CEE 2015/2016 – Exam – Solution

P1) a)

$$Y = \frac{1}{s^2+3s+2} U \qquad x_1 = y \quad x_2 = \dot{y}$$

$$s(sY) = -3sY - 2Y + U$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -2x_1 - 3x_2 + u$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $$\begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda^2 + 3\lambda + 2 = 0$$

$$\lambda_1 = -1 \qquad \lambda_2 = -2$$

$$\lambda_i v_1^i - v_2^i = 0 \qquad v_1^i = 1$$

$$v_2^i = \lambda_i$$

$$v^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad v^2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

c) $$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + K_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + K_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{cases} K_1 + K_2 = 1 \\ -K_1 - 2K_2 = 2 \end{cases}$$

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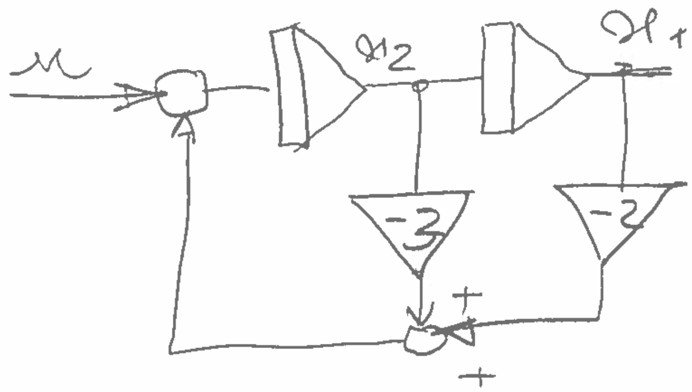
$$-K_2 = 3$$

$K_1 = 4 \quad K_2 = -3$

d)



P2)

a) $M\ddot{y} = -K(y - u)$

$x_1 = y \quad x_2 = \dot{y}$

$\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{K}{M}x_1 + \frac{K}{M}u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K}{M} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) a) $y = [1 \quad 0]$

$\mathcal{O} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ car $\mathcal{O} = 2 \Rightarrow$ observable, yes.

b. $y = \begin{bmatrix} 0 & 1 \end{bmatrix}$

$$\mathcal{O} = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{bmatrix} \quad \text{ran } \mathcal{O} = 2 \Rightarrow \text{observable, yes.}$$

c) Desired observer characteristic polynomial

$$\alpha_0(s) = (s+100)^2 + 100^2 = s^2 + 200s + 2\times 10^4.$$

$$A - LC = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -L_1 & 1 \\ -\frac{K}{M} - L_1 & 0 \end{bmatrix}$$

$$\alpha_L(s) = \det(sI - A + LC) =$$

$$= \begin{vmatrix} s + L_1 & -1 \\ \frac{K}{M} + L_2 & s \end{vmatrix} = s^2 + L_1 s + \frac{K}{M} + L_2$$

$$L_1 = 200$$

$$L_2 = 2\times 10^4 - \frac{K}{M} = 19100.$$

d) Desired controller characteristic polynomial

$$\alpha_c(s) = (s+20)^2 + 400 = s^2 + 40s + 800$$

$$A - bK = \begin{bmatrix} 0 & 1 \\ -\frac{K}{M} & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ \frac{K}{M} \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 \\ -(k_1+1)\frac{K}{M} & -\frac{K}{M}k_2 \end{bmatrix}$$

$$\alpha_K(s) = \det(sI - A + bK) =$$

$$= \begin{vmatrix} s & -1 \\ (k_1+1)\frac{K}{M} & s + \frac{K}{M}k_2 \end{vmatrix} =$$

$$= s^2 + \frac{K}{M}k_2 s + (k_1+1)\frac{K}{M}$$

$$\frac{K}{M}k_2 = 40 \qquad (k_1+1)\frac{K}{M} = 800$$

$$k_2 = \frac{40}{K/M} = \frac{4}{90} = 0{,}044$$

$$k_1 = \frac{800}{900} - 1 = -\frac{1}{9} = -0{,}11$$

e) No. The observer must be faster than the controller.

f) $\dot{\hat{x}} = A\hat{x} + bu + L(y - C\hat{x}) - Lr$

$\dot{\hat{x}} = (A - bK - LC)\hat{x} - Le$

$u = -Kx$

$$A - bK - LC = \begin{bmatrix} -L_1 & 1 \\ -(k_1+1)\frac{k}{H} - L_2 & -\frac{k}{H}k_2 \end{bmatrix}$$ 5/

P3) a)

$x_2 = 0$ and $-0{,}6x_2 - 3x_1 - x_1^2 = 0$

$(-3 - x_1)x_1 = 0$

Equilibrium points:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

b) $$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\substack{x_1=-3 \\ x_2=0}} = \begin{bmatrix} 0 & 1 \\ -3-2x_1 & -0.6 \end{bmatrix}_{\substack{x_1=-3 \\ x_2=0}}$$

$$A = \begin{bmatrix} 0 & 1 \\ 3 & -0{,}6 \end{bmatrix}$$

c) $$\begin{vmatrix} \lambda & -1 \\ -3 & \lambda+0{,}6 \end{vmatrix} = \lambda^2 + 0{,}6\lambda - 6$$

$$\lambda_{1,2} = \frac{-0{,}6 \pm \sqrt{0{,}36 + 24}}{2} = \begin{cases} 2{,}16 \\ -2{,}76 \end{cases}$$

Since there is a positive eigenvalue, the nonlinear system is unstable around $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ (saddle point).

P4) $f = u \quad f_x = u$

$$L = -\frac{1}{2}u^2 \qquad L_x = 0$$

$$-\dot{\lambda} = 0 \Rightarrow \lambda(t) = C$$

Since there is a terminal state constraint there is no terminal condition on λ.

$$H = Cu - \frac{1}{2}u^2$$

$$\frac{\partial H}{\partial u} = 0 \Longrightarrow u^* = C$$

$$x(t) = x(0) + Ct = Ct$$

$$x(T) = CT \Rightarrow C = \frac{P}{T}$$

$$u^*(t) = \frac{P}{T} \quad \forall_{t \in [0,T]}$$

P5)

a) $\dot{x} = -(K+a)x$

$$K + a = 1 \Rightarrow K = 1 - a$$

b) $K = 1 - \hat{a}$

$$\dot{x} = -ax - (1-\hat{a})x$$

$$\dot{x} = -x - \tilde{a}x$$

c) $\dot V = x\dot x + \frac{1}{\gamma}\tilde a \dot{\tilde a} =$

$$= -x^2 + \tilde a \underbrace{\left(\frac{1}{\gamma}\dot{\tilde a} - x^2\right)}_{=0}$$

$$\dot{\tilde a} = \gamma x^2$$

$$\dot{\hat a} = -\dot{\tilde a}$$

$$\dot{\hat a} = -\gamma x^2$$

$$\hat a(t) = \hat a(0) + \gamma \int_0^t x^2(\tau)\, d\tau$$

d) $\dot V = -x^2$

According to the IST all the trajectories will approach the largest invariant set inside the set in which $\dot V = 0$, for which $x^2 = 0$. Hence, all the trajectories will approach $x = 0$.

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