DM DEPARTAMENTO DE MATEMÁTICA
TÉCNICO LISBOA

Introduction to Stochastic Processes

MMA, LMAC

2nd. Semester – 2018/2019
2019/07/10 – 9:45AM, Room P12

Duration: 90 minutes **Test 2** (*Recurso*)

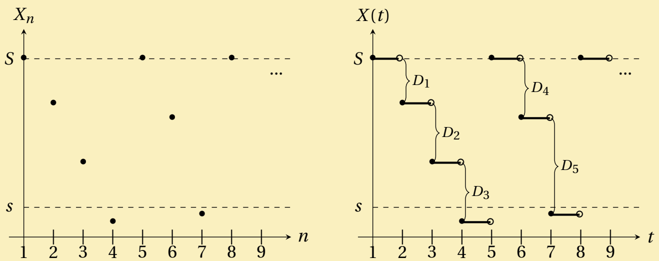
- Please justify all your answers.
- This test has TWO PAGES and THREE GROUPS. The total of points is 20.0.

Group 1 — Renewal Processes 2.0 points

Let X_n be the amount of inventory in a warehouse at the beginning of day n. Suppose $X_1 = S$, where S is a fixed positive number. Let D_n be the size of the demand on day n and assume that $\{D_n : n \in \mathbb{N}\}$ is a sequence of i.i.d. r.v. with common c.d.f. F. Admit that as soon as the inventory goes below s $(s < S)$, it is instantaneously replenished to S. Then $\{X_n : n \in \mathbb{N}\}$ is a regenerative process. (2.0)

Draw a sample path of $\{X_n : n \in \mathbb{N}\}$. Determine $\lim_{n\to+\infty} P(X_n \geq x)$ in terms of $m(t)$, the renewal function generated by the sequence on *inter-renewal times* $\{D_n : n \in \mathbb{N}\}$.

- **Regenerative process**
  $\{X_n : n \in \mathbb{N}\}$
  X_n = inventory level at the beginning of day n
  $X_1 = S$
  D_n = size of the demand on day n
  $D_n \overset{i.i.d.}{\sim} D \sim F, \quad n \in \mathbb{N}$
- **Obs.**
  $$X_{n+1} = \begin{cases} \max\{0, X_n - D_n\}, & s < X_n \leq S \\ \max\{0, S - D_n\}, & X_n \leq s \end{cases}$$
- **Sample path of** $\{X_n : n \in \mathbb{N}\}$ **and** $\{X(t) : t \geq 1\}$ (where $X(t)$ = inventory level at time t)



- **Auxiliary functions and r.v.**
  $m(t)$, renewal function generated by the sequence on *inter-renewal times* $\{D_n : n \in \mathbb{N}\}$
  S_1 = time between two days beginning with inventory level equal to S
  $E(S_1) = 1 + m(S - s)$ (check sample path)
  $E(S_1) < +\infty$ (the renewal function is finite in finite time)
  U_x = time spent in state $\leq x$ during $[0, S_1)$
  $E(U_x) = m(S - x)$ (check sample path)
- **Requested limit**
  $$\lim_{n\to+\infty} P(X_n \geq x) = \frac{E(U_x)}{E(S_1)} = \frac{m(S-x)}{1 + m(S-s)}.$$

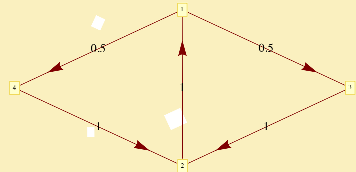
Group 2 — Discrete time Markov chains 9.0 points

1. Suppose a customer keeps switching between brands 1, 2, 3, and 4 every month according to the TPM

$$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

(a) Draw the associated transition diagram and classify the states of this DTMC. Are the states periodic? (1.5)

- **DTMC**
  $\{X_n : n \in \mathbb{N}\}$
  X_n = brand acquired on the n^{th} month
- **State space**
  $\mathcal{S} = \{1, 2, 3, 4\}$
- **TPM**
  $$\mathbf{P} = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$
- **Transition diagram**



- **Classification of the states of the DTMC**
  - Judging by the transition diagram, all states communicate with one another, thus $\mathcal{S} = \{1, 2, 3, 4\}$ is a finite single closed communicating class [hence the DTMC has a finite state space and is irreducible]. With that being said, [by Prop. 3.35,] all states are positive recurrent.
  - The transition diagram leads to the conclusion that we can only return to state 1 after $3, 6, \ldots$ transitions, thus $d(1) = gcd\left\{n \in \mathbb{N} : P_{11}^{n} > 0\right\} = 3$ and this state is aperiodic.
    The same holds for the remaining states of this closed (and positive recurrent) communicating class, $\mathcal{S} = \{1, 2, 3\}$.
    [After all (a)periodicity is a class property.]

