DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC, MECD

2nd. Semester – 2019/2020
2020/07/08 – 9:45AM

Duration: 90 minutes **Test 2** (*Recurso*)

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes 2.0 points

Consider a regenerative renewal process $\{X(t): t \geq 0\}$, where $X(t)$ represents the number of machines in working condition at time t. Admit that: the state space is $\mathscr{S} = \{0,1,2\}$; all operation (resp. repair) times are non-negative continuous i.i.d. r.v. with p.d.f. f_Y (resp. f_R); operation and repair times are independent r.v.; the times spent in states 0, 1, 2 during the first cycle $[0, S_1)$, are $U_0 = \max\{0, R-Y\}$, $U_1 = \min\{Y, R\}$, and $U_2 = \max\{0, Y-R\}$. (2.0)

Derive an expression for $P_1 = \lim_{t\to+\infty} P[X(t) = 1]$ in terms of f_Y and f_R.[1]

- **Regenerative process and space state**
  $\{X(t): t \leq 0\}$
  $X(t) =$ number of machines in working condition at time t
  $\mathscr{S} = \{0,1,2\}$
- **Auxiliary r.v.**
  $Y =$ operation time, with p.d.f. $f_Y \quad \perp\!\!\!\perp \quad R =$ repair time, with p.d.f. f_R
  $S_1 =$ duration of the first cycle
  $U_j =$ time spent in state j during the first cycle $[0, S_1)$, $\quad j \in \mathscr{S}$
  $U_0 = \max\{0, R-Y\}, \quad U_1 = \min\{Y, R\}, \quad U_2 = \max\{0, Y-R\}$
- **Requested limit**

$$P_1 = \lim_{t\to+\infty} P[X(t) = 1] \stackrel{form.}{=} \frac{E(U_1)}{E(S_1)} = \frac{E(U_1)}{E(U_0 + U_1 + U_2)},$$

where:

$$\begin{aligned}
E(U_1) &= E[\min\{Y,R\}] \\
&\stackrel{Y \perp\!\!\!\perp R}{=} \int_0^{+\infty}\int_0^{+\infty} \min\{y,r\} \times f_Y(y) \times f_R(r)\, dy\, dr \\
&= \int_0^{+\infty}\int_0^{r} y \times f_Y(y) \times f_R(r)\, dy\, dr + \int_0^{+\infty}\int_r^{+\infty} r \times f_Y(y) \times f_R(r)\, dy\, dr; \\
E(U_0 + U_1 + U_2) &\stackrel{hint}{=} E[\max\{Y,R\}] \\
&\stackrel{Y \perp\!\!\!\perp R}{=} \int_0^{+\infty}\int_0^{+\infty} \max\{y,r\} \times f_Y(y) \times f_R(r)\, dy\, dr \\
&= \int_0^{+\infty}\int_0^{r} r \times f_Y(y) \times f_R(r)\, dy\, dr + \int_0^{+\infty}\int_r^{+\infty} y \times f_Y(y) \times f_R(r)\, dy\, dr.
\end{aligned}$$

[This is valid as long as: $S_1 = \max\{Y,R\}$ is not lattice (this checks because Y and R are both continuous r.v.); $E(S_1) < +\infty$.]

- [**Note**: This regenerative process refers to a system described as follows. Although it needs only a single machine to function, the system maintains a spare machine as a backup. Any of these two machines while in use operate for a random time and then fail. If a machine fails while the other one is not being repaired, then the latter is immediately put in use and, simultaneously, repair begins on the one that just failed. If a machine fails while the other machine is in repair, then the newly failed machine waits until the repair is completed; at that time the repaired machine is put in use and, simultaneously, repair begins on the recently failed one.]

[1] **Hint**: $\max\{y,r\} = \max\{0, r-y\} + \min\{y,r\} + \max\{0, y-r\}$, for $y, r > 0$.

