DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC, MECD

2nd. Semester – 2019/2020
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Duration: 90 minutes **Test 1** (*Recurso*)

- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 0 — Introduction to Stochastic Processes 2.5 points

Let $\{X(t): t \geq 0\}$ be a Gaussian process,[1] with mean and autocovariance functions $\mu(t) = 0$ and $\gamma(s,t) = s\ (s \leq t)$, and α a positive real constant.

$\{V(t) = e^{-\frac{\alpha t}{2}} X(e^{\alpha t}) : t \geq 0\}$ is called the Ornstein-Uhlenbeck process and has been proposed as a model for describing the velocity of a particle immersed in a liquid or gas.

(a) Compute the mean function and the autocovariance function of $\{V(t): t \geq 0\}$ (1.0)

- **Auxiliary stochastic process; mean and autocovariance functions**
  $\{X(t): t \geq 0\}$ (Gaussian process)
  $\mu(t) = E[X(t)] = 0$
  $\gamma(s,t) = cov(X(s), X(t)) = s, \quad s \leq t$
- **Stochastic process**
  $\{V(t) = e^{-\frac{\alpha t}{2}} X(e^{\alpha t}) : t \geq 0\}, \quad \alpha > 0$
- **Mean function**

$$\begin{aligned} E[V(t)] &= E\left[e^{-\frac{\alpha t}{2}} X\left(e^{\alpha t}\right)\right] \\ &= e^{-\frac{\alpha t}{2}} E\left[X\left(e^{\alpha t}\right)\right] \\ &\overset{E[X(e^{\alpha t})]=0}{=} 0, \quad t \geq 0 \end{aligned}$$

- **Autocovariance function**
  Taking advantage of the properties of the covariance operator and on the fact that $cov(X(s), X(s')) = s,\ s \leq s'$, we get:

$$\begin{aligned} cov(V(t), V(t+h)) &= cov\left(e^{-\frac{\alpha t}{2}} X\left(e^{\alpha t}\right), e^{-\frac{\alpha(t+h)}{2}} X\left(e^{\alpha(t+h)}\right)\right) \\ &= e^{-\frac{\alpha t}{2}} e^{-\frac{\alpha(t+h)}{2}} cov\left(X\left(e^{\alpha t}\right), X\left(e^{\alpha(t+h)}\right)\right) \\ &= e^{-\alpha t - \frac{\alpha h}{2}} cov\left(X\left(e^{\alpha t}\right), X\left(e^{\alpha(t+h)}\right)\right) \\ &\overset{h,\alpha>0}{=} e^{-\alpha t - \frac{\alpha h}{2}} e^{\alpha t} \\ &= e^{-\frac{\alpha h}{2}}. \end{aligned}$$

(b) Show that $\{V(t): t \geq 0\}$ is not only a second order weakly stationary process, but also a strictly stationary process. (1.5)

- **Checking whether the process is second order weakly stationary**
  The mean function $E[V(t)]$ does not depend on t and the autocovariance $cov(V(t), V(t+h))$ only depends on the time lag h, hence $\{V(t): t \geq 0\}$ is a second order weakly stationary process.
- **Checking whether the process is strictly stationary**
  Since $X(t) \sim \text{normal}(0,t)$ and $(X(t_1), \ldots, X(t_n))$ has a multivariate normal distribution (for all $t_1, \ldots, t_n \geq 0$), we can successively conclude that:
  - $V(t) \sim \text{normal}(\mu(t) = 0, \gamma(0,0) = 1)$;
  - $(V(t_1), \ldots, V(t_n))$ has a multivariate normal distribution (for all $t_1, \ldots, t_n \geq 0$), i.e.,

$$\begin{aligned} F_{t_1,\ldots,t_n}(v_1,\ldots,v_n) &= P[V(t_1) \leq v_1, \ldots, V(t_n) \leq v_n] \\ &= F_{\mathcal{N}_n(\underline{\mu}_n, \underline{\Sigma}_n)}(x_1,\ldots,x_n), \quad (v_1,\ldots,v_n) \in \mathbb{R}^n, \end{aligned}$$

  where the mean vector equals

$$\underline{\mu}_n = [E[V(t_i)]]_{i=1,\ldots,n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

  and the covariance matrix

$$\begin{aligned} \underline{\Sigma}_n &= \left[cov\left(V(t_i), V(t_j)\right)\right]_{i,j=1,\ldots,n} \\ &= [\gamma(0, t_j - t_i)]_{i,j=1,\ldots,n} \\ &= \left[e^{-\frac{\alpha(t_j - t_i)}{2}}\right]_{i,j=1,\ldots,n}; \end{aligned}$$

