

# 1.State model equations of Continuous linear systems

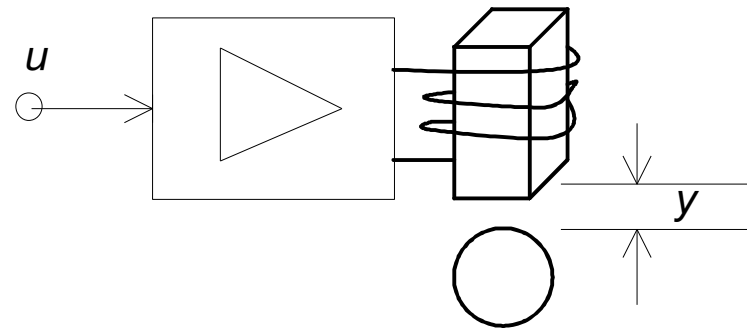
## ***Objective:***

*How to write state model equations for different systems.*

[JML-CEE2019] cap. 1, pp. 27-63



## Example: Magnetic suspension (simplified)



A model using Newton's law:

$$m \frac{d^2 y}{dt^2} = \sum \text{forces applied}$$

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The forces applied are the weight  $P$  and the electromagnetic force made of two parcels, one that compensates the weight and the other equal to the signal. If  $m = 1$  we have the model

$$\frac{d^2 y}{dt^2} = u$$

The model is described by a 2<sup>nd</sup> order ODE (ordinary differential equation).

It is not surprising that the magnetic suspension is described by a 2<sup>nd</sup> order ODE: We need to characterize the position and the velocity.

This suggests that we consider two coupled ODEs, one for each variable.

Take as variables the position

$$x_1(t) = y(t)$$

and the velocity

$$x_2(t) = \dot{y}(t)$$

The system can be described by the system of 2 1<sup>st</sup> order ODEs

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = u \end{cases}$$

The system of 2 1<sup>st</sup> order ODEs

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = u \end{cases}$$

Is equivalent to the 2<sup>nd</sup> order ODE

$$\frac{d^2 y}{dt^2} = u$$

In both cases 2 initial conditions must be specified.

$$\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = u \end{cases}$$

Equivalent matrix form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

This is the state model of the suspension and  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$  is the system state.

[JML-CEE2019] pp. 29, 30

## Standard form of the state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Define

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = 0$$

The state model is written in the standard form

$$\dot{x} = Ax + bu$$

$$y = Cx + Du$$

We will usually consider  $D = 0$  (systems with more poles than zeros; no instantaneous response).

## Definition of state

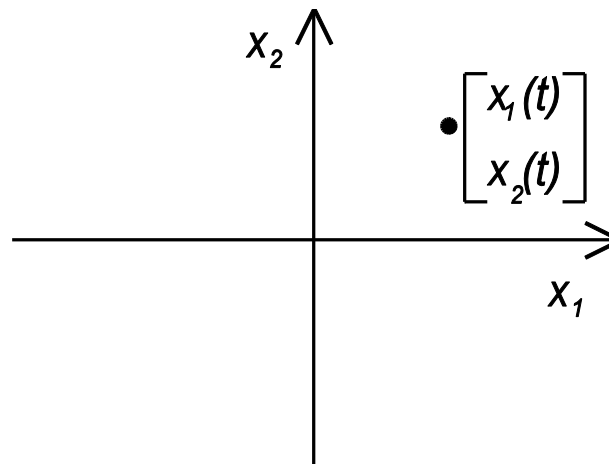
A vector of variables such that, if known at a certain time, and if the future values of the input variables are also known, then it is possible to compute all their future values, by integrating the state equations.

