FICHA 10 - SOLUÇÕES

AULA PRÁTICA

1. a) $P\left(\cos^2 x\right) = P\left(\dfrac{\cos(2x)+1}{2}\right) = \dfrac{1}{4}\operatorname{sen}(2x) + \dfrac{x}{2}$; OU por partes, com $u' = v = \cos x$.

   b) $P\left(\operatorname{sen}^3 x\right) = P(\operatorname{sen} x(1-\cos^2 x)) = -\cos x + \dfrac{1}{3}\cos^3 x$; OU por partes, com $u' = \operatorname{sen} x \Rightarrow u = -\cos x$ e $v = \operatorname{sen}^2 x \Rightarrow v' = 2\operatorname{sen} x \cos x$.

   c) $\frac{1}{3}\operatorname{sen}^3 x - \frac{1}{5}\operatorname{sen}^5 x$, escrevendo $\cos^2 x = 1 - \operatorname{sen}^2 x$; OU por partes, com $u' = \cos x \operatorname{sen}^2 x$, $v = \cos^2 x$.

   d) $\frac{1}{3}\operatorname{ch}^3 x$,

   e) $3\operatorname{sh} x + \operatorname{sh}^3 x$, escrevendo $\operatorname{ch}^2 x = \operatorname{sh}^2 x + 1$; OU por partes, com $u' = \operatorname{ch} x$, $v = \operatorname{ch}^2 x$.

   f) $\dfrac{1}{3}\operatorname{tg}^3 x$,

2. a) $-e^{-x}(x^3+3x^2+6x+6)$, b) $x \operatorname{arcsen} x + \sqrt{1-x^2}$, c) $\frac{2}{3}x^{\frac{3}{2}}\left(\ln x - \frac{2}{3}\right)$
   d) $\dfrac{1}{2}\left(-x + (x^2+1)\operatorname{arctg} x\right)$, e) $x\ln^2 x - 2x\ln x + 2x$, f) $\frac{1}{2}\operatorname{sen}(2x)\ln(\operatorname{tg} x) - x$,
   g) $\dfrac{1}{2}(\operatorname{sh} x \cos x + \operatorname{ch} x \operatorname{sen} x)$, h) $\frac{2}{3}x^{3/2}\operatorname{arctg}\sqrt{x} - \frac{1}{3}(x - \ln(1+x))$,
   i) $\dfrac{x}{2}(\cos(\ln x) + \operatorname{sen}(\ln x))$,

3. Sendo $P\left(\frac{1}{1+x}\right) = \ln(x+1)$, para todo o $x \in]-1,+\infty[$, temos $\psi'(x) = \ln(x+1) + C_1$. A condição $\psi'(0) = 1$ resulta em $C_1 = 1$. Usando primitivação por partes (verifique) temos $P\left(\ln(x+1)+1\right) = (x+1)\ln(x+1)$, ou seja $\psi(x) = (x+1)\ln(x+1) + C_2$. Dado que $\psi(0) = 1$, obtém-se o resultado $\psi(x) = (x+1)\ln(x+1) + 1$.

4. a) $P\left(\dfrac{x}{1+(x-1)^2}\right) = P\left(\dfrac{(x-1)+1}{1+(x-1)^2}\right) = \dfrac{1}{2}\ln(1+(x-1)^2) + \operatorname{arctg}(x-1)$,

   b) $P\left(\dfrac{1}{x^2+2x+2}\right) = P\left(\dfrac{1}{(x+1)^2+1}\right) = \operatorname{arctg}(x+1)$,

   c) $P\left(\dfrac{x+1}{(x+2)^3}\right) = P\left(\dfrac{(x+2)-1}{(x+2)^3}\right) = -\dfrac{1}{x+2} + \dfrac{1}{2(x+2)^2}$.