Group 2 — Discrete time Markov chains 8.5 points

1. Admit the base of a DNA nucleotide takes any of 4 values,[2] say in $\mathscr{S} = \{1,2,3,4\}$. A standard model for a mutational change of the base of a DNA nucleotide at a specific location is a DTMC governed as follows: from period to period, the base of the DNA nucleotide does not change with probability $(1-3\alpha)$, where $0 < \alpha < \frac{1}{3}$; if it changes then it is equally likely to change to any of the other 3 values.

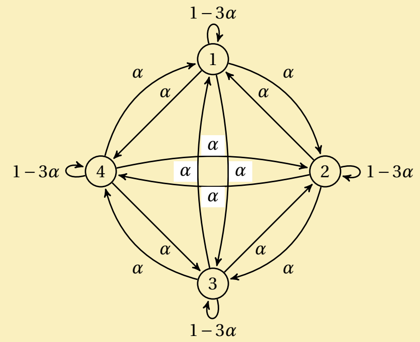
(a) Identify the TPM and draw the associated transition diagram. (1.0)

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  $X_n =$ value of the base of the DNA nucleotide at period n
- **State space**
  $\mathscr{S} = \{1,2,3,4\}$
- **TPM**
  Judging by the description above,

$$\mathbf{P} = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix},$$

where $0 < \alpha < \frac{1}{3}$.

- **Transition diagram**



(b) Resort to mathematical induction to show that the entries of the n−step TPM, $\mathbf{P}^n = [P^n_{ij}]_{i,j\in\mathscr{S}}$ ($n \in \mathbb{N}$), are equal to: $\quad P^n_{ij} = \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n$, for $j = i$; $\quad P^n_{ij} = \frac{1}{4} - \frac{1}{4}(1-4\alpha)^n$, for $j \neq i$. (1.5)

- **Requested proof by induction**

Induction hypothesis

$$P^n_{ij} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n, & j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^n, & j \neq i \end{cases} \qquad (1)$$

Base case
For $n = 1$ and $i, j \in \mathscr{S}$, we get

$$P^n_{ij} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha) = 1-3\alpha, & j = i \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha) = \alpha, & j \neq i. \end{cases} \quad \checkmark$$

[2] The DNA nucleotide consists of one of four bases (cytosine, thymine, adenine, guanine), a sugar (deoxyribose) and a phosphate.

Induction step

Invoking the Chapman-Kolmogorov equations, the induction hypothesis in (1) and the fact that $\#\mathscr{S} = 4$, we can successively write:

- for $j = i$,

$$\begin{aligned} P_{ii}^{n+1} &= \sum_{k\in\mathscr{S}} P_{ik}^n \times P_{ki} \\ &= P_{ii}^n \times P_{ii} + \sum_{k\neq i} P_{ik}^n \times P_{ki} \\ &= \left[\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right] \times (1-3\alpha) + \sum_{k\neq i} \left\{\left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] \times \alpha\right\} \\ &= \left[\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right] \times (1-3\alpha) + 3 \times \left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] \times \alpha \\ &= \frac{1}{4} \times (1-3\alpha) + \frac{3}{4}(1-4\alpha)^n \times (1-3\alpha) + \frac{3}{4} \times \alpha - \frac{3}{4}(1-4\alpha)^n \times \alpha \\ &= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n \times (1-3\alpha-\alpha) \\ &= \frac{1}{4} + \frac{3}{4}(1-4\alpha)^{n+1}; \quad \checkmark \end{aligned}$$

- for $j \neq i$,

$$\begin{aligned} P_{ij}^{n+1} &= \sum_{k\in\mathscr{S}} P_{ik}^n \times P_{kj} \\ &= P_{ii}^n \times P_{ij} + \sum_{k\neq i, k\neq j} P_{ik}^n \times P_{kj} + P_{ij}^n \times P_{jj} \\ &= \left[\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right] \times \alpha + \sum_{k\neq i, k\neq j} \left\{\left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] \times \alpha\right\} \\ &\quad + \left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] \times (1-3\alpha) \\ &= \left[\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right] \times \alpha + \left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] \times (2\alpha+1-3\alpha) \\ &= \frac{1}{4} \times (\alpha+1-\alpha) - \frac{1}{4}(1-4\alpha)^n \times (-3\alpha+1-\alpha) \\ &= \frac{1}{4} - \frac{1}{4}(1-4\alpha)^{n+1}. \quad \checkmark \end{aligned}$$