(b) Admit the initial brand X_1 is uniformly distributed in the state space. Calculate the probability that the customer acquires brand 4 at the end of the first trimester. (1.0)

- **Initial state**
  $X_1 \sim \text{Uniform}(\{1, 2, 3, 4\})$
- **Requested probability**
  Since the initial state of this DTMC is X_1 (instead of X_0) we have to adapt the results in the list of formulae:
  $$\begin{aligned} \underline{\alpha} &= [P(X_1 = i)]_{i\in\mathcal{S}} \\ &= [1/4 \quad 1/4 \quad 1/4 \quad 1/4] \\ \underline{\alpha}^n &= [P(X_{n+1} = i)]_{i\in\mathcal{S}} \\ &\overset{form.}{=} \underline{\alpha} \times \mathbf{P}^n. \end{aligned}$$

Thus,

$$\begin{aligned}
\underline{\alpha}^2 &= [P(X_{2+1}=i)]_{i\in\mathscr{S}} \\
&= \underline{\alpha}\times\mathbf{P}^2 \\
P(X_{2+1}=4) &= \underline{\alpha}\times\mathbf{P}\times \text{4th. column of } \mathbf{P} \\
&= \underline{\alpha}\times\begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}\times\begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
&= [1/4 \quad 1/4 \quad 1/4 \quad 1/4]\times\begin{bmatrix} 0 \\ 0.5 \\ 0 \\ 0 \end{bmatrix} \\
&= 0.125
\end{aligned}$$

(c) Given that $X_1 = 1$, calculate the expected number of months until brand 4 is acquired. (2.0)

Note: Check the footnote![1]

- **Important**
  Let us consider another DTMC with absorbing state/brand 4 and sub-stochastic matrix governing the transitions between the transient states ($T = \{1,2,3\}$) given by

$$\mathbf{Q} = \begin{bmatrix} 0 & 0 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

- **Requested expected value**
  $\tau = \inf\{n \in \mathbb{N} : X_n \not\in T\}$ is the number of monthly acquisitions until brand 4 is acquired. Furthermore [(see Prop. 3.116)], the result in the footnote yields

$$\begin{aligned}
[E(\tau \mid X_1 = i)]_{i\in T} &\overset{form.}{=} (\mathbf{I}-\mathbf{Q})^{-1}\times\underline{1} \\
&= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\right)^{-1}\times\underline{1} \\
&= \begin{bmatrix} 1 & 0 & -0.5 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1}\times\underline{1} \\
&= \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}\times\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}.
\end{aligned}$$

  Thus, the requested expected value is $E(\tau \mid X_1 = 1) = 4$.

(d) What is the probability that the customer acquires brand 4 before brand 3, given $X_1 = 2$? (2.0)

Note: You may have to consider states 3 and 4 absorbing, identify sub-stochastic matrices $\mathbf{Q}$ and $\mathbf{R}$ and calculate $(\mathbf{I}-\mathbf{Q})^{-1}\times\mathbf{R}$.

[1] The following result may come handy: $\begin{bmatrix} 1 & 0 & -0.5 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 2 & 2 & 2 \end{bmatrix}.$

- **Important**
  To calculate the requested probability, we have to consider another DTMC, whose states/brands 3 and 4 (*filling*) are absorbing and whose associated TPM is

$$\mathbf{P}^\star = \begin{bmatrix} 0 & 0 & 0.5 & 0.5 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

  The substochastic matrices governing the transitions between the transient states ($T = \{1,2\}$) of this DTMC and the transitions from the transient to the absorbing states ($\overline{T} = \{3,4\}$) are

$$\begin{aligned}
\mathbf{Q} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\
\mathbf{R} &= \begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

  respectively.