5. a) $\ln\left|\dfrac{x+2}{x+1}\right| - \dfrac{2}{x+2}$, b) $x + \dfrac{1}{4}\ln\left|\dfrac{x-1}{x+1}\right| - \dfrac{1}{2}\operatorname{arctg} x$,

   c) $\ln(x^2+2x+2) + \ln|x| - \operatorname{arctg}(x+1)$, d) $-\dfrac{2}{x-1} - \dfrac{1}{2(x-1)^2}$,

   e) $-\dfrac{1}{5}\ln|x-2| + \dfrac{1}{10}\ln(x^2+1) + \dfrac{2}{5}\operatorname{arctg}(x)$, f) $\ln|x| - \dfrac{2}{x-1}$,

6. a) O domínio de $\dfrac{x}{(x+1)(x+2)^2}$ é $\mathbb{R}\setminus\{-2,-1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln\left(\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_1, & \text{se } x > -1,\\ \ln\left(-\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_2, & \text{se } -2 < x < -1,\\ \ln\left(\frac{x+2}{x+1}\right) - \frac{2}{x+2} + C_3, & \text{se } x < -2,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

b) O domínio de $\dfrac{x^4}{x^4-1}$ é $\mathbb{R}\setminus\{-1,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} x + \frac{1}{4}\ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2}\operatorname{arctg} x + C_1, & \text{se } x > 1,\\ x + \frac{1}{4}\ln\left(-\frac{x-1}{x+1}\right) - \frac{1}{2}\operatorname{arctg} x + C_2, & \text{se } -1 < x < 1,\\ x + \frac{1}{4}\ln\left(\frac{x-1}{x+1}\right) - \frac{1}{2}\operatorname{arctg} x + C_3, & \text{se } x < -1,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

c) O domínio de $\dfrac{3x^2+2}{x(x^2+2x+2)}$ é $\mathbb{R}\setminus\{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln(x^2+2x+2) + \ln(x) - \operatorname{arctg}(x+1) + C_1, & \text{se } x > 0,\\ \ln(x^2+2x+2) + \ln(-x) - \operatorname{arctg}(x+1) + C_2, & \text{se } x < 0,\end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

As restantes alíneas fazem-se de forma semelhante.

7. a) $P\left(\dfrac{1+\sqrt{x}}{x(4-\sqrt{x})}\right) = 2P\left(\dfrac{1+t}{t(4-t)}\right) = \frac{1}{2}\ln\left|\dfrac{\sqrt{x}}{(4-\sqrt{x})^5}\right|$, fazendo $\sqrt{x} = t \Leftrightarrow x = t^2$, com $x > 0$, $x \neq 16$, e $t > 0$, $t \neq 4$;

b) $P\left(\dfrac{e^{4x}}{e^{2x}+1}\right) = \dfrac{1}{2}P\left(\dfrac{t}{t+1}\right) = \frac{1}{2}e^{2x} - \frac{1}{2}\ln\left(e^{2x}+1\right)$, fazendo $e^{2x} = t \Leftrightarrow x = \frac{1}{2}\ln t$, com $x \in \mathbb{R}$ e $t > 0$;

c) $P\left(\dfrac{2\ln x - 1}{x\ln x(\ln x - 1)^2}\right) = P\left(\dfrac{2t-1}{t(t-1)^2}\right) = \ln\left|\dfrac{\ln x - 1}{\ln x}\right| - \dfrac{1}{\ln x - 1}$, fazendo $\ln x = t \Leftrightarrow x = e^t$, com $x \in \mathbb{R}^+ \setminus \{1, e\}$ e $t \in \mathbb{R}\setminus\{0,1\}$;

d) $P\left(\dfrac{1-\operatorname{tg} x}{1+\operatorname{tg} x}\right) = P\left(\dfrac{1-t}{(1+t)(1+t^2)}\right) = \ln|\cos x| + \ln|\operatorname{tg} x + 1|$, fazendo $\operatorname{tg} x = t \Leftrightarrow x = \operatorname{arctg} t$, com $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, $x \neq -\frac{\pi}{4}$ e $t \neq -1$.
NOTA: Esta substituição é invertível para $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, mas como a função $\operatorname{tg} x$ é periódica de período π, o resultado é válido no domínio de $\operatorname{tg} x$.