  - $\{V(t): t \geq 0\}$ is also a Gaussian process.

  Moreover, the mean vector $\underline{\mu}_n$ is constant over time and the covariance matrix $\underline{\Sigma}_n$ only depends on the time lag, hence $(V(t_1+h), \ldots, V(t_n+h))$ also has the same multivariate normal distribution with parameters $\underline{\mu}_n$ and $\underline{\Sigma}_n$, i.e.,

$$\begin{aligned} F_{t_1+h,\ldots,t_n+h}(v_1,\ldots,v_n) &= F_{\mathcal{N}_n(\underline{\mu}_n, \underline{\Sigma}_n)}(v_1,\ldots,v_n) \\ &= F_{t_1,\ldots,t_n}(v_1,\ldots,v_n), \end{aligned}$$

  for any $n \in \mathbb{N}$. Thus, $\{V(t): t \geq 0\}$ is a strictly stationary process.

  [We could have invoked that: as the finite dimensional distributions of a Gaussian process (being multivariate normal) are determined by their means and covariance, it follows that a second order weakly stationary Gaussian process is strictly stationary.]

[1] That is, for all $t_1, \ldots, t_n$, each finite-dimensional vector $(X(t_1), \ldots, X(t_n))$ has a multivariate normal distribution $\mathcal{N}_n(\underline{\mu}, \underline{\Sigma})$ for some mean vector $\underline{\mu}$ and some covariance matrix $\underline{\Sigma}$ which may depend on $t_1, \ldots, t_n$.

Group 1 — Poisson Processes 8.5 points

1. Jobs are submitted for execution on a central computer and come from four independent sources. The interarrival times for the jobs submissions from source i are i.i.d. r.v. with exponential distributions with mean μ_i minutes $(i = 1,2,3,4)$, with $\mu_1 = 10$, $\mu_2 = 15$, $\mu_3 = 30$, $\mu_4 = 60$.

Let $N(t)$ be the total number of job submissions from all four sources in the interval $(0,t]$.

(a) Obtain $P[N(60) = 10, N(120) = 22]$. (1.5)

- **Auxiliary stochastic process**
  It is said that sources are independent and the interarrival times for the jobs submissions from source i are i.i.d. r.v. with an exponential distribution with mean μ_i, hence

$$\{N_i(t): t \geq 0\} \overset{indep.}{\sim} PP(\lambda_i = 1/\mu_i \text{ jobs per minute}), \quad i = 1,2,3,4,$$

  where: $N_i(t)$ = number of job submissions from source i in the interval $(0,t]$; $\lambda_1 = 1/10$, $\lambda_2 = 1/15$, $\lambda_3 = 1/30$, $\lambda_4 = 1/60$.
- **Stochastic process**
  $N(t) = \sum_{i=1}^4 N_i(t)$ = total number of job submissions in the interval $(0,t]$
  Since we are dealing with the merging of four independent PP,

$$\{N(t): t \geq 0\} \sim PP(\lambda),$$

  where $\lambda = \sum_{i=1}^4 \lambda_i = 13/60$ job submissions per minute (or 13 job submissions per hour).
- **Relevant distribution**
  $N(t) \sim \text{Poisson}(\lambda t)$
  $P[N(t) = x] = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x \in \mathbb{N}_0$

