CEE 2015/2016 - Test 2 - Solution

P 1 - a) The time derivative vanishes at an equilibrium point.

Point 1:

$$\left.\frac{dx_1}{dt}\right|_{\begin{bmatrix}0\\0\end{bmatrix}} = 0$$

$$\left.\frac{dx_2}{dt}\right|_{\begin{bmatrix}0\\0\end{bmatrix}} = 9\sin(0) = 0$$

Point 2:

$$\left.\frac{dx_1}{dt}\right|_{\begin{bmatrix}\pi\\0\end{bmatrix}} = 0$$

$$\left.\frac{dx_2}{dt}\right|_{\begin{bmatrix}\pi\\0\end{bmatrix}} = 9\sin(\pi) = 0$$

b) $$\begin{bmatrix}\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2}\\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}0 & 1\\ 9\cos(x_1) & 0\end{bmatrix}$$

Point 1:

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{\begin{bmatrix}0\\0\end{bmatrix}} = \begin{bmatrix}0 & 1\\ 9 & 0\end{bmatrix}$$

$$A_2 = \left.\frac{\partial f}{\partial x}\right|_{\begin{bmatrix}\pi\\0\end{bmatrix}} = \begin{bmatrix}0 & 1\\ -9 & 0\end{bmatrix}$$

c) Eigenvalues of the linearized systems

Point 1: $\begin{vmatrix}\lambda & -1\\ -9 & \lambda\end{vmatrix} = \lambda^2 - 9 = (\lambda+3)(\lambda-3) =$

Eigenvalues; $-3, +3$ (saddle point)

Since one eigenvalue has a positive real part, point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is unstable.

Point 2;

$$\begin{vmatrix} \lambda & -1 \\ 9 & \lambda \end{vmatrix} = \lambda^2 + 9 = 0$$

Eigenvalues: $\pm j3$ (centre)

Since the two eigenvalues are over the imaginary axis, nothing can be said about the nonlinear system.

P2 A)

a) Closed-loop model:

$$\dot{\theta} = \omega$$

$$\dot{\omega} = -g(\omega) - h(\theta)$$

$$V(0,0) = \frac{1}{2}0^2 + \int_0^0 h(\sigma)\, d\sigma = 0$$

$$V(\theta, \omega) = \frac{1}{2}\omega^2 + \int_0^\theta h(\sigma)\, d\sigma$$

>0 for $\omega \neq 0$ because it is a square

>0 for $\theta \neq 0$ because $h(\theta) > 0$ for $\theta > 0$ and $h(\theta) < 0$ for $\theta < 0$.

$$\dot{V} = \omega\dot{\omega} + h(\theta)\dot{\theta} =$$
$$= -\omega g(\omega) - \cancel{\omega h(\theta)} + \cancel{h(\theta)\omega}$$
$$= -\omega g(\omega) \leq 0 \quad \text{for } \begin{bmatrix}\omega\\ \theta\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$$

The standard Lyapunov theorem allows only to conclude that $\begin{bmatrix}0\\0\end{bmatrix}$ is stable at least.

By the set invariance theorem all the trajectories approach the set in which $\dot{V}=0$. Since $\dot{V} = -\omega g(\omega)$ this implies that all the trajectories will approach a set in which $\omega = 0$, and hence that $\dot{\omega} = 0$.

Since

$$\underset{\overset{\shortparallel}{0}}{\dot{\omega}} = -\underbrace{g(\underset{0}{\overset{\shortparallel}{\omega}})}_{=0} - h(\theta)$$