8. a) Fazendo a substituição $\sqrt{1+2x} = t \Leftrightarrow x = \frac{1}{2}(t^2-1)$, com $x > -\frac{1}{2}$ e $t > 0$, temos (verifique)

$$P\left(\frac{1}{x\sqrt{1+2x}}\right) = P\left(\frac{2}{t^2-1}\right) = \ln\left|\frac{\sqrt{1+2x}-1}{\sqrt{1+2x}+1}\right|.$$

b) Fazendo a substituição $x = t^6 \Leftrightarrow t = \sqrt[6]{x}$, para $x > 0$, $t > 0$, temos

$$P\left(\frac{1}{\sqrt{x}+\sqrt[3]{x}}\right) = P\left(\frac{6t^5}{t^3+t^2}\right) = P\left(\frac{6t^3}{t+1}\right) = P\left(t^2 - t + 1 - \frac{1}{t+1}\right).$$

Logo,

$$P\left(\frac{1}{\sqrt{x}+\sqrt[3]{x}}\right) = \frac{\sqrt{x}}{3} - \frac{\sqrt[3]{x}}{2} + \sqrt[6]{x} - \ln(\sqrt[6]{x}+1).$$

c) $P\left(\dfrac{e^{2x}}{(e^{2x}-1)(1+e^x)}\right) = \dfrac{1}{4}\ln\left|\dfrac{e^x-1}{e^x+1}\right| - \dfrac{1}{2(1+e^x)}.$

d) Fazendo a substituição $t = \sqrt{1+e^x} \Leftrightarrow x = \ln(t^2-1)$, para $x \in \mathbb{R}$, $t > 1$, temos

$$P\left(\frac{1}{\sqrt{1+e^x}}\right) = P\left(\frac{1}{t}\,\frac{2t}{t^2-1}\right) = P\left(\frac{2}{(t-1)(t+1)}\right).$$

e assim (verifique)

$$P\left(\frac{1}{\sqrt{1+e^x}}\right) = \ln\left|\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right|.$$

e) Fazendo a substituição $x = \cos t \Leftrightarrow t = \arccos x$, com $t \in]0,\pi[$, $t \neq \pi/2$, temos

$$P\left(\frac{\sqrt{1-x^2}}{x^4}\right) = P\left(-\frac{\operatorname{sen}^2 t}{\cos^4 t}\right) = P\left(-\operatorname{tg}^2 t\,\frac{1}{\cos^2 t}\right) = -\frac{1}{3}\operatorname{tg}^3 t.$$

Logo, notando que se $t = \arccos x$ então de $1+\operatorname{tg}^2 t = \frac{1}{\cos^2 t}$ vem $\operatorname{tg} t = \sqrt{\frac{1}{\cos^2 t} - 1} = \sqrt{\frac{1}{x^2} - 1}$ temos

$$P\left(\frac{\sqrt{1-x^2}}{x^4}\right) = -\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{3/2}.$$

f) $P\left(\dfrac{1}{\sqrt{x(1-x)}}\right) = 2\operatorname{arcsen}\sqrt{x}.$

g) $P\left(\dfrac{1}{2+\operatorname{tg} x}\right) = P\left(\dfrac{1}{(2+t)(1+t^2)}\right) = \dfrac{1}{5}P\left(\dfrac{1}{2+t} + \dfrac{-t+2}{1+t^2}\right) = \dfrac{1}{5}\ln|2+\operatorname{tg} x| - \dfrac{1}{10}\ln(1+\operatorname{tg}^2 x) + \dfrac{2}{5}x.$

h) Fazendo a substituição $\operatorname{sen} x = t \Leftrightarrow x = \operatorname{arcsen} t$, para $x \in]-\frac{\pi}{2},\frac{\pi}{2}[$, $x \neq 0$, obtém-se (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x\cos x}\right) = P\left(\frac{\cos x}{\operatorname{sen}^2 x(1-\operatorname{sen}^2 x)}\right) = P\left(\frac{1}{t^2(1-t^2)}\right).$$