[We could have invoked the symmetry of $\mathbf{P}$ to conclude that $P_{ij}^n = \frac{1}{3}(1-P_{ii}^n)$, $i,j \in \mathscr{S}$, $j \neq i$, and as a result we would have immediately obtained $P_{ij}^{n+1} = \frac{1}{3}(1-P_{ii}^{n+1}) = \frac{1}{3}(1-\frac{1}{4}-\frac{3}{4}(1-4\alpha)^{n+1}) = \frac{1}{4}-\frac{1}{4}(1-4\alpha)^{n+1}$, for $j \neq i$.]

(c) Classify the states of this DTMC. Are the states periodic? What is the long-run proportion of time the chain is in each state? (1.5)

- **Classification of the states of the DTMC**
  - Judging by the transition diagram, all states communicate with one another, thus $\mathscr{S} = \{1,2,3,4\}$ is a single closed communicating class. Thus, the DTMC has a finite state space and is irreducible. As a result, all states are positive recurrent.
  - The transition diagram leads to the conclusion that we can return to state 1 after $1,2,3,\ldots$ transitions, thus $d(1) = gcd\left\{n \in \mathbb{N} : P_{11}^n > 0\right\} = 1$ and this state is aperiodic.
    The same holds for the remaining states of this irreducible DTMC. [After all, periodicity is a class property.]
- **Requested long-run fraction of time**
  Since the DTMC is irreducible, positive recurrent and aperiodic we can obtain the long-run fraction of time that the chain is in each state equals, by calculating the following non-negative limit probabilities:

$$\lim_{n\to+\infty} P_{ii}^n \overset{(a)}{=} \lim_{n\to+\infty} \left[\frac{1}{4} + \frac{3}{4}(1-4\alpha)^n\right] = \frac{1}{4}, \quad i \in \mathscr{S};$$

$$\lim_{n\to+\infty} P_{ij}^n \overset{(a)}{=} \lim_{n\to+\infty} \left[\frac{1}{4} - \frac{1}{4}(1-4\alpha)^n\right] = \frac{1}{4}, \quad i,j \in \mathscr{S} \quad (j \neq i).$$

That is, the limit distribution is $\underline{\pi} = [1/4 \quad 1/4 \quad 1/4 \quad 1/4]$.

[$\mathbf{P}$ is a doubly stochastic TPM ($\sum_{k\in\mathscr{S}} P_{ik} = \sum_{k\in\mathscr{S}} P_{kj} = 1, i,j \in \mathscr{S}$), hence the limit distribution is uniform in $\mathscr{S}$, as have just we shown.]

(d) Now, consider the DTMC $\{X_m : m \in \mathbb{Z}\}$ governed by the same TPM $\mathbf{P}$. Are we dealing with a time reversible DTMC? (0.5)

- **New DTMC and associated TPM**
  $\{X_m : m \in \mathbb{Z}\}$

$$\mathbf{P} = \begin{bmatrix} 1-3\alpha & \alpha & \alpha & \alpha \\ \alpha & 1-3\alpha & \alpha & \alpha \\ \alpha & \alpha & 1-3\alpha & \alpha \\ \alpha & \alpha & \alpha & 1-3\alpha \end{bmatrix} \quad (0 < \alpha < \tfrac{1}{3})$$

- **Checking time reversibility**
  $\{X_m : m \in \mathbb{Z}\}$ is time reversible iff the detailed balance equations

$$\pi_i \times P_{ij} = \pi_j \times P_{ji}, \quad i,j \in \mathscr{S},$$

  are verified.
  $\mathbf{P}$ is a symmetric TPM ($P_{ij} = P_{ji}, i,j \in \mathscr{S}$), thus a doubly stochastic matrix ($\sum_{k\in\mathscr{S}} P_{ik} = \sum_{k\in\mathscr{S}} P_{kj} = 1, i,j \in \mathscr{S}$). As a result, the stationary distribution is uniform in $\{1,2,3,4\}$ (i.e., $\pi_j = \frac{1}{4}, j \in \mathscr{S}$), as seen in (c), and the detailed balance equations are immediately verified.