[JML-CEE2019] p. 31



## The state space of the magnetic suspension

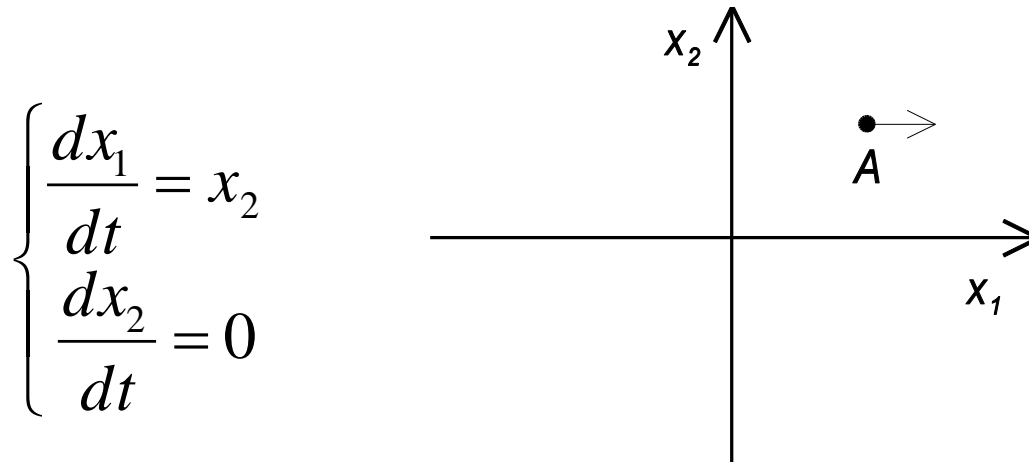
An advantage of the state model is that we can consider the geometric evolution is the space  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .



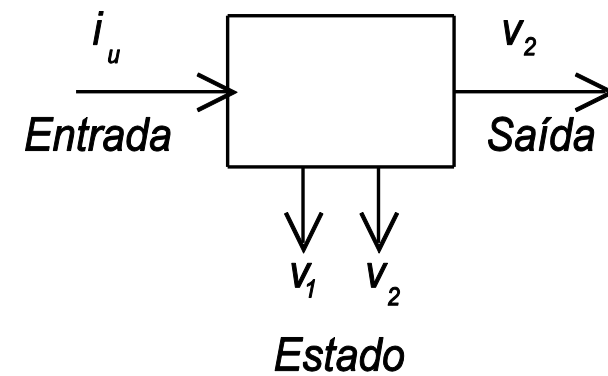
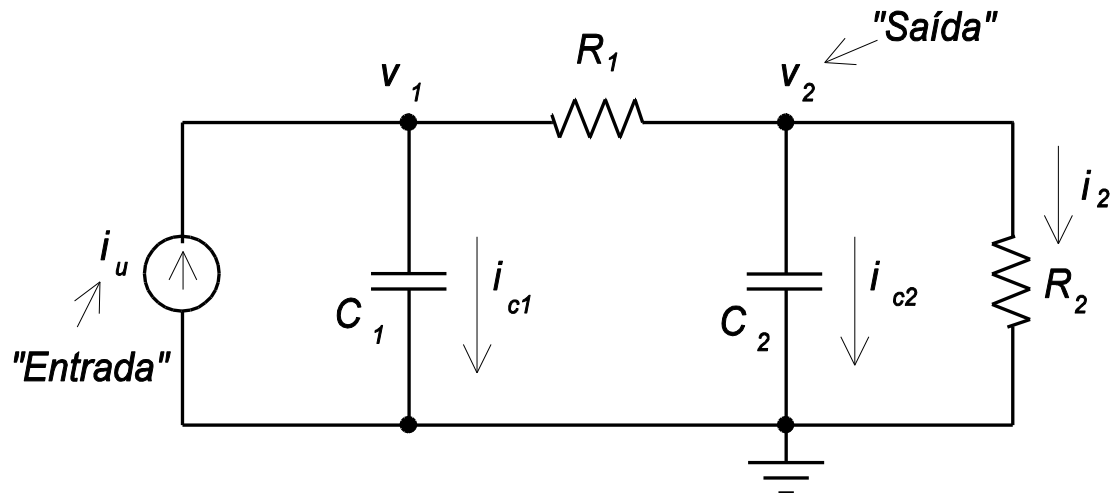
This space is the **state space**.

The evolution of the state variables can be considered as trajectories in the space  $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ .

For instance, if the sphere is initially in state A and  $u = 0$ , when time passes, it will move in the direction of the arrow



## Example: Electrical circuit



$$\begin{cases} C_1 \frac{dv_1}{dt} = i_u - \frac{v_1 - v_2}{R_1} \\ C_2 \frac{dv_2}{dt} = \frac{v_1 - v_2}{R_1} - \frac{v_2}{R_2} \end{cases}$$

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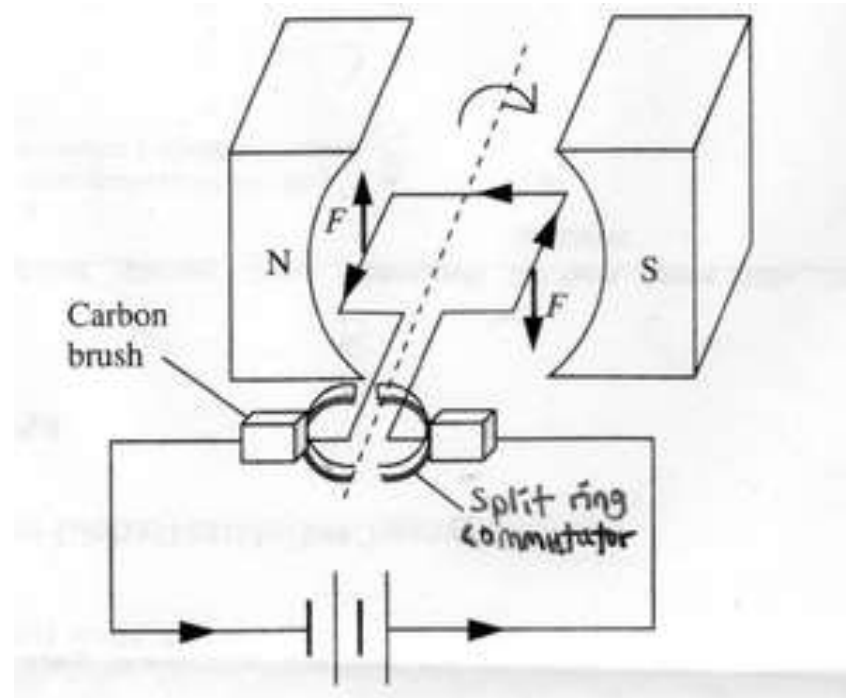
Define

$$A = \begin{bmatrix} -\frac{1}{C_1 R_1} & \frac{1}{C_1 R_1} \\ \frac{1}{C_2 R_1} & -\frac{1}{C_2 R_1} - \frac{1}{C_2 R_2} \end{bmatrix} \quad B = \begin{bmatrix} 1/C_1 \\ 0 \end{bmatrix} \quad C = [0 \quad 1] \quad D = 0 \quad u = i_u \quad x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \quad y = v_2$$

The state model of the circuit is written in the standard form

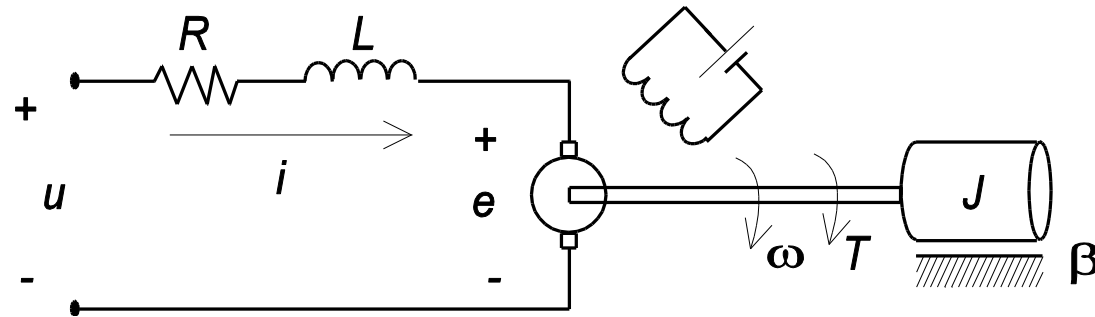
$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

## DC motor – physical structure



[JML-CEE2019] pp. 35, 36

## DC servomotor



Motor torque:

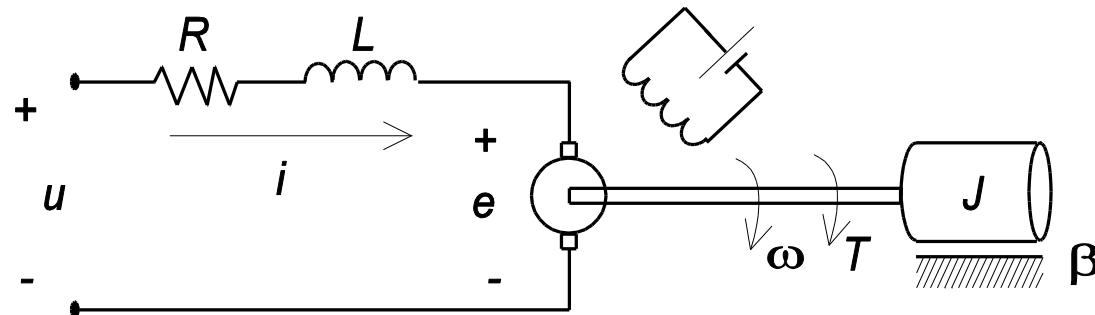
$$T(t) = K' \phi(t) i(t)$$

Assume that the flux  $\phi$  created by the armature circuit is constant,

$$T(t) = Ki(t)$$

Electrical tension at the rotor terminals

$$e = K_b \omega$$



Motor rotor circuit:

$$L \frac{di}{dt} + R i + e = u$$

Shaft rotation:

$$J \frac{d\omega}{dt} = T(t) - \beta \omega$$

## State variables

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \omega \\ i \end{bmatrix}$$

State variables, considering as output the velocity  $\omega$  :

$$\dot{x} = \begin{bmatrix} -\frac{\beta}{J} & \frac{K}{J} \\ -\frac{K_b}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

If we want to model the shaft rotation angle, an extra state is needed.



## State model – general linear case

**State equation** (ODE that relates the input  $u$  with the state  $x$ ):

$$\dot{x}(t) = Ax(t) + Bu(t)$$

State initial condition

$$x(0) = x_0$$

State equation (algebraic eq. That relates the state  $x$  with the output  $y$ ):

$$y(t) = Cx(t) + Du(t)$$

Dimensions:

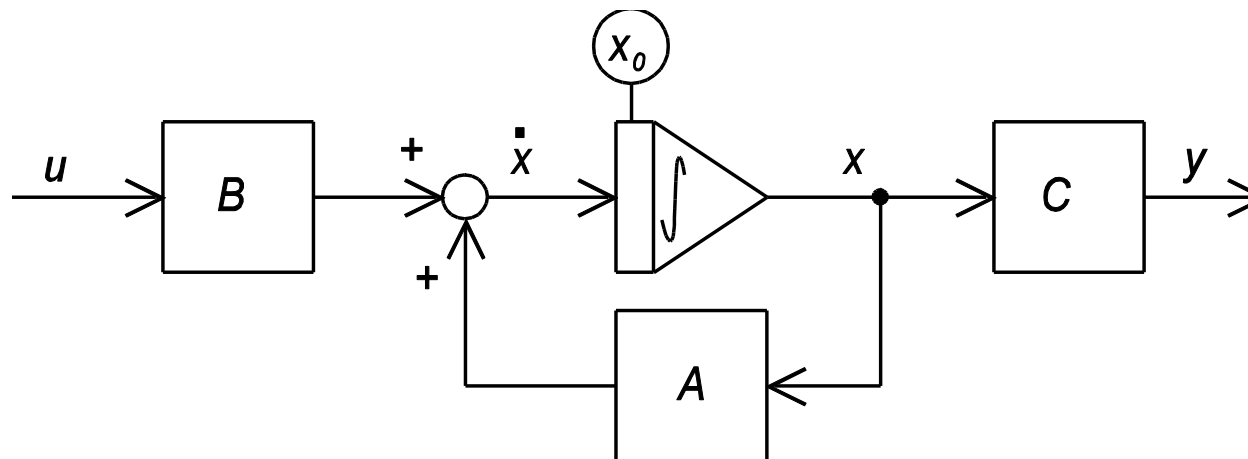
$$x(t) \in R^n, \quad u(t) \in R^m, \quad y(t) \in R^p \quad A[n \times n] \quad B[n \times m] \quad C[p \times n] \quad D[p \times m]$$