Logo (verifique)

$$P\left(\frac{1}{\operatorname{sen}^2 x\cos x}\right) = -\frac{1}{\operatorname{sen} x} + \frac{1}{2}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|.$$

SUPLEMENTARES

1. a) $P\left(\operatorname{sen}^3 x \cos^4 x\right) = P\left(\operatorname{sen} x(1-\cos^2 x)\cos^4 x\right) = P\left(\operatorname{sen} x(\cos^4 x - \cos^6 x)\right) =$
$= -\dfrac{1}{5}\cos^5 x + \dfrac{1}{7}\cos^7 x$; OU por partes, com $u' = \operatorname{sen} x \cos^4 x \Rightarrow u = -\frac{1}{5}\cos^5 x$, $v = \operatorname{sen}^2 x$.

   b) $\dfrac{1}{4}\operatorname{sen}^4 x - \dfrac{1}{6}\operatorname{sen}^6 x$ (como c) e d));

   c) $P(4\cos^2 x \operatorname{sen}^2 x) = P(\operatorname{sen}^2(2x)) = \dfrac{x}{2} - \dfrac{1}{8}\operatorname{sen}(4x)$;

   d) $\frac{1}{2}(\operatorname{sh} x \operatorname{ch} x + x)$, por partes, com $u' = v = \operatorname{ch} x$ e usar $\operatorname{ch}^2 x - \operatorname{sh}^2 x = 1$.

   e) $P(\operatorname{tg}^3 x + \operatorname{tg}^4 x) = P((\sec^2 x - 1)\operatorname{tg} x) + P((\sec^2 x - 1)\operatorname{tg}^2 x) = \dfrac{1}{2}\operatorname{tg}^2 x + \ln|\cos x| + \dfrac{1}{3}\operatorname{tg}^3 x - \operatorname{tg} x + x$;

   f) $\operatorname{tg} x + \dfrac{1}{3}\operatorname{tg}^3 x$, escrevendo $\sec^2 x = 1 + \operatorname{tg}^2 x$, OU por partes, $u' = \sec^2 x \Rightarrow u = \operatorname{tg} x$, $v = \sec^2 x \Rightarrow v' = 2\sec^2 x \operatorname{tg} x$.

2.

   a) $e^x(e^x + x - 1) - e^{2x}/2$, b) $-x\cos x + \operatorname{sen} x$, c) $e^x(\operatorname{sen} x - \cos x)/2$,

   d) $P(x^3 e^{x^2}) = P(x^2 \cdot x e^{x^2}) = (x^2-1)\frac{e^{x^2}}{2}$, e) $(x^2+2)\operatorname{ch} x - 2x \operatorname{sh} x$,

   f) $x \operatorname{arctg} x - \frac{1}{2}\ln(1+x^2)$, g) $\frac{1}{4}(1+x^2)^2 \operatorname{arctg} x - x/4 - x^3/12$,

   h) $x\ln|1/x+1| + \ln|x+1|$, i) $\frac{x^3}{3}\ln^2 x - \frac{2}{9}x^2 \ln x + \frac{2}{27}x^3$,

   j) $x(\ln^3 x - 3\ln^2 x + 6\ln x - 6)$, k) $-\frac{1}{x}\operatorname{sen}\frac{1}{x} - \cos\frac{1}{x}$,

   l) $-\cos x \ln(1+\operatorname{sen} x) + x + \cos x$, m) $-(1-x^2)^{3/2}\operatorname{arcsen} x + x - x^3/3$,

   n) $-\dfrac{\ln x}{1+x} + \ln\left|\dfrac{x}{1+x}\right|$, o) $\dfrac{1}{1+\ln^2 3}3^x(\operatorname{sen} x + \ln 3 \cos x)$,

   p) $\frac{2}{3}x^{3/2}\operatorname{arctg}(1/\sqrt{x}) + \frac{1}{3}(x - \ln(x+1))$,

   q) $P(x^n \ln x) = \frac{1}{n+1}x^{n+1}\ln x - P\left(\frac{1}{n+1}x^{n+1}\frac{1}{x}\right) = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1}$.