- **Requested probability**
  [We want to obtain the probability that a total of 10 job submissions occur in the first hour and a total of 22 job submissions occur in the first two hours,] $P[N(60) = 10, N(120) = 22] = \star$, where

$$\begin{aligned}
\star &= P[N(60)=10, N(120)-N(60) = 22-10] \\
&\overset{indep.incr.}{=} P[N(60)=10] \times P[N(120)-N(60)=12] \\
&\overset{station.incr.}{=} P[N(60)=10] \times P[N(120-60)=12] \\
&= P[N(60)=10] \times P[N(60)=12] \\
&\overset{N(60)\sim Poi(\frac{13}{60}\times 60=13)}{=} \frac{e^{-13}\times 13^{10}}{10!} \times \frac{e^{-13}\times 13^{12}}{12!} \\
&= \frac{e^{-26}\times 13^{22}}{10!\times 12!} \\
&\simeq 0.009441.
\end{aligned}$$

(b) Compute the probability that the second job submission from source 1 occurs before the arrival of the third job submission from source 2. (1.0)

- **Relevant r.v.**
  $S_n^{(1)}$ = time of arrival of the n^{th} job submission from source 1
  $S_m^{(2)}$ = time of arrival of the m^{th} job submission from source 2
- **Requested probability**
  The job submissions from sources 1 and 2 are governed by two independent PP, therefore

$$\begin{aligned}
P\left[S_n^{(1)} < S_m^{(2)}\right] &= 1 - F_{binomial(n+m-1,\lambda_1/(\lambda_1+\lambda_2))}(n-1) \\
&\overset{n=2,m=3,\lambda_1=\frac{1}{10},\lambda_1=\frac{1}{15}}{=} 1 - F_{binomial(2+3-1,\frac{1}{10}/(\frac{1}{10}+\frac{1}{15}))}(2-1) \\
&= 1 - F_{binomial(4,\frac{3}{5})}(1) \qquad [= 1 - \sum_{i=0}^{1}\binom{4}{i} 0.6^i (1-0.6)^{4-i}] \\
&= F_{binomial(4,1-\frac{3}{5})}(4-1-1) \\
&= F_{binomial(4,0.4)}(2) \\
&\overset{tables}{=} 0.8208.
\end{aligned}$$

(c) Each job submission from source i takes, on average, ξ_i minutes to complete. What is the mean completion time of a typical job submission if $\xi_1 = 3,\ \xi_2 = 4,\ \xi_3 = 6,\ \xi_4 = 8$?[2] (1.0)

- **Auxiliary r.v.**
  Z = source of a typical job submission
  $P(Z=i) \overset{hint}{=} \frac{\lambda_i}{\sum_{k=1}^4 \lambda_k}, \quad i = 1,\dots,4$
- **Requested mean**

$$\begin{aligned}
\xi &= \sum_{i=1}^4 \xi_i \times P(Z=i) \\
&= \sum_{i=1}^4 \xi_i \times \frac{\lambda_i}{\sum_{k=1}^4 \lambda_k} \\
&= \frac{3\times\frac{1}{10} + 4\times\frac{1}{15} + 6\times\frac{1}{30} + 8\times\frac{1}{60}}{\frac{13}{60}} \\
&= \frac{18+16+12+8}{13} \\
&= \frac{54}{13} \\
&\simeq 4.153846.
\end{aligned}$$

[2] **Hint**: If $Z_n = i$ in case the n^{th} event in $\{N(t): t \geq 0\}$ comes from the Poisson process $\{N_i(t): t\geq 0\}$ then $\{Z_n : n \in \mathbb{N}\}$ is a sequence of i.i.d. r.v. with common p.f. $P(Z_n = i) = \lambda_i / \sum_{k=1}^r \lambda_k$, $i = 1,\dots,r$.

2. Admit that hormonal secretion bursts occur according to a non-homogeneous Poisson process with intensity function: $\lambda(t) = \lambda$, for $t \in [2i, 2i+1]$ $(i \in \mathbb{N}_0)$; $\lambda(t) = 0$, for $t \in (2i+1, 2i+2)$ $(i \in \mathbb{N}_0)$.

