## P1

$$|\lambda I - A| = \begin{vmatrix} \lambda & -1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$$

$\lambda_1 = 1, \quad \lambda_2 = -1$

Eigenvectors

$v^i_1 = 1$

$\lambda_i v^i_1 - v^i_2 = 0 \longrightarrow v^i_2 = \lambda_i$

$\lambda_1 = 1 \qquad v^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2 = -1 \qquad v^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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## P2 Modal decomposition

$$\begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + K_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

$$K_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + K_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{cases} K_1 + K_2 = 1 \longrightarrow \\ K_1 - K_2 = 0 \end{cases}$$

$2K_1 = 1 \longrightarrow K_1 = \frac{1}{2}, \quad K_2 = \frac{1}{2}$

$$x^1(t) = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} \\ e^t - e^{-t} \end{bmatrix}$$

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P3

$$\begin{aligned} K_1 + K_2 &= 0 \\ K_1 - K_2 &= 1 \end{aligned}$$

$$2K_1 = 1 \longrightarrow K_1 = \frac{1}{2}$$

$$K_2 = -K_1 \longrightarrow K_2 = -\frac{1}{2}$$

$$x^2(t) = \frac{1}{2}\begin{bmatrix} e^t - e^{-t} \\ e^t + e^{-t} \end{bmatrix}$$

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P4

$$e^{At} = \left[ x^1(t) \,\vdots\, x^2(t) \right]$$

$$e^{At} = \frac{1}{2}\begin{bmatrix} e^t + e^{-t} & e^t - e^{-t} \\ e^t - e^{-t} & e^t + e^{-t} \end{bmatrix}$$

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P5

$$\det(e^{At}) = \frac{1}{4}\left[ (e^t + e^{-t})^2 - (e^t - e^{-t})^2 \right] =$$

$$= \frac{1}{4}\left(e^t + \cancel{e^{-t}} + e^t - \cancel{e^{-t}}\right)\left(\cancel{e^t} + e^{-t} - \cancel{e^t} + e^{-t}\right)$$

$$= e^t e^{-t} = 1 \neq 0 \Rightarrow \text{inVertible}$$

P6

$$x(2) = e^{A(2-1)} x(1) = e^{A} x(1)$$

$$x(2) = \frac{1}{2}\begin{bmatrix} e + \frac{1}{e} & e - \frac{1}{e} \\ e - \frac{1}{e} & e + \frac{1}{e} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x(2) = \frac{1}{2}\begin{bmatrix} e + \cancel{\frac{1}{e}} + e - \cancel{\frac{1}{e}} \\ e - \cancel{\frac{1}{e}} + e + \cancel{\frac{1}{e}} \end{bmatrix} = e\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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P7 $\mathcal{C} = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} c \\ cA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(A, b)$ is controllable
$(A, c)$ is observable

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P8 $\alpha_c(s) = (s+2)(s+3) = s^2 + 5s + 6$

$$A - bk = \begin{bmatrix} -k_1 & 1 + k_2 \\ 1 & 0 \end{bmatrix}$$

$$|\lambda - A + bk| = \begin{vmatrix} \lambda + k_1 & k_2 - 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + k_1\lambda + k_2 - 1$$

$k_1 = 5$
$k_2 - 1 = 6$ $\rightarrow$ $k_1 = 5, \quad k_2 = 7$

P9 $\alpha_0(s) = (s+5)^2 = s^2 + 10s + 25$

$$A - LC = \begin{bmatrix} -L_1 & 1 \\ 1 - L_2 & 0 \end{bmatrix}$$

$$|\lambda I - A + LC| = \begin{vmatrix} \lambda + L_1 & -1 \\ L_2 - 1 & \lambda \end{vmatrix} =$$

$$= \lambda^2 + L_1 \lambda + L_2 - 1$$

$L_1 = 10 \qquad L_2 = 26.$

P10 Controller

$$\dot{\hat{x}} = (A - bK - LC)\hat{x} + Ly$$

$$u = -K\hat{x}$$

$$\Phi = A - bK - LC = \begin{bmatrix} -K_1 - L_1 & 1 - K_2 \\ 1 - L_2 & 0 \end{bmatrix}$$