Usually we will consider SISO systems  $D = 0, \quad m = 1, \quad p = 1.$

## Bloc diagram of the state model

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t)$$



[JML-CEE2019] pp.38-41

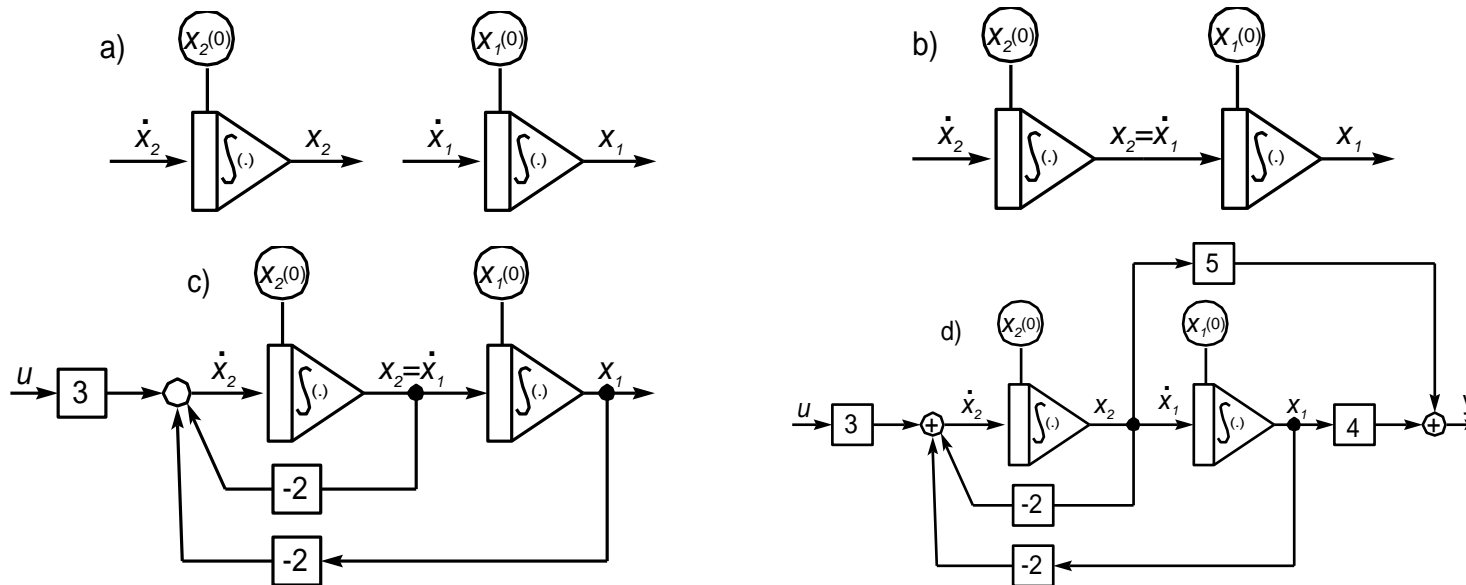
## Block diagrams of the state equations – Example

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 3 \end{bmatrix} u,$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -2x_1 - 2x_2 + 3u$$

$$y = \begin{bmatrix} 4 & 5 \end{bmatrix} x$$



## Selection of the state variables

State variables are a set of variables such that, if the initial conditions and the control variables are known, they can be known at every future instant by integrating the state equations.

Types of state variables

- Physical variables (position, velocity, temperature, ...)
- Defined mathematically (the output and its first  $n$  derivatives).

State variables are not unique and they don't have to be selected in the minimum number.

## State plane

State space for dimension 2.