2. A study of social mobility of families across generations was recently carried out and three socio-economic classes were considered: *upper class* (state 1); *middle class* (state 2); *lower class* (state 3). The TPM of the associated DTMC, $\{X_n : n \in \mathbb{N}\}$, was estimated to be

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}.$$

(a) Determine $f_{ii}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$, for $i,n = 1,2,3$ and $j = 2$. (2.0)

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  X_n = socio-economic class of the family in generation n
- **State space**
  $\mathscr{S} = \{1,2,3\}$
  1 = *upper* class
  2 = *middle* class
  3 = *lower* class
- **TPM**

$$\mathbf{P} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0.1 & 0.5 & 0.4 \end{bmatrix}$$

- **Requested probabilities**
  Let:
  i) $f_{ij}^n = P(X_n = j, X_{n-1} \neq j, \ldots, X_1 \neq j \mid X_0 = i)$ be the probability of reaching state j for the first time starting from state i, for $i,j \in \mathscr{S}$ and $n \in \mathbb{N}$;
  ii) $\underline{f}_j^n = [f_{ij}^n]_{i\in\mathscr{S}}$ be the associated vector, for fixed $j \in \mathscr{S}$ and $n \in \mathbb{N}$.
  According to the formulae,

$$\underline{f}_j^n = \begin{cases} \underline{f}_j^1 = [P_{ij}]_{i\in\mathscr{S}}, & n=1 \\ {}^{(j)}\mathbf{P} \times \underline{f}_j^{n-1} = \left[{}^{(j)}\mathbf{P}\right]^{n-1} \times \underline{f}_j^1, & n=2,3,\ldots, \end{cases}$$

where ${}^{(j)}\mathbf{P}$ is obtained by setting all the entries of the j^{th} column of $\mathbf{P}$ equal to 0.
When $j=2$, we get

$$\begin{aligned} {}^{(2)}\mathbf{P} &= \begin{bmatrix} 0.4 & 0 & 0.1 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.4 \end{bmatrix} \\ \underline{f}_2^1 &= [P_{i1}]_{i\in\mathscr{S}} \\ &= \begin{bmatrix} 0.5 \\ 0.7 \\ 0.5 \end{bmatrix} \\ \underline{f}_2^2 &= {}^{(2)}\mathbf{P} \times \underline{f}_2^1 \\ &= \begin{bmatrix} 0.4 & 0 & 0.1 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.5 \\ 0.7 \\ 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 0.25 \\ 0.15 \\ 0.25 \end{bmatrix} \\ \underline{f}_2^3 &= {}^{(2)}\mathbf{P} \times \underline{f}_2^2 \\ &= \begin{bmatrix} 0.4 & 0 & 0.1 \\ 0.1 & 0 & 0.2 \\ 0.1 & 0 & 0.4 \end{bmatrix} \times \begin{bmatrix} 0.25 \\ 0.15 \\ 0.25 \end{bmatrix} \\ &= \begin{bmatrix} 0.125 \\ 0.075 \\ 0.125 \end{bmatrix}. \end{aligned}$$

(b) Obtain the expected number of generations it takes a family to reach state 3, starting from state 1. (2.0)

- **Initial state**
  $X_0 = i$
- **Important**
  To obtain the expected number of generations until a family reaches state 3, given $X_0 = i$, we have to consider another DTMC whose state 3 is absorbing. The associated TPM is

$$\mathbf{P}' = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.1 & 0.7 & 0.2 \\ 0 & 0 & 1 \end{bmatrix}.$$

