P1. a) $\dot{x}_1 = \alpha x_2 \leftarrow X_1 = \frac{\alpha}{s} X_2$

$X_2 = \frac{1}{s+1} U \qquad \dot{x}_2 = -x_2 + u$

$y = x_1 + x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \alpha \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) the model is both controllable and observable if there are no zero-pole cancellations.

Transfer function:

$$\left(1 + \frac{\alpha}{s}\right) \frac{1}{s+1} = \frac{s+\alpha}{s(s+1)}$$

there are cancellations for $\alpha = 0$ and $\alpha = 1$.

c) $$\mathcal{O} = \begin{bmatrix} C \\ \hline CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & \alpha - 1 \end{bmatrix}$$

$\det \mathcal{O} = \alpha - 1 = 0$ for $\alpha = 1$

Hence looses observability for $\alpha = 1$

d) $$\mathcal{C} = [b \,\vdots\, Ab] = \begin{bmatrix} 0 & \alpha \\ 1 & -1 \end{bmatrix}$$

$\det \mathcal{C} = -\alpha$

Looses controllability for $\alpha = 0$.

P2 a) $C\frac{dv_c}{dt} = i_L \rightarrow \frac{dv_c}{dt} = \frac{1}{C} i_L$

$u = L\frac{di_L}{dt} + v_c \rightarrow \frac{di_L}{dt} = -\frac{1}{L} v_c + \frac{1}{L} u$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

b) $\lambda I - A = \begin{bmatrix} \lambda & -\frac{1}{C} \\ \frac{1}{L} & \lambda \end{bmatrix}$

Eigenvalues

$\lambda^2 + \frac{1}{LC} = 0 \qquad \lambda_1 = j\frac{1}{\sqrt{LC}} \qquad \lambda_2 = -j\frac{1}{\sqrt{LC}}$

$\lambda_i - \frac{1}{C} v_2^i = 0 \rightarrow v_2^i = \lambda_i C$

$\lambda_1 = j\frac{1}{\sqrt{LC}} \qquad v^1 = \begin{bmatrix} 1 \\ j\sqrt{\frac{C}{L}} \end{bmatrix}$

$\lambda_2 = -j\frac{1}{\sqrt{LC}} \qquad v^2 = \begin{bmatrix} 1 \\ -j\sqrt{\frac{C}{L}} \end{bmatrix}$

c) $G(s) = C(sI - A)^{-1} b$

$(sI - A)^{-1} = \begin{bmatrix} \phi_{11}(s) & \phi_{12}(s) \\ \phi_{21}(s) & \phi_{22}(s) \end{bmatrix}$

$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \phi_{11}(s) & \phi_{12}(s) \\ \phi_{21}(s) & \phi_{22}(s) \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} =$

$$G(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\phi_{12}(s)}{L} \\ \frac{\phi_{22}(s)}{L} \end{bmatrix} = \frac{1}{L}\phi_{12}(s)$$ 3/

We only need to compute $\phi_{12}(s)$!

$$(sI-A)^{-1} = \frac{1}{s^2+\frac{1}{LC}} \begin{bmatrix} s & -\frac{1}{L} \\ \frac{1}{C} & s \end{bmatrix}^T =$$

$$= \frac{1}{s^2+\frac{1}{LC}} \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s \end{bmatrix}$$

$$G(s) = \frac{\frac{1}{LC}}{s^2+\frac{1}{LC}} = \frac{1}{LCs^2+1}$$

The static gain is 1, which is in agreement with the fact that, in steady state, i_L is constant and there is no drop of tension on the inductor, implying that $v_c = u$ (in steady state!).