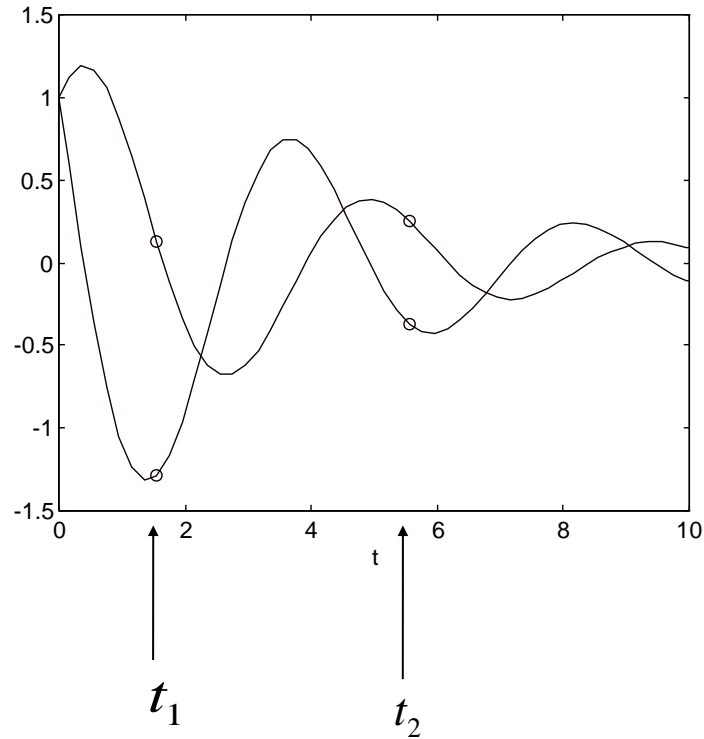
### Example

$$\frac{dx_1}{dt} = x_2$$

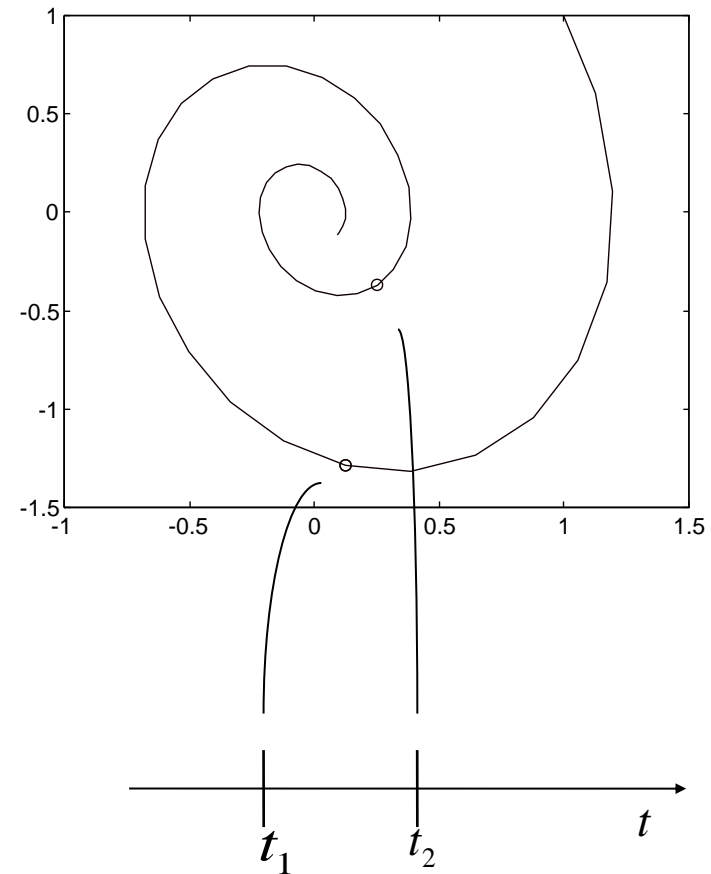
$$\frac{dx_2}{dt} = -2x_1 - 2x_2$$

With initial condition  $x_1(0) = 1$   $x_2(0) = 1$ . The time solution and the corresponding orbit(or trajectory) in the state plane.