(a) Derive: the mean value function; and the survival function of the time S_1 of the occurrence of the first hormonal secretion burst. Obtain $P(S_1 \in (2,3])$, when $\lambda = 1$. (1.5)

- **Stochastic process**
  $\{N(t): t > 0\} \sim NHPP$
  $N(t)$ = number of hormonal secretion bursts until time t
- **Intensity function**

$$\lambda(t) = \begin{cases} \lambda, & t \in [2i, 2i+1] \quad (i \in \mathbb{N}_0) \\ 0, & t \in (2i+1, 2i+2) \quad (i \in \mathbb{N}_0) \end{cases}$$

- **Requested mean value function**

$$m(t) = \begin{cases} \lambda \times t, & t \in [0,1] \\ \lambda, & t \in (1,2) \\ \lambda + \lambda \times (t-2) = \lambda \times (t-1), & t \in [2,3] \\ 2\lambda, & t \in (3,4) \\ \lambda + \lambda \times (t-4) = \lambda \times (t-2), & t \in [4,5] \\ 3\lambda, & t \in (5,6) \\ \vdots \end{cases} = \begin{cases} \lambda \times (t-i), & t \in [2i, 2i+1] \quad (i \in \mathbb{N}_0) \\ \lambda \times (i+1), & t \in (2i+1, 2i+2) \quad (i \in \mathbb{N}_0) \end{cases}$$

- **Relevant r.v.**
  S_1 = time of the first hormonal secretion burst
- **Requested survival function**

$$\begin{aligned}
P(S_n > t) &= 1 - F_{S_n}(t) \\
&\overset{form.}{=} 1 - [1 - F_{Poisson(m(t))}(n-1)] \\
&\overset{n=1}{=} F_{Poisson(m(t))}(0) \\
&= e^{-m(t)}
\end{aligned}$$

- **Requested probability**

$$\begin{aligned}
P(S_1 \in (2,3]) &= P(S_1 \leq 3) - P(S_1 \leq 2) \\
&= P(S_1 > 2) - P(S_1 > 3) \\
&= e^{-m(2)} - e^{-m(3)} \\
&= e^{-\lambda\times(2-1)} - e^{-\lambda\times(3-1)} \\
&\overset{\lambda=1}{=} e^{-1} - e^{-2} \\
&\simeq 0.232544.
\end{aligned}$$

(b) Suppose a hormonal secretion burst occurring at time s is registered by an electronic device with probability $p(s)$, independently of everything else. (1.5)
Let $N_R(t)$ be the number of registered hormonal secretion bursts up to time t. Show that $N_R(t) \sim \text{Poisson}(\int_0^{+\infty} \lambda(s) \times p(s)\, ds)$, for any intensity function $\lambda(s)$.

- **Relevant r.v.**
  $N_R(t) =$ number of registered hormonal secretion bursts up to time t
- **Requested proof**
  $N_R(t)$ results from a non-homogenous Bernoulli splitting of $\{N(t): t \geq 0\} \sim NHPP(\lambda(t))$, thus we call follow closely the proof found in Kulkarni (1995, pp. 220–221).
  According to the formulae, since we are dealing with a $NHPP(\lambda(t))$, the event times $S_1, \dots, S_2$ behave such as the order statistics of the random sample $(Y_1, \dots, Y_n)$ from the population Y with c.d.f. $F_Y(s) = m(s)/m(t)$, $0 \leq s \leq t$. Hence the p.d.f. of Y is given by

$$\begin{aligned}
f_Y(s) &\overset{form.}{=} \frac{dm(s)/ds}{m(t)} \\
&= \frac{\lambda(s)}{m(t)}, \quad 0 \leq s \leq t.
\end{aligned}$$

Now, assume an event is generated in $(0,t)$ according to the distribution of Y. This event will be registered by the electronic device with probability

$$\begin{aligned}\alpha(t) &\overset{total\,prob.law}{=} \int_0^t p(s)\times f_Y(s)\,ds\\ &= \frac{1}{m(t)}\int_0^t \lambda(s)\times p(s)\,ds.\end{aligned}$$