$$\Longrightarrow h(\theta) = 0 \Longrightarrow \theta = 0.$$

$\square$

P3) $f = -0{,}2x + u$ $\qquad f_x = -0{,}2$

$L = x - u$ $\qquad L_x = 1$

$\varphi(x(5)) = 0$

$\varphi_x\big|_{x=x(5)} = 0$

$-\dot{\lambda} = -0{,}2\lambda + 1$

$\dot{\lambda} = 0{,}2\lambda - 1$

$\lambda(5) = 0$

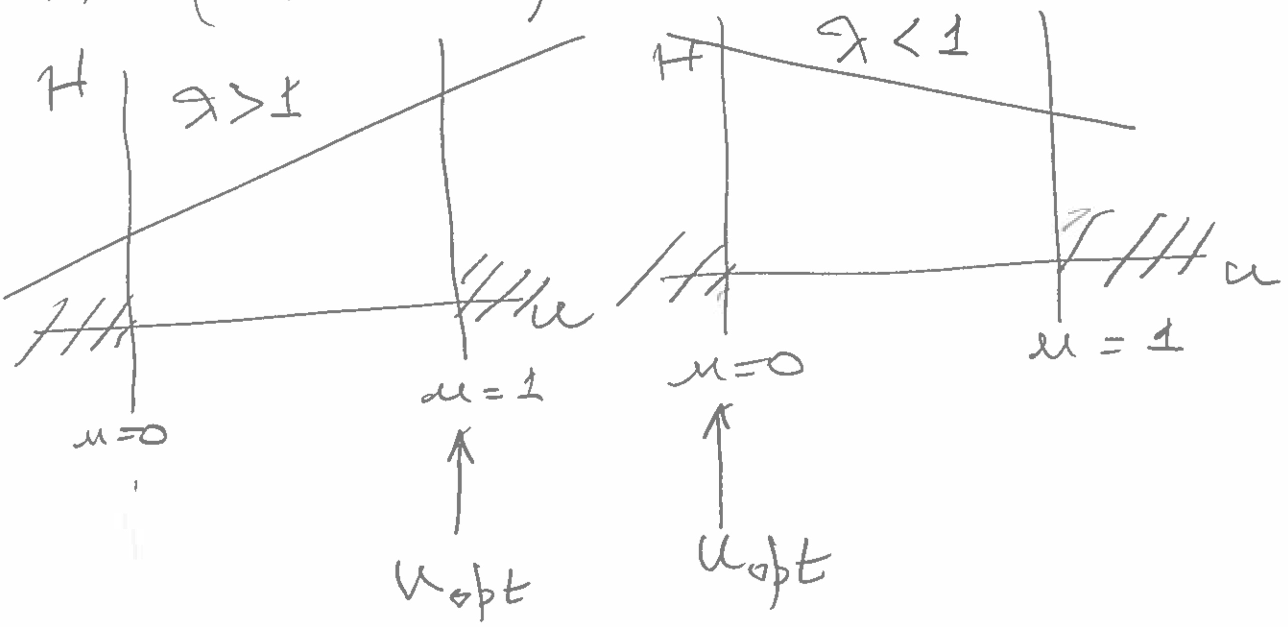
$\lambda(t) = 5 + C e^{0{,}2t}$

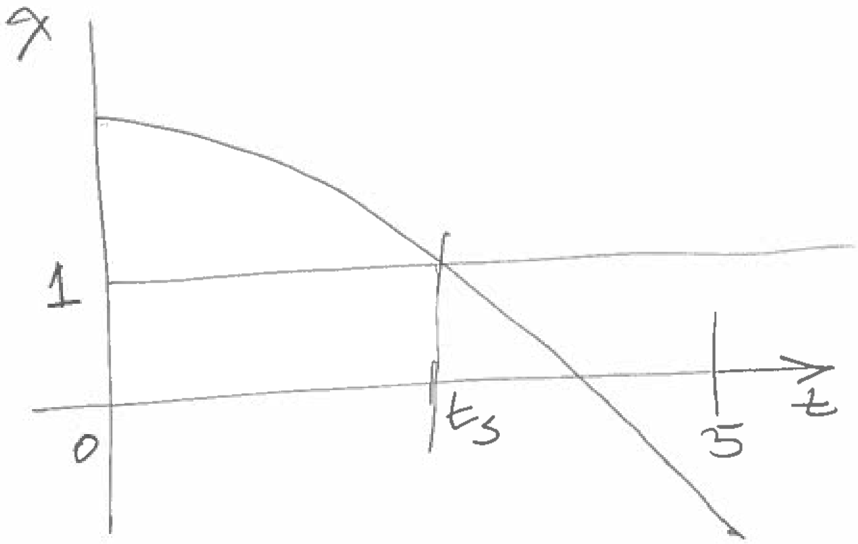
$0 = 5 + C e$ $\qquad C = -\frac{5}{e} \approx -1{,}84$

