Mestrado Integrado em

Engenharia Electrotécnica e de Computadores

Controlo Em Espaço de Estados

2016/2017

Second Test

31 de Maio de 2017, 18h30 horas – salas V1.23 a 26

Duration 2 hours

Document consultation not allowed

Grades: P1-a)2 b) 3 c) 2 P2-a)2 b)2 c)1 P2A-3 P3-a)3 b)2 P4-3

P1. Consider the nonlineare state model without input

$$\frac{dx_1}{dt} = -x_1 + x_2$$

$$\frac{dx_2}{dt} = x_1 x_2 - 1$$

a) Find **all** the equilibrium points.

b) Obtain the dynamic matrices of the linearized system around each of the equilibrium points.

c) Using the results of b), what can you say about the stability of these equilibrium points?

P2. In this problem you have 2 choices, named A and B. Choice A is more complicated, but it has a higher value (5 points). Alternative B has the smaller value of 3 points. You must indicate your choice. If you answer both choices, only choice A is considered.

A) Consider the block diagram of a feedback servomechanism shown in figure P2-1. In this position control system the input signal u of a direct current motor is applied by a power amplifier, the characteristic of which is described by the nonlinear function f, known, that has as argument the tracking error e.

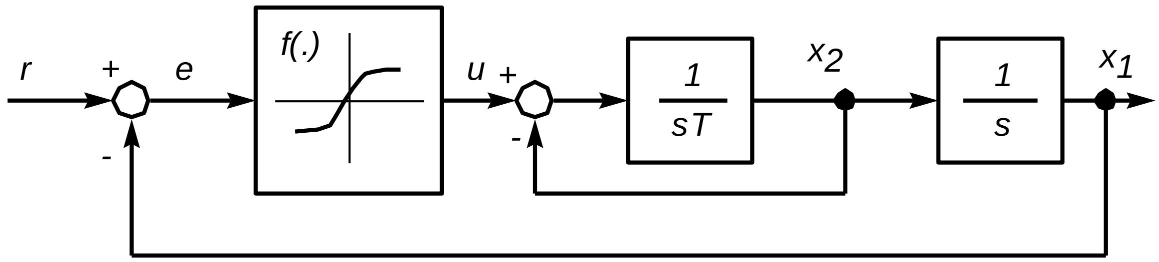

Fig. P2-1 Feedback servomechanism with a nopnlinear actuator.

This function is such that

$$f(e) > 0 \text{ for } e > 0; \quad f(e) = 0 \text{ for } e = 0; \quad f(e) < 0 \text{ for } e < 0$$

These properties mean that

$$\int_0^e f(\sigma)\,d\sigma > 0 \quad \text{for} \quad e \neq 0$$

The reference r is constant in time.

Variables x_1 and x_2 are, respectively, the angular position and the angular velocity of the shaft motor. Parameter $T > 0$ is the motor time constant.

Answer the following questions:

a) Consider the state defined by the tracking e and the angular velocity x_2. Write the corresponding nonlinear state equations. These equations depend on the function f.

b) Show that

$$V(e, x_2) = \frac{T}{2} x_2^2 + \int_0^e f(\sigma)\,d\sigma$$

Is a Lyapunov function for the origin. State the conclusions that you can obtrain wsing the standard Lyapunov theorem.

c) State the conclusions that you can obtain from the invariant set theorem.

B) Consider

$$\frac{dx_1}{dt} = -x_1 - x_1 x_2^2$$

$$\frac{dx_2}{dt} = -x_2 - x_1^2 x_2$$

For this system the origin ($x_1 = 0,\ x_2 = 0$) is an equilibrium point. In relation to this equilibrium consider the Lyapunov function

$$V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2).$$

Question: What can you say about the stability properties of the origin? Show all your calculations.