Furthermore, since the $Y_i \overset{i.i.d.}{\sim} Y$, $i=1,2\ldots,n$, we obtain:

$$\begin{aligned}N_R(t)\mid N(t)=n &\overset{form.}{\sim} \text{binomial}(n,\alpha(t));\\ P[N_R(t)=k] &\overset{total\,prob.law}{=} \sum_{n=k}^{+\infty} P[N_R(t)=k\mid N(t)=n]\times P[N(t)=n]\\ &= \sum_{n=k}^{+\infty}\frac{n!}{k!(n-k)!}[\alpha(t)]^k[1-\alpha(t)]^{n-k}\times\frac{e^{-m(t)}[m(t)]^n}{n!}\\ &= \frac{e^{-m(t)}[\alpha(t)\,m(t)]^k}{k!}\times\sum_{n=k}^{+\infty}\frac{\{[1-\alpha(t)]\,m(t)\}^{n-k}}{(n-k)!}\\ &= \frac{e^{-m(t)}[\alpha(t)\,m(t)]^k}{k!}\times e^{[1-\alpha(t)]\,m(t)}\\ &= \frac{e^{-\alpha(t)\,m(t)}[\alpha(t)\,m(t)]^k}{k!},\quad k\in\mathbb{N}_0.\end{aligned}$$

Thus, $N_R(t)\sim \text{Poisson}(\int_0^{+\infty}\lambda(s)\times p(s)\,ds)$. ✓

3. An actuary admits that the number of claims up to time t forms a conditional Poisson process, $\{N(t): t\geq 0\}$, with random rate Λ (in claims per month) with a negative binomial distribution with fixed parameters r ($r\in\mathbb{N}$) and p ($p\in(0,1)$, and p.f. $P(\Lambda=\lambda)=\binom{\lambda-1}{r-1}p^r(1-p)^{\lambda-r}$, for $\lambda=r,r+1,\ldots$[3] (2.0)

Derive a general expression for the following *a posteriori* probability: $P[\Lambda=\lambda\mid N(t)=0]$.

Compute this probability when $r=2$, $p=0.1$, $\lambda=2$, $t=1$.

- **Stochastic process**

  $\{N(t): t\geq 0\}\sim CondPP(\text{NegativeBin}(r,p))$

  $N(t)=$ number of claims up to time t

- **Random arrival rate**

  $\Lambda\sim \text{NegativeBin}(r,p)$, $\quad r\in\mathbb{N}$, $p\in(0,1)$

  $P(\Lambda=\lambda)=\binom{\lambda-1}{r-1}p^r(1-p)^{\lambda-r}$, $\lambda=r,r+1,\ldots$

  $G(\lambda)=P(\Lambda\leq\lambda)$

- **Requested general expression and probability**

  Since

$$\begin{aligned}P[N(t)=n\mid\Lambda=\lambda] &= e^{-\lambda t}\frac{(\lambda t)^n}{n!},\quad n\in\mathbb{N}_0\\ P[N(t)=n] &= \int_0^{+\infty}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\,dG(\lambda),\quad n\in\mathbb{N}_0\\ P[N(t)=0] &= \int_0^{+\infty} e^{-\lambda t}\,dG(\lambda)\\ &= M_\Lambda(-t)\\ &\overset{form.}{=} \left[\frac{p\,e^{-t}}{1-(1-p)\,e^{-t}}\right]^r,\end{aligned}$$

  we obtain

$$\begin{aligned}P[\Lambda=\lambda\mid N(t)=0] &\overset{T.Bayes}{=} \frac{P[N(t)=0\mid\Lambda=\lambda]\times P(\Lambda=\lambda)}{P[N(t)=0]}\\ &= \frac{e^{-\lambda t}\times\binom{\lambda-1}{r-1}p^r(1-p)^{\lambda-r}}{\left[\frac{p\,e^{-t}}{1-(1-p)\,e^{-t}}\right]^r}\\ &\overset{r=2,p=0.1,\lambda=2,t=1}{=} \frac{e^{-2\times 1}\times\binom{2-1}{2-1}0.1^2(1-0.1)^{2-2}}{\left[\frac{0.1\,e^{-1}}{1-(1-0.1)\,e^{-1}}\right]^2}\\ &\simeq 0.447439.\end{aligned}$$

[3] An actuary is a *person who compiles and analyses statistics and uses them to calculate insurance risks and premiums.* In insurance mathematics the conditional PP is often used to model the *accident proneness* of the members of a collective of risks.