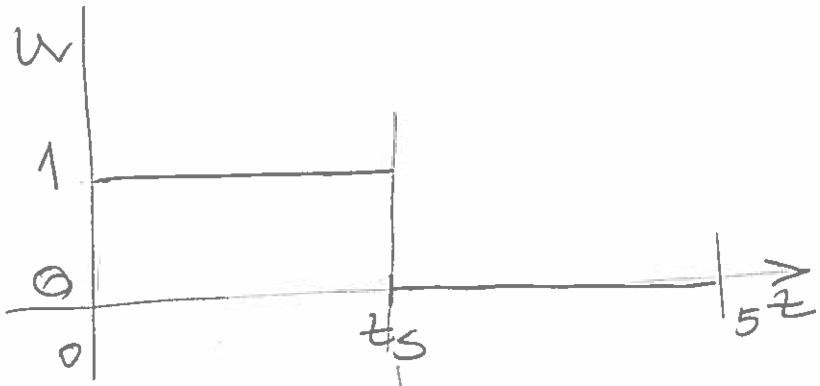
$\lambda(t) = 5 - 1{,}84\, e^{0{,}2t}$

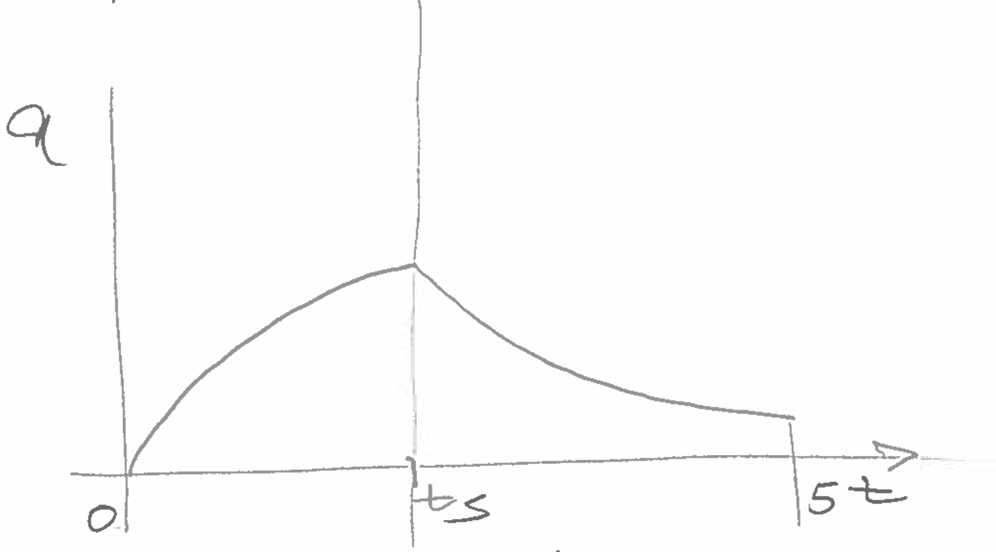
$H = \lambda(-0{,}2x + u) + x - u$

$H = (-0{,}2\lambda + 1)x + (\lambda - 1)u$









$$5 - 1{,}84\, e^{0{,}2\, t_s} = 1$$

$$\frac{4}{1{,}84} = e^{0{,}2\, t_s} \qquad t_s = 5 \log\left(\frac{4}{1{,}84}\right)$$

$$t_s = 3{,}88$$

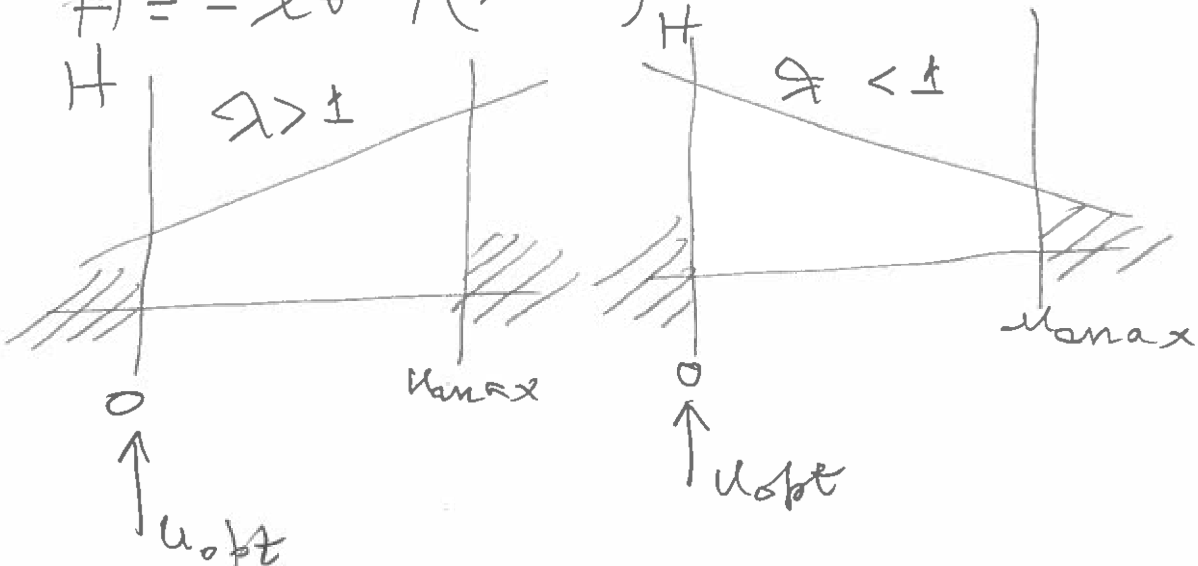
P4.