- **Requested expected value**
  Let

$$\mathbf{Q} = \begin{bmatrix} 0.4 & 0.5 \\ 0.1 & 0.7 \end{bmatrix}$$

be the substochastic matrix governing the transitions between the states in $T = \{1,2\}$, the class of transient states of this new DTMC, and

$$\tau = \inf\{n \in \mathbb{N}_0 : X_n \notin T\}$$

be the number of generations until a family reaches state 3. Then, by capitalizing on the fact that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we obtain

$$\begin{aligned} [E(\tau \mid X_0 = i)]_{i\in T} &= (\mathbf{I}-\mathbf{Q})^{-1} \times \underline{1} \\ &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.4 & 0.5 \\ 0.1 & 0.7 \end{bmatrix} \right)^{-1} \times \underline{1} \\ &= \begin{bmatrix} 0.6 & -0.5 \\ -0.1 & 0.3 \end{bmatrix}^{-1} \times \underline{1} \\ &= \frac{1}{0.6\times 0.3 - (-0.5)\times(-0.1)} \begin{bmatrix} 0.3 & 0.5 \\ 0.1 & 0.6 \end{bmatrix} \times \underline{1} \\ &= \frac{1}{0.13} \begin{bmatrix} 0.8 \\ 0.7 \end{bmatrix} \\ &\simeq \begin{bmatrix} 6.153846 \\ 5.384615 \end{bmatrix}. \end{aligned}$$

Thus, the requested expected value equals

$$E(\tau \mid X_0 = 1) \simeq 6.153846.$$

Group 3 — Continuous time Markov chains	9.5 points

1. Consider an $M/M/1$ queueing system with an harmonic discouragement of arrivals with respect to the number present in the system, i.e., the birth rates are equal to $\lambda_i = \frac{\lambda}{i+1}$, $i \in \mathbb{N}_0$, and the death rates are given by $\mu_i = \mu$, $i \in \mathbb{N}$.

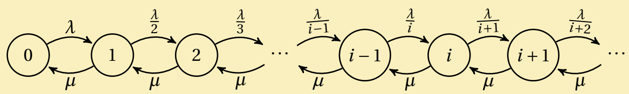
Let $X(t)$ be the number of customers at this queueing system at time t.

(a) Draw the rate diagram and identify the infinitesimal generator $\mathbf{R}$ of the CTMC $\{X(t) : t \geq 0\}$. (1.5)

- **CTMC**
  $\{X(t) : t \geq 0\}$
  $X(t)$ = number of customers in the queueing system at time t
- **State space**
  $\mathscr{S} = \mathbb{N}_0$
- **Birth/death rates**

$$\begin{aligned} \lambda_i &= \frac{\lambda}{i+1}, \; i \in \mathbb{N}_0 \\ \mu_i &= \mu, \; i \in \mathbb{N} \end{aligned}$$

- **Rate diagram**
  [Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it. These rates coincide with the birth and death rates...]



- **Infinitesimal generator**
  This matrix has entries

$$r_{ij} = \begin{cases} q_{ij}, & i \neq j \\ -\nu_i = -\sum_{m\in\mathscr{S}} q_{im}, & j = i \end{cases}$$

and in this case $\mathbf{R} = [r_{ij}]_{i,j\in\mathscr{S}}$ is equal to

$$\begin{bmatrix} -\lambda & \lambda & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \mu & -(\frac{\lambda}{2}+\mu) & \frac{\lambda}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \mu & -(\frac{\lambda}{3}+\mu) & \frac{\lambda}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \mu & -(\frac{\lambda}{i-1}+\mu) & \frac{\lambda}{i-1} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \mu & -(\frac{\lambda}{i}+\mu) & \frac{\lambda}{i} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \mu & -(\frac{\lambda}{i+1}+\mu) & \frac{\lambda}{i+1} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \mu & -(\frac{\lambda}{i+2}+\mu) & \frac{\lambda}{2+1} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}.$$

(b) Write the Kolmogorov's forward differential equations in terms of $P_j(t) \equiv P_{0j}(t) = P[X(t) = j \mid X(0) = 0]$, for $j \in \mathbb{N}_0$. (Do not try to solve the differential equations!) (1.5)