Group 2 — Renewal Processes	9.0 points

1. Suppose a part in a machine is available from two different suppliers, A and B. When the part fails it is replaced by a new one from supplier A (resp. B) with probability 0.3 (resp. 0.7). A part from supplier A (resp. B) lasts for an exponential amount of time with a mean of 8 days (resp. 5 days), and it takes exactly one day (resp. half a day) to install it. Assume that a failure has occurred at time 0^- and $N(t)$ represents the number of failures in the interval $(0,t]$.[4]

(a) Verify that the inter-renewal distribution of $\{N(t): t\geq 0\}$ has expected value and variance equal to $\mu=6.55$ and $\sigma^2=39.2725$. (2.0)

- **Renewal process**

  $\{N(t): t\geq 0\}$

  $N(t)=$ number of failures in the interval $(0,t]$

- **Inter-renewal times**

  $X_i\overset{i.i.d.}{\sim} X$, $i\in\mathbb{N}$

  $$X\sim\begin{cases}1+Y_A, & \text{with probability } p_A\\ 1+Y_B, & \text{with probability } p_B\end{cases}$$

  where: $Y_A\sim$ exponential$(1/8)$, $E(Y_A)=8$, $V(Y_A)=8^2$, $p_A=0.3$; $Y_B\sim$ exponential$(1/5)$, $E(Y_B)=5$, $V(Y_B)=5^2$, $p_B=0.7$.

- **Requested mean and variance of the IRT**

  Since X is a mixture of the r.v. $(1+Y_A)$ and $(0.5+Y_B)$, its expected value is given by the following convex linear combination of expected values:

$$\begin{aligned}\mu &= E(X)\\ &= p_A\times E(1+Y_A)+p_B\times E(0.5+Y_B)\\ &= p_A\times[1+E(Y_A)]+p_B\times[0.5+E(Y_B)]\\ &= 0.3\times(1+8)+0.7\times(0.5+5)\\ &= 6.55.\quad ✓\end{aligned}$$

  Similarly, the second moment of X is a convex linear combination of second moments:

$$\begin{aligned}E(X^2) &= p_A\times E[(1+Y_A)^2]+p_B\times E[(0.5+Y_B)^2]\\ &= p_A\times[1+2\,E(Y_A)+E(Y_A^2)]+p_B\times[0.25+E(Y_B)+E(Y_B^2)]\\ &= p_A\times[1+2\,E(Y_A)+V(Y_A)+E^2(Y_A)]+p_B\times[0.25+E(Y_B)+V(Y_B)+E^2(Y_B)]\\ &= 0.3\times(1+2\times 8+8^2+8^2)+0.7\times(0.25+5+5^2+5^2)\\ &= 0.3\times 145+0.7\times 55.25\\ &= 82.175.\end{aligned}$$

  As a result,

$$\begin{aligned}\sigma^2 &= V(X)\\ &= E(X^2)-E^2(X)\end{aligned}$$

[4] Hence we are not counting the failure at time 0^-.

$$\begin{aligned}\sigma^2 &= 82.175 - 6.55^2 \\ &= 39.2725. \quad \checkmark\end{aligned}$$

(b) Calculate an approximate value of the probability that at least 52 failures occur in the first year. (1.0)

- **[Inter-renewal times**
  $X_i \stackrel{i.i.d.}{\sim} X,\ i \in \mathbb{N}$
  $\mu = E(X) \stackrel{(a)}{=} 6.55$
  $\sigma^2 = V(X) \stackrel{(a)}{=} 39.2725]$