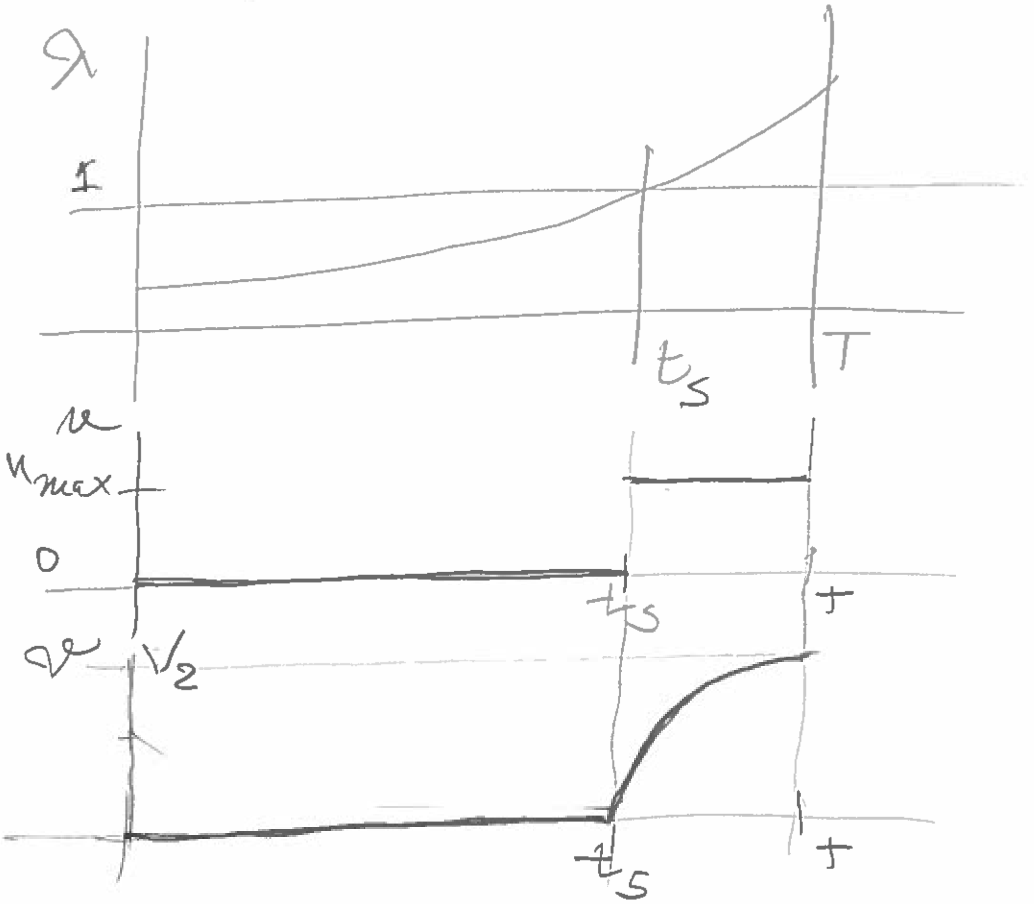
a) $\dot{\lambda} = \lambda$

The terminal condition on λ is unknown since there the final value of the state is constrained ($v(T) = V_2$).

$\lambda(t) = C_1 e^{t-T}$

$H = -\lambda v + (\lambda - 1) u$





From $t=0$ until $t=t_s$ $u=0$

$$\dot{v} = -v \quad v(0)=0$$

$$\Longrightarrow v(t)=0 \quad 0 \leq t \leq t_s$$

From $t=t_s$ up to $t=t=T$

$$\dot{v} = -v + u_{max}$$

$$v(t_s)=0$$

$$v(t) = \left(1 - e^{-(t-t_s)}\right) u_{max}$$

$$V_2 = \left(1 - e^{t_s - t}\right) u_{max}$$

$$t_s = T + \log\left(1 - \frac{V_2}{u_{max}}\right)$$

b) For a given u_{max} and T, the maximum value of V_2 is $u_{max}T$. Hence, the problem is solvable if

$$u_{max}T \geq V_2$$

$\boxtimes$