Denominator of the controller transfer function

$$\det(sI - \Phi) = s^2 + (K_1 + L_2)s + (1 - K_2)(L_2 - 1)$$

P11 Equilibria

$$\begin{cases} \bar{x}_1 = \bar{x}_2 \\ \bar{x}_1 \bar{x}_2 - 4 = 0 \end{cases} \rightarrow x^A = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad x^B = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Jacobian matrix

$$\frac{\partial f}{\partial x} = \begin{bmatrix} x_2 & x_1 \\ 1 & -1 \end{bmatrix}$$

Linearization about $x^A$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^A} = \begin{bmatrix} 2 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\begin{vmatrix} \lambda-2 & -2 \\ -1 & \lambda-1 \end{vmatrix} = (\lambda-2)(\lambda+1)-2 =$$

$$= \lambda^2 - \lambda - 4$$

$\mathbb{C}$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1+16}}{2}$$

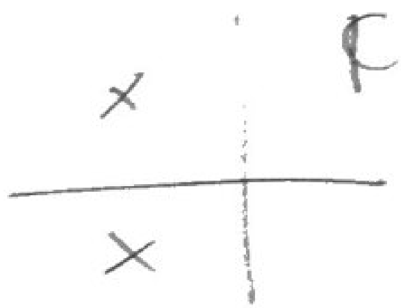
$\lambda_1 = 2{,}56 \qquad \lambda_2 = -1{,}56$

Linearization about $x^B$

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^B} = \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$\lambda^2 + 3\lambda + 4$$

$\mathbb{C}$



$$\lambda_{1,2} = -\frac{3}{2} \pm j\frac{\sqrt{7}}{2}$$

P12

$$\dot V = x\dot x + \tilde a \dot{\tilde a} =$$

$$= x(ax - kx) + \tilde a \dot{\tilde a} =$$

$$= x(ax - \hat a x + a_m x) + \tilde a \dot{\tilde a} =$$

$$= a_m x^2 - \tilde a x^2 + \tilde a \dot{\tilde a} =$$

$$= a_m x^2 + \tilde a(\dot{\tilde a} - x^2)$$

$$\dot{\tilde a} = x^2 \qquad \tilde a = \hat a - a$$

$$\dot{\hat a} = x^2$$

$$\hat a(t) = \hat a(0) + \int_0^t x^2(\tau)\, d\tau$$

$$\dot V = a_m x^2 \leq 0$$

7.13 $f = x + u, \quad f_x = 1$

$$L = -\frac{1}{2}\left[(r-x)^2 + \rho u^2\right], \quad L_x = r - x$$

$$\dot{\lambda} = -\lambda - r + x$$

$$H = \lambda(x+u) - \frac{1}{2}\left[(r-x)^2 + \rho u^2\right]$$

$$\frac{\partial H}{\partial u} = \lambda - \rho u = 0 \rightarrow u = \frac{1}{\rho}\lambda$$

Assume $\quad p, g$ const.

$$\lambda = -px + g$$

$$u = -\frac{p}{\rho}x + \frac{1}{\rho}g$$

$$\dot{\lambda} = -p\dot{x}$$

$$-\lambda - r + x = -p(x+u)$$

$$\left[\frac{p^2}{\rho} - 2p - 1\right]x = \left(\frac{p}{\rho} - 1\right)g - r$$

$$p^2 - 2p\rho - \rho = 0 \quad ; \quad g = \frac{\rho}{p-\rho}$$

$$u^*(t) = -\frac{p}{\rho}x(t) + \frac{1}{p-\rho}r$$

$$p = \rho + \sqrt{\rho^2 + \rho}$$

closed-loop

$$\dot{x} = -\sqrt{1+\frac{1}{\rho}}\,x + \frac{1}{\sqrt{1+\frac{1}{\rho}}}$$

Equilibrium

$$\bar{x} = \frac{1}{1+\rho} x$$

Relative error

$$\frac{x - \bar{x}}{x} = 1 - \frac{1}{x}\bar{x} =$$

$$= 1 - \frac{1}{1+\rho} = \frac{\rho}{1+\rho} \leq \frac{M}{100}$$

$$\rho \leq \frac{M/100}{1 - M/100}$$

$$M = 5\% \rightarrow \rho \leq \frac{5}{95} \simeq 0{,}05263$$

$$\Rightarrow M \leq 5\%$$

$$\Rightarrow \left\{ \rho \leq 0{,}052 \Rightarrow M \leq 5\% \right\}$$