P3. The novel *Madame de Bovary*, published in 1857 by Gustave Flaubert describes a grave digger that plants potatoes in the part not occupied by tombs. The grave digger had a double income: from the graves that he was opening (for which he was paid, but that they were progressively reducing the space available for potatoes), and from the potatoes that he was planting in the remaining space. If one reads *Madame de Bovary*, the following problem immediately arises: what is the optimal strategy of the grave digger to accept or refuse funerals in order to maximize his profit during the period of time in which he was in function (this problem is not treated in Flaubert's book, which is not a wonder since Pontryagin's Principle was discovered only 100 years later). To solve this problem, consider the fillowing mathematical formulation(this part in the novel):

Let A be the total area of the cemetery. Of this area, an area x is occupied by tombs, and the remnaining area, $A - x$, is occupied by potatoes. ocupada pela plantação de batatas. In order to be able to use a model in the form of a differential equation, it is assumed x is a real variable that may assume any value between 0 and A, and that the rate u with which the grave digger opens the graves may also take any value between 0 and $\bar{u}$. Therefore

$$\frac{dx}{dt} = u \qquad \text{(P3-1)}$$

Sice when the grave digger started his activity there were no graves, and therefore we have the following initial condition

$$x(0) = 0 \qquad \text{(P3-2)}$$

It is assumed that the grave digger is active during a fixed period of T years, and that after this period, there is still room for more graves, meaning that $x(T) > 0$.

The problem consists in finding the rate of work u of the grave digger such as to maximize

$$J = \int_0^T \left(A - x(t) + \rho\, u(t)\right) dt \qquad \text{(P3-3)}$$

wheree $\rho > 0$ is a constant parameter, while satisfying the constraint

$$0 \le u \le \bar{u}$$

Answer the following questions:

a) Using Pontryagin's Principle, find the function $u(t)$ in the interval of time $t \in [0, T]$, that maximizes J given by (P3-3).

b) Sketch the functions $u(t)$ and $x(t)$ for $t \in [0, T]$ when using the optimal control..

P4. After a hard life of work, miss Aulíria has retired and wants to decid a plan of optimal savings and expenditure during an interval of time that starts at the current moment, $t = 0$, and ends in a future time $t = T$.

Miss Aulíria has no other source of income besides her savings. Being $x_1(t)$ the value of savings at time t, this variable satisfies the differential equation

$$\dot{x}_1 = \alpha x_1(t) - u(t), \qquad \text{(P4-1)}$$

With the initial condition $x_1(0) = x_0$, in which $\alpha > 0$ is a constant parameter that represents the rate of valorization of the savings, and $u(t)$ represents the rate of expenditure of miss Aulíria at time t.

At each time t, it is assumed that the rate of expenditure $u(t)$ has an utility (that is to say, the pleasure joyed by miss Aulíria from it) given by $\sqrt{u}$. Since the utility of future expenditures made at time t has less value then the present ones (made at time 0) it is assumed that the utility decays exponentially in time. The functional to maximize is therefore

$$J = \int_0^T e^{-\beta t} \sqrt{u}(t) dt, \qquad \text{(P4-2)}$$

where β is a positive parameter.

Question: Using Pontryagin's maximum principle, find the control law that maximizes the functional J given by (P4-2), subject to the dynamics (P4-1) and the terminal condition

$$x_1(T) = 0. \tag{3}$$

Help: Observe that the lagrangian function in (P4-2) depends on time. The form of Pontryagin's Maximum Principle that you have studied does not include this case. Show that this difficulty can be easily overcome by the introduction of an additional state variable. Define this extra state variable and reformulate the problem in a way that the lagrangian does not depend on t.

Helpful hints

$$\boxed{\begin{array}{l} \dot{x} = -ax + b \quad a, b \text{ constant} \\ \qquad x(t) = \dfrac{b}{a} + Ce^{-at} \end{array}}$$

$$\frac{dx}{dt} = f(x,u) \qquad x(0) = x_0 \qquad J(u) = \Psi(x(T)) + \int_0^T L(x,u)dt$$

$$-\left(\frac{d\lambda}{dt}\right)' = \lambda'(t) f_x(x(t),u(t)) + L_x(x(t),u(t)) \qquad \lambda'(T) = \Psi_x(x(T))$$

$$H(\lambda, x, u) = \lambda' f(x,u) + L(x,u)$$

$$f_x = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} \qquad L_x = \begin{bmatrix} \dfrac{\partial L}{\partial x_1} & \dfrac{\partial L}{\partial x_2} \end{bmatrix} \qquad \Psi_x = \begin{bmatrix} \dfrac{\partial \Psi}{\partial x_1} & \dfrac{\partial \Psi}{\partial x_2} \end{bmatrix}$$

Laplace transforms

$$1 \ \rightarrow \ \frac{1}{s}, \qquad e^{-at} \ \rightarrow \ \frac{1}{s+a}, \qquad 1 - e^{-at} \ \rightarrow \ \frac{a}{s(s+a)}$$