- **Kolmogorov's forward differential equations**
  These equations can be written in matrix form:

$$\frac{d\mathbf{P}(t)}{dt} = \left[\frac{dP_{ij}(t)}{dt}\right]_{i,j\in\mathscr{S}} \stackrel{form.}{=} \mathbf{P}(t)\times\mathbf{R}.$$

  Since $i = 0$, we are only interested in the first row of the previous matrix. Hence the following Kolmogorov's forward differential equations

$$\frac{dP_j(t)}{dt} \stackrel{form.}{=} P_{j-1}(t)\times\lambda_{j-1} - P_j(t)\times(\lambda_j+\mu_j) + P_{j+1}(t)\times\mu_{j+1}, \quad j\in\mathscr{S}.$$

  They read as follows:

$$\begin{aligned} \frac{dP_0(t)}{dt} &= -P_0(t)\times\lambda + P_1(t)\times\mu \\ \frac{dP_j(t)}{dt} &= P_{j-1}(t)\times\frac{\lambda}{j} - P_j(t)\times\left(\frac{\lambda}{j+1}+\mu\right) + P_{j+1}(t)\times\mu, \quad j\in\mathbb{N}. \end{aligned}$$

(c) After checking the ergodicity condition, derive the equilibrium probabilities $P_j = \lim_{t\to+\infty} P_j(t)$ for this birth and death process. (2.0)

- **Ergodicity condition**
  It reads as follows: $\exists k_0 \in \mathbb{N}: \forall k \geq k_0,\ \frac{\lambda_k}{\mu_k} < 1.$
  This condition is verified as long as $\frac{\lambda}{\mu} < +\infty$. Indeed,

$$\frac{\lambda_k}{\mu_k} < 1 \quad\Leftrightarrow\quad \frac{\lambda}{(k+1)\,\mu} < 1 \quad\Leftrightarrow\quad k > \frac{\lambda}{\mu} - 1,$$

  thus $k_0 = \left\lceil \frac{\lambda}{\mu} - 1 \right\rceil + 1$ certainly leads to the verification of the ergodicity condition.
- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P_j(t)$

$$\begin{aligned} P_0 &= \left[1 + \sum_{n=1}^{+\infty}\left(\prod_{i=0}^{n-1}\frac{\lambda_i}{\mu_{i+1}}\right)\right]^{-1} \\ &= \left\{1 + \sum_{n=1}^{+\infty}\left[\prod_{i=0}^{n-1}\frac{\lambda}{(i+1)\,\mu}\right]\right\}^{-1} \\ &= \left[1 + \sum_{n=1}^{+\infty}\left(\frac{\lambda}{\mu}\right)^n \frac{1}{n!}\right]^{-1} \\ &= e^{-\frac{\lambda}{\mu}} \\ P_j &= \left(\prod_{i=0}^{j-1}\frac{\lambda_i}{\mu_{i+1}}\right)\times P_0 \end{aligned}$$

$$\begin{aligned} P_j &= \left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!}\times P_0 \\ &= e^{-\frac{\lambda}{\mu}}\left(\frac{\lambda}{\mu}\right)^j \frac{1}{j!}, \quad j\in\mathbb{N}_0. \end{aligned}$$

[That is, the limiting probabilities coincide with the p.f. of a r.v. with a Poisson($\frac{\lambda}{\mu}$) distribuition, such as the p.f. of $L_s^{M/M/\infty}$.]

2. Consider a 24/7 automobile emission inspection station with four inspection stalls, each with room for only one car. Cars arrive to the station according to a Poisson process with a rate of 6 cars per hour. The service times are i.i.d. r.v. exponentially distributed with mean equal to 20 minutes. The station cannot accommodate any arriving cars for which no inspection stall is available.

(a) Obtain and interpret $P(L_s = 4)$. (2.0)
What is the expected number of cars per hour that are blocked from entering the station?