3. Use primitivação por partes, com $u' = 1$, $v = \ln^n|x|$. Temos $P(\ln^2|x|) = x\ln^2|x| - 2x\ln|x| + 2x$ em $\mathbb{R}\setminus\{0\}$. Logo

$$F(x) = \begin{cases} x\ln^2|x| - 2x\ln|x| + 2x + C, & \text{se } x > 0 \\ x\ln^2|x| - 2x\ln|x| + 2x - C, & \text{se } x < 0 \end{cases}$$

com $C \in \mathbb{R}$, já que somando constantes C em $x > 0$ e C' em $x < 0$ temos $F(1) = 2 + C = -F(-1) = -(-2 + C')$ logo $C' = -C$.

4.

a) $P\left(\frac{1}{1-x}\right) = -\ln|1-x|$, b) $P\left(\frac{1}{(x-3)^3}\right) = -\frac{1}{2(x-3)^2}$,

c) $P\left(\frac{x+1}{x^2+1}\right) = \frac{1}{2}\ln(x^2+1) + \operatorname{arctg} x$, d) $P\left(\frac{2x+1}{x^2+4}\right) = \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)$,

e) $P\left(\frac{x+1}{a^2+x^2}\right) = \frac{1}{2}\ln(a^2+x^2) + \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right)$,

f) $P\left(\frac{1}{x^2+x+1}\right) = P\left(\frac{1}{\left(x+\frac{1}{2}\right)^2+3/4}\right) = \frac{4}{3}P\left(\frac{1}{\left(\frac{2x+1}{\sqrt{3}}\right)^2+1}\right) = \frac{2\sqrt{3}}{3}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$.

5.

a) $\ln|x| - \ln|x+1| = \ln\left|\frac{x}{x+1}\right|$, b) $\ln|x| - \ln|x-1| - \frac{2}{x-1}$,

c) $-\ln|x| + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right)$, d) $2\ln|x-1| - \ln|x| + \frac{1}{x}$,

e) $\frac{x^4}{4} + \frac{x^2}{2} + \frac{1}{2}\ln|x^2-1|$, f) $\frac{x^2}{2} + \ln|x+1| + \frac{1}{x+1}$,

g) $x + \frac{1}{4}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{2}\operatorname{arctg} x$, h) $\frac{1}{2}\ln(x^2+4) + \operatorname{arctg}\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left|\frac{x-2}{x+2}\right|$,

i) $\ln|x-3| + \ln|x+1| - \frac{1}{x+1}$, j) $\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| - \frac{1}{x+1}$, k) $\ln|x-3| - \frac{1}{x+1}$,

l) $\ln|1+x| - \frac{1}{2}\ln(x^2+2x+2)$, m) $7\ln|x+1| - 2\ln(x^2+x+1) - \frac{8}{\sqrt{3}}\operatorname{arctg}\left(\frac{2x+1}{\sqrt{3}}\right)$.