- **Requested probability**
  Keeping in mind that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

  we get

$$\begin{aligned}
\mathbf{U} &= [u_{ik}]_{i\in T, k\in\overline{T}} \\
&= [P(\text{reach absorbing state } k \mid X_1 = i)]_{i\in T, k\in\overline{T}} \\
&\overset{form}{=} (\mathbf{I}-\mathbf{Q})^{-1}\times\mathbf{R} \\
&= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)^{-1}\times\mathbf{R} \\
&= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}^{-1}\times\mathbf{R} \\
&= \frac{1}{1\times 1-(-1)\times 0}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\times\begin{bmatrix} 0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}
\end{aligned}$$

  Therefore the probability that the customer acquires brand 4 before brand 3, given $X_1 = 2$, is equal to

$$\begin{aligned}
u_{24} &= P(X_\tau = 4 \mid X_1 = 2) \\
&= 0.5.
\end{aligned}$$

2. Let $\{X_n : n \in \mathbb{N}_0\}$ be a branching process describing the size of generation n. Admit that $X_0 = 1$ and the number of offspring per individual has p.g.f. given by

$$P(s) = 0.15 + 0.05\,s + 0.03\,s^2 + 0.07\,s^3 + 0.4\,s^4 + 0.25\,s^5 + 0.05\,s^6, \quad s \in [0,1].$$

(a) Verify that the extinction probability is equal to $\pi \simeq 0.159293$. (1.5)

- **Branching process**
  $\{X_n : n \in \mathbb{N}_0\}$
  $X_n =$ size of generation n
  $X_0 = 1$
  $X_n = \sum_{l=1}^{X_{n-1}} Z_l,\ n \in \mathbb{N}$

- **Number of offspring per individual**
  $Z_l \equiv Z_{l,n}$ = number of offspring of the l^{th} individual of generation n
  $Z_l \overset{i.i.d.}{\sim} Z,\ l \in \mathbb{N}$
- **P.g.f. of the number of offspring per individual**
  $P(s) = E(s^Z) = \sum_j s^j \times P(Z=j) = 0.15+0.05\,s+0.03\,s^2+0.07\,s^3+0.4\,s^4+0.25\,s^5+0.05\,s^6,\ s\in[0,1]$
- **Probability of extinction**
  Since

$$\begin{aligned} E(Z) &\overset{form.}{=} \left.\frac{dP(s)}{ds}\right|_{s=1} \\ &= \left(0.05+0.06s+0.21s^2+1.6s^3+1.25s^4+0.3s^5\right)\big|_{s=1} \\ &= 0.05+0.06+0.21+1.6+1.25+0.3 \\ &= 3.47 \\ &> 1 \end{aligned}$$

the probability of extinction, $\pi \overset{form.}{=} \lim_{n\to+\infty} P(X_n=0 \mid X_0=1)$, is the smallest positive number satisfying

$$\begin{aligned} \pi &\overset{form.}{=} \sum_{j=0}^{+\infty} \pi^j \times P_j \\ &= P(\pi). \end{aligned}$$

Indeed, we have

$$\begin{aligned} 0.159293 &= P(0.159293) \\ &\simeq 0.15+0.05\times0.159293+0.03\times0.159293^2+0.07\times0.159293^3+0.4\times0.159293^4 \\ &\quad +0.25\times0.159293^5+0.05\times0.159293^6 \\ &\simeq 0.159293 \end{aligned}$$

(b) Obtain the probability that the process is extinct in the second generation. (1.0)

- **Requested probability**

$$\begin{aligned} \pi_2 &= P(X_2=0 \mid X_0=1) \\ &= P_2(0) \\ &\overset{form.}{=} P[P(0)] \\ &= P\left[0.15+0.05\times0+0.03\times0^2+0.07\times0^3+0.4\times0^4+0.25\times0^5+0.05\times0^6\right] \\ &= P(0.15) \\ &= P\big(0.15+0.05\times0.15+0.03\times0.15^2+0.07\times0.15^3+0.4\times0.15^4+0.25\times0.15^5 \\ &\quad +0.05\times0.15^6\big) \\ &= 0.158633. \end{aligned}$$

Group 3 — Continuous time Markov chains	9.0 points

1. Let $X(t)$ be the state of a machine ($0\equiv$ on; $1\equiv$ off) at time t. Admit $\{X(t): t\geq 0\}$ is a CTMC with transition probability matrix given by

$$\mathbf{P}(t) = \begin{bmatrix} 0.6+0.4\,e^{-5t} & 0.4-0.4\,e^{-5t} \\ 0.6-0.6\,e^{-5t} & 0.4+0.6\,e^{-5t} \end{bmatrix}.$$

(a) Determine $P[X(3.4)=1, X(3.8)=0 \mid X(0)=0]$. (2.0)

- **CTMC**
  $\{X(t): t\geq 0\}$
  $X(t)$ = state of a machine at time t
- **State space**
  $\mathscr{S} = \{0,1\}$
  $0\equiv$ on; $1\equiv$ off