- **Birth-death queueing system**
  $M/M/m/m$
  $m = 4$
- **State space**
  $\mathscr{S} = \{0, 1, \dots, m\}$
- **Birth/death rates**
  $\lambda_k = \lambda = 6,\ k \in \{0, 1, \dots, m-1\}$
  $\mu_k = k\,\mu = (1/3)^{-1}\,k = 3\,k,\ k \in \{1, 2, \dots, m\}$
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{6}{4\times 3} = 1/2 < +\infty$
- **Performance measure (in the long-run)**
  L_s = number of rented cars

$$P(L_s = k) = \begin{cases} \dfrac{\frac{(m\rho)^k}{k!}}{\sum_{j=0}^{m}\frac{(m\rho)^j}{j!}}, & k = 0, 1, \dots, m \\ 0, & k = m+1, m+2, \dots \end{cases}$$

- **Requested probability and interpretation**

$$\begin{aligned} P(L_s = m) &= B(m, m\rho) \\ &= \frac{\frac{(m\rho)^m}{m!}}{\sum_{j=0}^{m}\frac{(m\rho)^j}{j!}} \\ &\stackrel{m=4,\lambda=6,\mu=3}{=} \frac{\frac{2^4}{4!}}{\sum_{j=0}^{4}\frac{2^j}{j!}} \\ &= \frac{2}{21} \\ &\simeq 0.095238. \end{aligned}$$

  $P(L_s = m)$ represents the fraction of cars that are blocked from entering the station because it attained full capacity (all 4 stalls are busy).
- **Requested expected number**
  The expected number of cars per hour that are blocked from entering the station equals

$$\begin{aligned} \lambda\times P(L_s = m) &= 6\times\frac{2}{21} \\ &\simeq 0.571429. \end{aligned}$$

(b) Admit that costs (personnel, equipment, etc.) amount to 20 euro per inspection stall (per hour) and each car inspection is priced at 50 euro. Obtain the mean hourly profit. (1.0)

- **R.v.**
  $C^{M/M/m/m}$ = hourly profit = $50 \times L_s - 20 \times m$

- **Mean hourly profit**

$$\begin{aligned} E\left(C^{M/M/m/m}\right) &= 50 \times E(L_s) - 20 \times m \\ &\stackrel{form}{=} 50\, m\rho \times [1 - B(m, m\rho)] - 20\, m \\ &\stackrel{(a), m\rho = \lambda/\mu = 2}{\simeq} 50 \times 2 \times (1 - 0.095238) - 20 \times 4 \\ &\simeq 10.4762. \end{aligned}$$

(c) What would be the value of $P(W_q > 0)$ and $E(L_q)$ if any arriving car could eventually wait for inspection?[3] (1.5)

- **New birth-death queueing system**
  $M/M/m$ because we admit that *any arriving car could eventually wait for inspection.*
  $m = 4$ as before.

- **Requested probability**
  The probability that an arriving car has to wait for its inspection to begin is equal to

$$\begin{aligned} P(W_q > 0) &= 1 - F_{W_q}(0) \\ &\stackrel{form.}{=} C(m, m\rho) \\ &\stackrel{hint}{=} \frac{m \times B(m, m\rho)}{m - m\rho \times [1 - B(m, m\rho)]} \\ &\stackrel{(b)}{\simeq} \frac{4 \times 0.095238}{4 - 4 \times \frac{1}{2} \times (1 - 0.095238)} \\ &\simeq 0.173913. \end{aligned}$$

- **Requested expected value**
  The expected number of cars that have to wait for their inspection to begin is given by

$$\begin{aligned} E(L_q) &\stackrel{form.}{=} \frac{\rho}{1-\rho} C(m, m\rho) \\ &\simeq \frac{1/2}{1 - 1/2} \times 0.173913 \\ &\simeq 0.173913. \end{aligned}$$

[3] **Hint**: $C(m, m\rho) = \frac{m \times B(m, m\rho)}{m - m\rho \times [1 - B(m, m\rho)]}$.