The poles are the roots of the characteristic polynomial, being given by $\pm j\frac{1}{\sqrt{LC}}$.

d) $\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = K_1 \begin{bmatrix} 1 \\ j \end{bmatrix} e^{jt} + K_2 \begin{bmatrix} 1 \\ -j \end{bmatrix} e^{-jt}$

$K_1 + K_2 = 1$

$jK_1 - jK_2 = 0 \longrightarrow K_2 = K_1$

$K_1 = \frac{1}{2} \qquad K_2 = \frac{1}{2}$

$x_1(t) = \frac{1}{2}\left(e^{jt} + e^{-jt}\right) = \cos t$

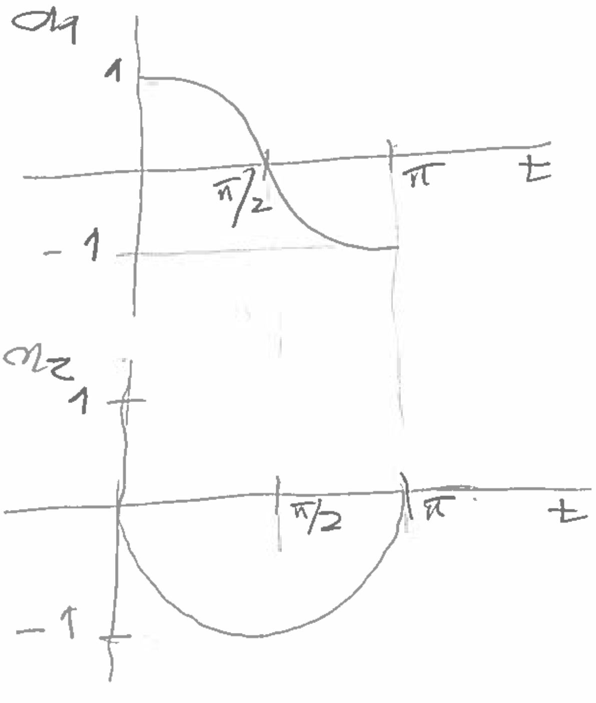
$x_2(t) = \frac{1}{2} j \left(e^{jt} - e^{-jt}\right) =$

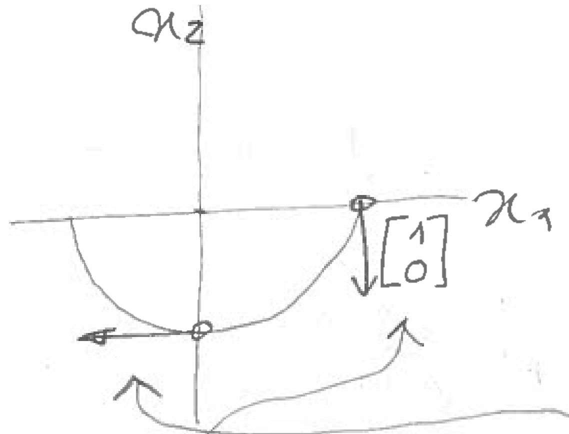
$= \frac{1}{2} j \left(\cancel{\cos t} + j \sin t - \cancel{\cos t} + j \sin t\right)$

$= \sin t$

$x(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$

e) f)





$x = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \qquad Ax = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$x = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad Ax = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

P3

a) $\alpha_c(s) = (s+2)^2 + 4 = s^2 + 4s + 8$

$$A - bk = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} =$$

$$= \begin{bmatrix} 0 & 1 \\ -2+k_1 & -3+k_2 \end{bmatrix}$$

$\det(sI - A + bk) =$

$$= \begin{vmatrix} s & -1 \\ 2-k_1 & s+3-k_2 \end{vmatrix} = s^2 + (3-k_2)s + 2 - k_1$$

$3 - k_2 = 4 \longrightarrow \boxed{k_2 = -1}$

$2 - k_1 = 8 \longrightarrow \boxed{k_1 = -6}$

b) $\alpha_o(s) = (s+10)^2 + 100 = s^2 + 20s + 200$

$$A - LC = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} -L_1 & 1 \\ -2-L_2 & -3 \end{bmatrix}$$

$\det(sI - A + LC) =$

$$= \begin{vmatrix} s+L_1 & -1 \\ 2+L_2 & s+3 \end{vmatrix} =$$

$$= s^2 + (3+L_1)s + 3L_1 + 2 + L_2$$

$3 + L_1 = 20 \longrightarrow \boxed{L_1 = 17}$

$3 \times 17 + 2 + L_2 = 200 \longrightarrow \boxed{L_2 = 147}$

c) No. The observer poles must be faster than the controller poles.