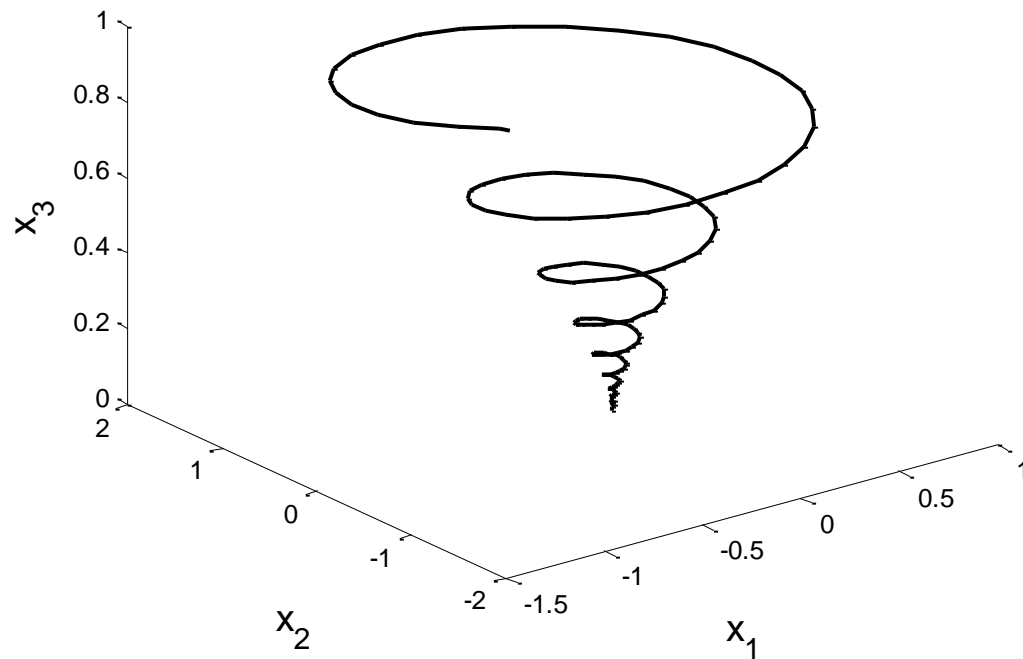
Resposta no tempo



Trajectória correspondente no plano de estado



An example of an orbit for  $n = 3$



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 - 0,3x_2$$

$$\dot{x}_3 = -0,1x_3$$

## 2. Conversion between the state model and the Transfer function

### ***Objective:***

*After studying this module, the student will be able to obtain the matrices of the state model for linear systems given a transfer function and vice-versa.*

[JML-CEE2019] pp. 41-50



## Obtaining the transfer function from the state model

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = Cx(t) \end{cases}$$

Take the Laplace transform with zero initial conditions:

$$\begin{aligned} sX(s) &= AX(s) + bU(s) \\ Y(s) &= CX(s) \end{aligned} \quad X(s) = TL(x) \quad U(s) = TL(u)$$

From which

$$(sI - A)X(s) = bU(s) \quad \rightarrow \quad X(s) = (sI - A)^{-1}bU(s)$$

or

$$Y(s) = C(sI - A)^{-1}b U(s)$$

$$Y(s) = C(sI - A)^{-1}b U(s)$$

The transfer function is given by

$$G(s) = C(sI - A)^{-1}b$$

Since

$$(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)}$$

the transfer function is written

$$G(s) = \frac{C \text{adj}(sI - A)b}{\det(sI - A)}$$

$\det(sI - A)$  is the **characteristic polynomial** of  $A$ .

## A note on linear algebra – adjoint of a matrix

The **adjoint** (or adjunct) of a matrix  $M = \begin{bmatrix} m_{ij} \end{bmatrix}$  is given by

$$\text{adj}(M) = \left[ M_{ij} \right]^T$$

where  $M_{ij}$  is the **cofactor**  $m_{ij}$ , i. e. is the determinant of the matrix obtained by eliminating the row  $i$  and the column  $j$ , multiplied by  $-1^{i+j}$ .

Example: 
$$\text{adj} \left( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

### Adjoint of a matrix – Example

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 0 & 6 & 1 \end{bmatrix} \quad \text{adj}(M) = \begin{bmatrix} 0 & -5 & 30 \\ 16 & 1 & -6 \\ 0 & 15 & -10 \end{bmatrix}^T = \begin{bmatrix} 0 & 16 & 0 \\ -5 & 1 & 15 \\ 30 & -6 & -10 \end{bmatrix}$$

To check the result:

$$M \frac{\text{adj}(M)}{\det(M)} = \frac{1}{80} \begin{bmatrix} 1 & 2 & 3 \\ 5 & 0 & 0 \\ 0 & 6 & 1 \end{bmatrix} \begin{bmatrix} 0 & 16 & 0 \\ -5 & 1 & 15 \\ 30 & -6 & -10 \end{bmatrix} = I_3$$

Reference: G. Strang, *Linear Algebra and its Applications*, 2<sup>a</sup> ed., p 170.

## Poles and zeros

$$G(s) = \frac{C \operatorname{adj}(sI - A)b}{\det(sI - A)}$$

The **poles** are the roots of the **characteristic polynomial** of matrix  $A$ , given by

$$\det(sI - A)$$

The zeros are the roots of the polynomial

$$C \operatorname{adj}(sI - A)b$$

## Transfer function from the state model – Example

$$A = \begin{bmatrix} -5 & -6 \\ 1 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C = [0 \quad 1]$$

$$sI - A = \begin{bmatrix} s+5 & 6 \\ -1 & s \end{bmatrix} \quad (sI - A)^{-1} = \frac{1}{s(s+5)+6} \begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix}$$

$$G(s) = \frac{1}{s(s+5)+6} [0 \quad 1] \begin{bmatrix} s & -6 \\ 1 & s+5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{(s+2)(s+3)}$$

## Obtaining the state model from the TF – Systems without zeros

Given the transfer function with no zeros

$$G(s) = \frac{b_0}{s^3 + a_1s^2 + a_2s + a_3}$$

Find an equivalent state model.