- **TPM**

$$\mathbf{P}(t) = \left[P[X(t)=j \mid X(0)=i]\right]_{i,j\in\mathscr{S}} = \begin{bmatrix} 0.6+0.4\,e^{-5t} & 0.4-0.4\,e^{-5t} \\ 0.6-0.6\,e^{-5t} & 0.4+0.6\,e^{-5t} \end{bmatrix}$$

- **Requested probability**

$$\begin{aligned} P[X(3.4)=1, X(3.8)=0 \mid X(0)=0] &= P[X(3.4)=1 \mid X(0)=0] \\ &\quad \times P[X(3.8)=0 \mid X(3.4)=1, X(0)=0] \\ &\overset{Markov\,prop.}{=} P[X(3.4)=1 \mid X(0)=0] \\ &\quad \times P[X(3.8)=0 \mid X(3.4)=1] \\ &\overset{time\,homog.}{=} P[X(3.4)=1 \mid X(0)=0] \\ &\quad \times P[X(0.4)=0 \mid X(0)=1] \\ &= \left(0.4-0.4\,e^{-5\times3.4}\right)\times\left(0.6-0.6\,e^{-5\times0.4}\right) \\ &\simeq 0.207520. \end{aligned}$$

(b) Derive the infinitesimal generator $\mathbf{R}$ of this CTMC and draw the associated rate diagram. (1.5)

- **Infinitesimal generator**
  This matrix has entries

$$r_{ij} = \begin{cases} q_{ij}, & i\neq j \\ -\nu_i = -\sum_{m\in\mathscr{S}} q_{im}, & j=i \end{cases}$$

where

$$\begin{aligned} q_{ij} &= \lim_{h\to0^+}\frac{P_{ij}(h)}{h},\ i\neq j \\ \nu_i &= \lim_{h\to0^+}\frac{1-P_{ii}(h)}{h}. \end{aligned}$$

In this case, we have

$$\begin{aligned} q_{01} &= \lim_{h\to0^+}\frac{P_{01}(h)}{h} \\ &= \lim_{h\to0^+}\frac{0.4-0.4\,e^{-5h}}{h} \\ &\overset{L'H\hat{o}pital\,rule}{=} \lim_{h\to0^+}\frac{0.4\times5\times e^{-5h}}{1} \\ &= 0.4\times5 \\ &= 2 \end{aligned}$$

$$\begin{aligned} q_{10} &= \lim_{h\to0^+}\frac{P_{10}(h)}{h} \\ &= \lim_{h\to0^+}\frac{0.6-0.6\,e^{-5h}}{h} \\ &\overset{L'H\hat{o}pital\,rule}{=} \lim_{h\to0^+}\frac{0.6\times5\times e^{-5h}}{1} \\ &= 0.6\times5 \\ &= 3 \end{aligned}$$

$$\begin{aligned} -\nu_0 &= -q_{01} \\ &= -2 \end{aligned}$$

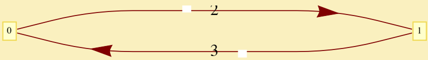
$$\begin{aligned} -\nu_1 &= -q_{10} \\ &= -3. \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{R} &= [r_{ij}]_{i,j\in\mathscr{S}} \\ &= \begin{bmatrix} -2 & 2 \\ 3 & -3 \end{bmatrix}. \end{aligned}$$

[Alternatively, we could find $\mathbf{R}: \frac{dP(t)}{dt} = P(t) \times \mathbf{R} = \mathbf{R} \times P(t)$.]

Rate diagram
Recall that the rate diagram of a CTMC is a directed graph — with no loops — in which each state is represented by a node and there is an arc going from node i to node j (if $q_{ij} > 0$) with q_{ij} written on it.



(c) Obtain the equilibrium probabilities $P_j = \lim_{t\to+\infty} P[X(t) = j \mid X(0) = 0]$, for $j = 0, 1$. (1.0)

- **Equilibrium probabilities** $P_j = \lim_{t\to+\infty} P[X(t) = j \mid X(0) = 0]$

$$\begin{aligned} P_0 &= \lim_{t\to+\infty} P[X(t) = 0 \mid X(0) = 0] \\ &= \lim_{t\to+\infty} P_{00}(t) \\ &= \lim_{t\to+\infty} \left(0.6 + 0.4\, e^{-5t}\right) \\ &= 0.6 \end{aligned}$$

$$\begin{aligned} P_1 &= \lim_{t\to+\infty} P[X(t) = 1 \mid X(0) = 0] \\ &= \lim_{t\to+\infty} P_{01}(t) \\ &= \lim_{t\to+\infty} \left(0.4 - 0.4\, e^{-5t}\right) \\ &= 0.4. \end{aligned}$$

2. A small crude-oil unloading port has four berths. When all unloading berths are full, arriving tankers have to wait. Tankers arrive according to a Poisson process with a rate of one tanker every 2 hours. The unloading time per tanker follows an exponential distribution with mean equal to 4 hours.