- **Requested approximate probability**

$$\begin{aligned}P[N(t) \geq n] &= 1 - P[N(t) < n] \\ &\stackrel{form.}{\simeq} 1 - \Phi\left(\frac{n - t/\mu}{\sqrt{t\sigma^2/\mu^3}}\right) \\ &\stackrel{n=52,\, t=365,\, \mu=6.55,\, \sigma^2=39.2725}{=} 1 - \Phi\left(\frac{52 - 365/6.55}{\sqrt{365 \times 39.2725/6.55^3}}\right) \\ &\simeq 1 - \Phi(-0.52) \\ &= \Phi(0.52) \\ &\stackrel{tables}{=} 0.6985.\end{aligned}$$

(c) Provide an approximate value of the expected number of failures bound to occur in the first two weeks of the second year. (1.5)

- **Requested expected value**
  Since the inter-renewal distribution is non-lattice, we can apply Blackwell's theorem and state that, for very large t (such as $t = 365$ days), the expected number of failures in the interval $(t, t+a]$ (for $a = 14$ days), $E[N(t+a) - N(t)]$, can be approximated as follows:

$$\begin{aligned}E[N(t+a) - N(t)] &\simeq \lim_{z \to +\infty} [m(z+a) - m(z)] \\ &\equiv \frac{a}{\mu} \\ &= \frac{14}{6.55} \\ &\simeq 2.137405.\end{aligned}$$

(d) Admit an inspection was made on February 1 of the second year. Obtain an approximation to the probability that the first failure after this inspection occurred after February 2 of that same year. (2.5)

- **C.d.f. of the IRT**
  Since X is a mixture of the r.v. $(1 + Y_A)$ and $(0.5 + Y_B)$, its c.d.f. is a convex linear combination of c.d.f. of these two r.v.:

$$\begin{aligned}F(x) &= P(X \leq u) \\ &= p_A \times P(1 + Y_A \leq u) + p_B \times P(1 + Y_B \leq u) \\ &= p_A \times F_{Y_A}(u-1) + p_B \times F_{Y_B}(u - 0.5),\end{aligned}$$

  where

$$F_{Y_A}(u-1) = \begin{cases} 0, & u < 1 \\ 1 - e^{-\frac{u-1}{8}}, & u \geq 1 \end{cases}$$

$$F_{Y_B}(u-0.5) = \begin{cases} 0, & u < 0.5 \\ 1 - e^{-\frac{u-0.5}{5}}, & u \geq 0.5 \end{cases}$$

- **Recurrence time**
  $Y(t) \stackrel{form}{=} S_{N(t)+1} - t =$ time until the first failure after the inspection at time t

- **Requested probability** (approximate value)
  We can once again invoke that $t = 365 + 31 + 1$ days is sufficiently large and provide the following approximate value

$$\begin{aligned}P[S_{N(t)+1} > t + x] &= 1 - P[S_{N(t)+1} - t \leq x] \\ &= 1 - F_{Y(t)}(x) \\ &\simeq 1 - \lim_{z \to +\infty} P[Y(z) \leq x] \\ &\stackrel{form.}{=} 1 - \frac{\int_0^x \{1 - F(u)\}\, du}{E(X)} \\ &\stackrel{x=1}{=} 1 - \frac{1}{E(X)} \times \int_0^1 [1 - 0.3 \times F_{Y_A}(u-1) - 0.7 \times F_{Y_B}(u-0.5)]\, du \\ &= 1 - \frac{1}{E(X)} \times \left[1 - 0.7 \times \int_{0.5}^1 F_{Y_B}(u - 0.5)\, du\right] \\ &= 1 - \frac{1}{E(X)} \times \left[1 - 0.7 \times \int_0^{0.5} F_{Y_B}(u)\, du\right] \\ &= 1 - \frac{1}{E(X)} \times \left[1 - 0.7 \times \int_0^{0.5} \left(1 - e^{-\frac{u}{5}}\right) du\right] \\ &= 1 - \frac{1}{E(X)} \times \left[1 - 0.7 \times 0.5 - 0.7 \times 5 \times e^{-\frac{u}{5}} \Big|_0^{0.5}\right] \\ &= 1 - \frac{1}{6.55} \times \left[1 - 0.35 - 3.5 \times \left(e^{-0.1} - 1\right)\right] \\ &\simeq 0.849913.\end{aligned}$$