6. a) O domínio de $\frac{1}{x^2+x}$ é $\mathbb{R}\setminus\{-1,0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x+1) + C_1, & \text{se } x > 0,\\ \ln(-x) - \ln(x+1) + C_2, & \text{se } -1 < x < 0,\\ \ln(-x) - \ln(-x-1) + C_3, & \text{se } x < -1,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

b) O domínio de $\frac{x+1}{x(x-1)^2}$ é $\mathbb{R}\setminus\{0,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} \ln x - \ln(x-1) - \frac{2}{x-1} + C_1, & \text{se } x > 1,\\ \ln x - \ln(-x+1) - \frac{2}{x-1} + C_2, & \text{se } 0 < x < 1,\\ \ln(-x) - \ln(-x+1) - \frac{2}{x-1} + C_3, & \text{se } x < 0,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

c) O domínio de $\frac{x^2+x-4}{x(x^2+4)}$ é $\mathbb{R}\setminus\{0\}$. A forma geral das primitivas desta função é:

$$\begin{cases} -\ln x + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_1, & \text{se } x > 0,\\ -\ln(-x) + \ln(x^2+4) + \frac{1}{2}\operatorname{arctg}\left(\frac{x}{2}\right) + C_2, & \text{se } x < 0,\end{cases}$$

em que C_1, C_2 são constantes reais arbitrárias.

d) O domínio de $\frac{x^2+1}{x^2(x-1)}$ é $\mathbb{R}\setminus\{0,1\}$. A forma geral das primitivas desta função é:

$$\begin{cases} 2\ln(x-1)-\ln x+1/x+C_1, & \text{se } x>1,\\ 2\ln(1-x)-\ln x+1/x+C_2, & \text{se } 0<x<1,\\ 2\ln(1-x)-\ln(-x)+1/x+C_3, & \text{se } x<0,\end{cases}$$

em que C_1, C_2, C_3 são constantes reais arbitrárias.

As restantes alíneas fazem-se de forma semelhante.

7. a) $\frac{1}{2}e^{x^2+2x}+C$, com $C\in\mathbb{R}$.

b)
$$P\left(\frac{x+3}{x^4-x^2}\right)=-\ln|x|+\frac{3}{x}+2\ln|x-1|-\ln|x+1|=\frac{3}{x}+\ln\frac{(x-1)^2}{|x(x+1)|}.$$

A forma geral da primitiva em $]1,+\infty[$ é $G(x)=\frac{3}{x}+\ln\frac{(x-1)^2}{x(x+1)}+K$, com $K\in\mathbb{R}$. Tem-se
$$\lim_{x\to+\infty}G(x)=\lim_{x\to+\infty}\frac{3}{x}+\ln\frac{(x-1)^2}{x(x+1)}+K=\ln(1)+K=K,$$
logo $\lim_{x\to+\infty}G(x)=3\Leftrightarrow K=3$.

8. c) $P\left(\frac{1}{(1+x^2)^2}\right)=\frac{x}{2(1+x^2)}+\frac{1}{2}\operatorname{arctg}x$.

$P\left(\frac{1}{(1+x^2)^3}\right)=\frac{x}{4(1+x^2)^2}+\frac{3x}{8(1+x^2)}+\frac{3}{8}\operatorname{arctg}x$.

9. a) $P\left(\dfrac{1}{\sqrt[3]{x}(1+\sqrt[3]{x^4})}\right)=P\left(\dfrac{3t^2}{1+t^4}\right)=\frac{3}{2}\operatorname{arctg}\sqrt[3]{x^2}$, fazendo $\sqrt[3]{x}=t\Leftrightarrow x=t^3$;

b) $P\left(\dfrac{\sqrt{x-1}}{x}\right)=P\left(\dfrac{2t^2}{t^2+1}\right)=2\sqrt{x-1}-2\operatorname{arctg}\sqrt{x-1}$, fazendo $\sqrt{x-1}=t\Leftrightarrow x=t^2+1$, com $x>1$ e $t>0$;

c) $P\left(\dfrac{1}{1+e^{2x}}\right)=P\left(\dfrac{1}{1+t}\cdot\dfrac{1}{2t}\right)=\frac{1}{2}\ln\left|\dfrac{t}{1+t}\right|=\frac{1}{2}\ln\left|\dfrac{e^{2x}}{1+e^{2x}}\right|$, fazendo $e^{2x}=t\Leftrightarrow x=\frac{1}{2}\ln t$, com $x\in\mathbb{R}$ e $t>0$;