(a) Find the probability that an arriving tanker has to wait for an available unloading berth. (1.5)

- **Birth and death queueing system**
  $M/M/m$
  $m = 4$
- **Birth/death rates**
  $\lambda_k = \lambda = \frac{1}{2}, \quad k \in \mathbb{N}_0$
  $\mu_k = \begin{cases} k\mu = k \times \frac{1}{4}, & k \in \{1, 2, \ldots, m-1\} \\ m\mu = m \times \frac{1}{4} & k \in \{m, m+1, \ldots,\} \end{cases}$
- **Traffic intensity/ergodicity condition**
  $\rho = \frac{\lambda}{m\mu} = \frac{\frac{1}{2}}{4 \times \frac{1}{4}} = 0.5 < +\infty$
- **Performance measure (in the long-run)**
  L_s = number of tankers at the port
- **Requested probability**
  An arriving tanker has to wait for an available unloading berth with probability

$$\begin{aligned} P(L_s \geq m) &= C(m, m\rho) \\ &= \frac{\frac{(m\rho)^m}{m!(1-\rho)}}{\sum_{j=0}^{m-1} \frac{(m\rho)^j}{j!} + \frac{(m\rho)^m}{m!(1-\rho)}} \\ &\overset{m=4,\lambda=0.5,\mu=0.25,\rho=0.5}{=} \frac{\frac{(4\times 0.5)^4}{4!(1-0.5)}}{\sum_{j=0}^{4-1} \frac{(4\times 0.5)^j}{j!} + \frac{(4\times 0.5)^4}{4!(1-0.5)}} \\ &= \frac{\frac{4}{3}}{1 + 2 + 2 + \frac{4}{3} + \frac{4}{3}} \end{aligned}$$

$$\begin{aligned} P(L_s \geq m) &= \frac{\frac{4}{3}}{1 + 2 + 2 + \frac{4}{3} + \frac{4}{3}} \\ &= \frac{4}{23} \\ &\simeq 0.173913. \end{aligned}$$

(b) On average, how many tankers are at the port and how long does a tanker spend at the port? (1.5)

- **Performance measures (in the long-run)**
  L_s = number of tankers at the port
  W_s = time (in hours) spent by an arriving tanker at the port
- **Requested expected value**

$$\begin{aligned} E(L_s) &= m\rho + \frac{\rho}{1-\rho} C(m, m\rho) \\ &\overset{(a)}{\simeq} 4 \times 0.5 + \frac{0.5}{1-0.5} \times 0.173913 \\ &= 2.173913 \quad \text{(tankers)} \end{aligned}$$

$$\begin{aligned} E(W_s) &= \frac{1}{\mu} + \frac{C(m, m\rho)}{m\mu(1-\rho)} \\ &\simeq \frac{1}{0.25} + \frac{0.173913}{4 \times 0.25 \times (1-0.5)} \\ &= 4.347826 \quad \text{(hours)}. \end{aligned}$$

[Alternatively, by invoking Little's law, $E(L_s) = \lambda E(W_s) \Leftrightarrow E(W_s) = \frac{E(L_s)}{\lambda} \simeq \frac{2.173913}{0.5} = 4.347826$.]

(c) The port management is considering building another unloading berth if tankers spend more than 4 hours at the port with a probability larger than 0.5. Is this the case? (1.5)

- **Performance measure (in the long-run)**
  W_s = time (in hours) an arriving tanker spends at the port
- **Requested probability**
  Since $\rho = 0.5 \neq \frac{m-1}{m} = 0.75$

$$\begin{aligned} P(W_s > t) &\overset{form}{=} \left[1 + \frac{e^{\mu[1-m(1-\rho)]t}}{1 - m(1-\rho)} \times C(m, m\rho)\right] e^{-\mu t} \\ &\overset{t=4,m=4,\lambda=0.5,\mu=0.25,\rho=0.5,etc.}{\simeq} \left[1 + \frac{e^{0.25\times[1-4\times(1-0.5)]\times 4}}{1 - 4\times(1-0.5)} \times 0.173913\right] e^{-0.25\times 4} \\ &\simeq 0.344343. \end{aligned}$$

- **Comment**
  Since $P(W_s > 4) \simeq 0.344343 \not> 0.5$ there is no need to build an additional unloading berth.