2e^{0,5(T-1-t)}$$





$$\min_{u} \; x(2) + \int_0^2 (1-u)\, x \, dt$$

s.t.

$\dot{x} = 0,5\, u x$

$x(0) > 0$

$0 \leq u(t) \leq 1 \quad \forall t$

Costate

$\lambda(t) = 2e^{0,5(1-t)} \qquad 0 \leq t \leq 1$

$\lambda(t) = 3 - t \qquad 1 \leq t \leq 2$

Optimal control

$u^*(t) = 1 \qquad 0 \leq t \leq 1$

$u^*(t) = 0 \qquad 1 < t \leq 2$