d) $P\left(\dfrac{e^{3x}}{(1+e^{2x})(e^x-1)^2}\right)=P\left(\dfrac{t^2}{(1+t^2)(t-1)^2}\right)=-\dfrac{1}{4}\ln(1+e^{2x})+\dfrac{1}{2}\ln|e^x-1|-\dfrac{1}{2}\dfrac{1}{e^x-1}$, fazendo $e^x=t\Leftrightarrow x=\ln t$, com $x\in\mathbb{R}\setminus\{0\}$ e $t>0$, $t\neq1$;

e) $P\left(\dfrac{\ln x}{x(\ln x-1)^2},\right)=P\left(\dfrac{t}{(t-1)^2},\right)=\ln|\ln x-1|-\frac{1}{\ln x-1}$, fazendo $\ln x=t\Leftrightarrow x=e^t$, com $x\in\mathbb{R}^+\setminus\{e\}$ e $t\in\mathbb{R}\setminus\{1\}$.

10. a) Fazendo a substituição $\sqrt{x}=t\Leftrightarrow x=t^2$, com $x>0$ e $t>0$, temos (verifique)
$$P\left(\frac{5}{2(x+1)(\sqrt{x}+2)}\right)=P\left(\frac{5t}{(t^2+1)(t+2)}\right)=\ln(x+1)+\operatorname{arctg}\sqrt{x}-2\ln(\sqrt{x}+1).$$

b) $P\left(\dfrac{1}{(2-x)\sqrt{1-x}}\right) = -2\operatorname{arctg}\sqrt{1-x}$

c) Fazendo a substituição $\sqrt[4]{1+x} = t \Leftrightarrow x = t^4 - 1$, com $x > -1$ e $t > 0$, temos

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = P\left(\frac{1}{(t^4-1)t}\,4t^3\right) = P\left(\frac{4t^2}{t^4-1}\right) = P\left(\frac{4t^2}{(t-1)(t+1)(t^2+1)}\right)$$

e assim (verifique),

$$P\left(\frac{1}{x\sqrt[4]{1+x}}\right) = \ln\left|\frac{\sqrt[4]{1+x}-1}{\sqrt[4]{1+x}+1}\right| + 2\operatorname{arctg}\sqrt[4]{1+x}.$$

d) $P\left(\dfrac{e^{x/2}}{\sqrt{1-e^x}},\right) = -2\operatorname{arcsen}\sqrt{1-e^x}$ (pode fazer também $t = e^{x/2}$ - mais simples).

e) Fazendo a substituição $\ln x = t \Leftrightarrow x = e^t$, para $x \in \mathbb{R}^+ \setminus \{e^{-2}, e^2\}$, $t \neq \pm 2$, temos

$$P\left(\frac{1}{x(4-\ln^2(x))}\right) = P\left(\frac{1}{e^t(4-t^2)}\,e^t\right) = P\left(\frac{1}{(2-t)(2+t)}\right).$$

e assim (verifique)

$$P\left(\frac{1}{x(4-\ln^2(x))}\right) = \frac{1}{4}\ln\left|\frac{2+\ln x}{2-\ln x}\right|.$$

f) Fazendo a substituição $\ln x = t \Leftrightarrow x = e^t$, para $x \in \mathbb{R}^+ \setminus \{1, e\}$, $t \in \mathbb{R} \setminus \{0, 1\}$, temos

$$P\left(\frac{1}{x\ln x(1-\ln x)}\right) = P\left(\frac{1}{t(1-t)}\right) = \ln\left|\frac{\ln x}{1-\ln x}\right|.$$

g) Fazendo a substituição $t = \operatorname{sen} x \Leftrightarrow x = \operatorname{arcsen} t$, com $x \in]-\frac{\pi}{2}, \frac{\pi}{2}[$, temos