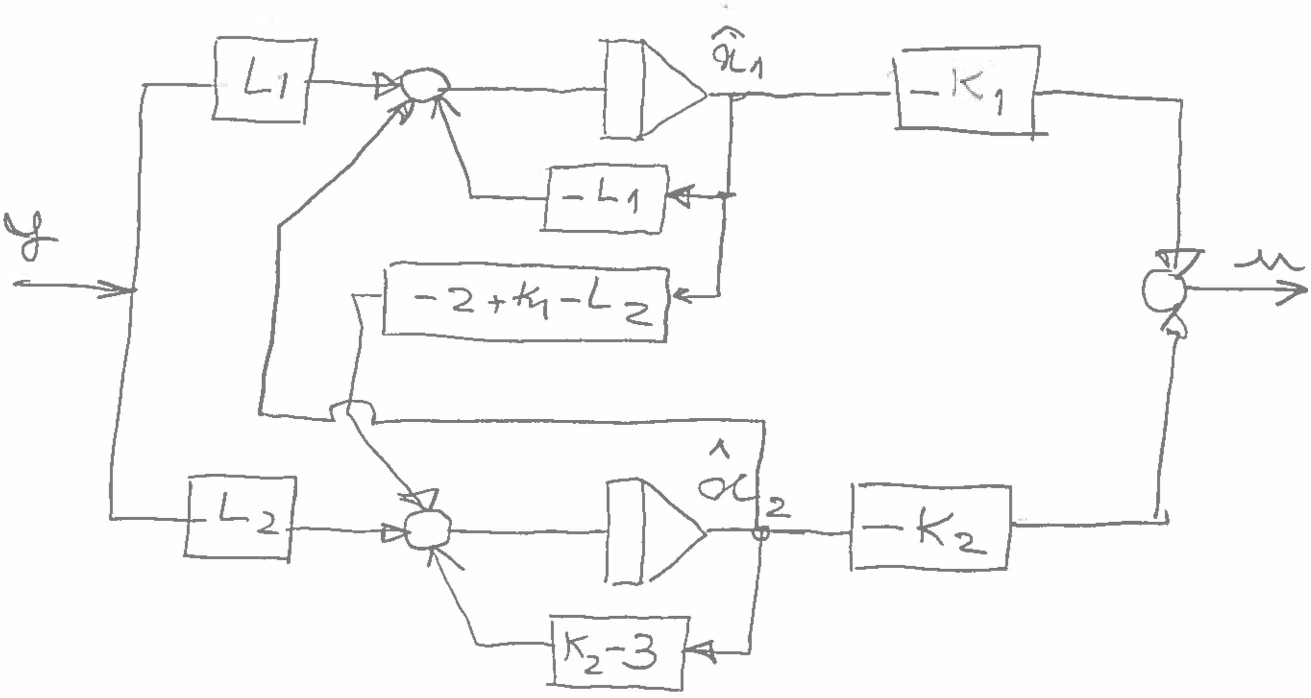
d) $\dot{\hat{x}} = A\hat{x} + bu + L(y - C\hat{x})$

$u = -k\hat{x}$

$\dot{\hat{x}} = (A - bk - LC)\hat{x} + Ly$

$$A - bk - LC = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -L_1 & 1 \\ -2 + k_1 - L_2 & -3 + k_2 \end{bmatrix}$$



P4.

$$u(k) = \sum_{i=0}^{N} u_i(k)$$

$$u_i(k) = u(i)\,\delta(k-i)$$

Let x^i denote the response to $u_i(k)$

$$x^i(k) = 0 \qquad \forall k : k \leq i$$

$$x^i(i+1) = A \underbrace{x^i(i)}_{=0} + b u^i(i) = b u^i(i)$$

$$x^i(N) = A^{N-(i+1)} x^i(i) = A^{N-i-1} b u(i)$$

By the Principle of Superposition

$$x(N) = \sum_{i=1}^{N} x^i(N) = \sum_{i=1}^{N} A^{N-i-1} b u(i)$$