2. Consider a renewal process $\{N(t) : t \geq 0\}$ associated with inter-renewal times and rewards $\{(X_i, R_i) : i \in \mathbb{N}\} \stackrel{i.i.d.}{\sim} (X, R)$ and event times $S_n = \sum_{i=1}^n X_i$ $(n \in \mathbb{N})$. (2.0)

Let $D(t) = \sum_{n=1}^{N(t)} e^{-\alpha S_n} \times R_n$ be the total discounted reward earned up to time t, where α is a positive discount factor. Show that $\lim_{t \to +\infty} E[D(t)] = \frac{E(e^{-\alpha X} R)}{1 - E(e^{-\alpha X})}$.[5]

- **Renewal process**
  $\{N(t) : t \geq 0\}$
  $N(t) =$ number of events up to time t

- **Auxiliary r.v.**
  $X_i =$ inter-renewal time i, $\quad i \in \mathbb{N}$
  $X_i \stackrel{i.i.d.}{\sim} X, \quad i \in \mathbb{N}$
  $S_n = \sum_{i=1}^n X_i =$ time of event n, $\quad n \in \mathbb{N}$
  $(X_i, R_i) \stackrel{i.i.d.}{\sim} (X, R),\ i \in \mathbb{N}$

- **Relevant r.v.**
  $D(t) = \sum_{n=1}^{N(t)} e^{-\alpha S_n} \times R_n$, where α represents a positive discount factor.

  Requested proof

$$\begin{aligned}E[D(t)] &= E\left\{E\left[\sum_{n=1}^{N(t)} e^{-\alpha S_n} \times R_n \mid N(t)\right]\right\} \\ &= E(\star).\end{aligned}$$

  Recall that: $X_i \stackrel{i.i.d.}{\sim} X,\ i \in \mathbb{N}$; $R_i \stackrel{i.i.d.}{\sim} R,\ i \in \mathbb{N}$; $X_1, \ldots, X_{n-1}$ are independent of R_n. Consequently, $\star$ is a r.v. that takes value

[5] **Note**: Invoking Jensen's inequality leads to $E(e^{-\alpha X}) \in (0, 1)$.

$$\begin{aligned}
E\left[\sum_{n=1}^{N(t)} e^{-\alpha S_n} \times R_n \mid N(t)=m\right] &= E\left[\sum_{n=1}^{m} e^{-\alpha(X_1+\cdots+X_{n-1})} \times e^{-\alpha X_n} R_n\right] \\
&= \sum_{n=1}^{m} E\left[e^{-\alpha(X_1+\cdots+X_{n-1})}\right] \times E\left(e^{-\alpha X_n} R_n\right) \\
&= \sum_{n=1}^{m} \left[E\left(e^{-\alpha X}\right)\right]^{m-1} \times E\left(e^{-\alpha X} R\right) \\
&= E\left(e^{-\alpha X} R\right) \times \frac{1-\left[E\left(e^{-\alpha X}\right)\right]^{m}}{1-E\left(e^{-\alpha X}\right)}
\end{aligned}$$

with probability $P[N(t)=m]$. Finally,

$$\begin{aligned}
\lim_{t\to+\infty} E[D(t)] &= \lim_{t\to+\infty} E\left\{E\left(e^{-\alpha X} R\right) \times \frac{1-\left[E\left(e^{-\alpha X}\right)\right]^{N(t)}}{1-E\left(e^{-\alpha X}\right)}\right\} \\
&= \frac{E\left(e^{-\alpha X} R\right)}{1-E\left(e^{-\alpha X}\right)} \times \lim_{t\to+\infty} E\left\{1-\left[E\left(e^{-\alpha X}\right)\right]^{N(t)}\right\} \\
&= \frac{E\left(e^{-\alpha X} R\right)}{1-E\left(e^{-\alpha X}\right)},
\end{aligned}$$

because $N(t)\to+\infty$, as $t\to+\infty$, and according to the note $E(e^{-\alpha X})\in(0,1)$.