$$P(\sec x) = P\left(\frac{\cos x}{1-\operatorname{sen}^2 x}\right) = P\left(\frac{1}{1-t^2}\right) = \frac{1}{2}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| = \ln|\sec x + \operatorname{tg} x|.$$

h) $P\left(\sec^3 x\right) = \dfrac{1}{4}\ln\left|\dfrac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right| + \dfrac{1}{4(1-\operatorname{sen} x)} - \dfrac{1}{4(1+\operatorname{sen} x)} = \dfrac{1}{2}\ln\left|\dfrac{1+\operatorname{sen} x}{\cos x}\right| + \dfrac{\operatorname{sen} x}{2\cos^2 x} =$
$\dfrac{1}{2}\ln|\sec x + \operatorname{tg} x| + \dfrac{1}{2}\sec x \operatorname{tg} x.$

i) $P\left(\dfrac{\cos x}{1+\operatorname{sen} x - \cos^2 x}\right) = P\left(\dfrac{\cos x}{\operatorname{sen} x + \operatorname{sen}^2 x}\right) = \ln\left|\dfrac{\operatorname{sen} x}{1+\operatorname{sen} x}\right|.$

j) Temos

$$P\left(\frac{\operatorname{sen} x}{\operatorname{sen}^2(x) + 3(\cos x - 1)}\right) = P\left(\frac{\operatorname{sen} x}{1-\cos^2(x) + 3(\cos x - 1)}\right) = P\left(\frac{-\operatorname{sen} x}{\cos^2(x) - 3\cos x + 2}\right)$$

Fazendo a substituição $t = \cos x \Leftrightarrow x = \arccos t$, com $x \in]0, \pi[$, temos

$$P\left(\frac{-\operatorname{sen} x}{\cos^2(x) - 3\cos x + 2}\right) = P\left(\frac{1}{t^2 - 3t + 2}\right)$$

e assim (verifique)

$$P\left(\frac{\operatorname{sen} x}{\operatorname{sen}^2(x)+3(\cos x-1)}\right)=\ln\left|\frac{\cos x-2}{\cos x-1}\right|.$$

k) Fazendo a substituição $t=\cos x \Leftrightarrow x=\arccos t$, com $x\in]0,\pi[$, temos

$$P\left(\frac{1}{\operatorname{sen} x(1+\cos x)}\right)=P\left(\frac{-\operatorname{sen} x}{(\cos x-1)(1+\cos x)^2}\right)=P\left(\frac{1}{(t-1)(1+t)^2}\right)$$

e obtém-se (verifique)

$$P\left(\frac{1}{\operatorname{sen} x(1+\cos x)}\right)=\frac{1}{4}\ln\left|\frac{\cos x-1}{\cos x+1}\right|+\frac{1}{2(1+\cos x)}.$$

l) Como na alínea anterior, fazendo a substituição $t=\operatorname{sen} x \Leftrightarrow x=\operatorname{arcsen} t$, com $x\in]-\frac{\pi}{2},\frac{\pi}{2}[$, temos

$$P\left(\frac{1}{\cos x(1-\operatorname{sen} x)}\right)=P\left(\frac{\cos x}{(1+\operatorname{sen} x)(1-\operatorname{sen} x)^2}\right)=P\left(\frac{1}{(1+t)(1-t)^2}\right).$$

e assim

$$P\left(\frac{1}{\cos x(1-\operatorname{sen} x)}\right)=\frac{1}{4}\ln\left|\frac{1+\operatorname{sen} x}{1-\operatorname{sen} x}\right|+\frac{1}{2(1-\operatorname{sen} x)}.$$

11. b) $P\left(\dfrac{1}{1+\operatorname{sen} x+\cos x}\right)=\ln\left|\operatorname{tg}\frac{x}{2}\right|-\ln\left|1+\operatorname{tg}\frac{x}{2}\right|,$

$P\left(\dfrac{\operatorname{sen} x}{1-\operatorname{sen} x}\right)=\dfrac{2}{1-\operatorname{tg}(x/2)}-x.$