This state model is **not** unique.

**Phase variables:** the output and its first  $n - 1$  derivatives.

In this example  $n = 3$ .

Obtain the differential equation

$$G(s) = \frac{b_0}{s^3 + a_1s^2 + a_2s + a_3}$$

$$s^3Y(s) + a_1s^2Y(s) + a_2sY(s) + a_3Y(s) = b_0U(s)$$

The differential equation is therefore

$$\ddot{y}(t) + a_1\dot{y}(t) + a_2y(t) + a_3y(t) = b_0u(t)$$



$$\ddot{y}(t) + a_1 \dot{y}(t) + a_2 y(t) + a_3 y(t) = b_0 u(t)$$

State variables (the output and its derivatives up to order  $n - 1 = 2$ ):

$$x_1 = y$$

$$x_2 = \dot{y} = \dot{x}_1$$

$$x_3 = \ddot{y} = \dot{x}_2$$

Obtain the equation for  $x_3$  from the ODE and the definition of the state variables

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + b_0 u(t)$$

The state model is

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_1 x_3 - a_2 x_2 - a_3 x_1 + b_0 u(t)$$

Or, using matrix notation

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ b_0 \end{bmatrix} u$$

$$y = [1 \ 0 \ 0]x$$

The matrix of dynamics

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix}$$

Is said in the **companion form** (forma companheira).

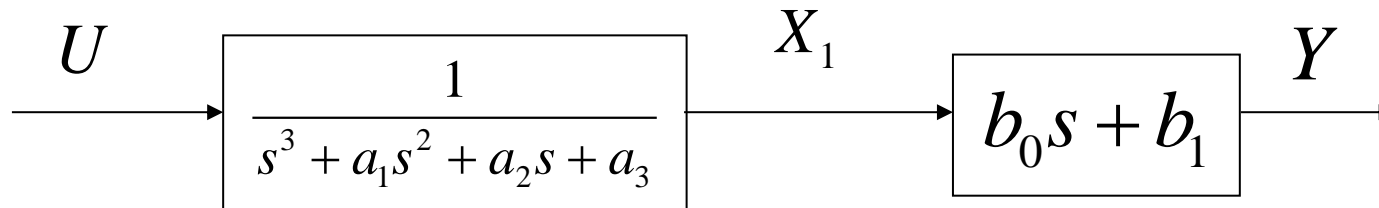
It consists of an identity of order  $n - 1$  in the upper right corner, a first column of zeros of length  $n - 1$  and the last row is the symmetric of the coefficients of the characteristic matrix.

## Systems with zeros

$$G(s) = \frac{b_0s + b_1}{s^3 + a_1s^2 + a_2s + a_3}$$

Caveat: If the previous technique is applied there is a derivative of  $u$

Solution (there are other possibilities: Break the system in zeros and poles. Take as state variables the output of the part only with poles and its derivatives.



The dynamic equation is the same.

The output equation is changed to

$$y = b_0\dot{x}_1 + b_1x_1 = b_0x_2 + b_1x_1$$

$$y = [b_1 \quad b_0 \quad 0]x$$

## 3.Change of coordinates

### ***Objective:***

*Given a state model and a linear transformation of the state variables,  
compute the model equations in the new coordinates.*

[JML-CEE2019] p. 50, 51

## Coordinate transform in the state model

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$y(t) = Cx(t)$$

Coordinate transform

$$z(t) = Tx(t)$$

with  $T$  square and invertible.

*What is the state model verified by  $z(t)$  ?*

Suggestion: Differentiate  $z(t) = Tx(t)$

$$z(t) = Tx(t)$$

Differentiate

$$\dot{z}(t) = T\dot{x}(t)$$

Use the state model for  $x(t)$ :

$$\dot{z}(t) = T(Ax(t) + bu(t))$$

Use the inverse transformation:

$$\dot{z}(t) = TAT^{-1}z(t) + Tbu(t)$$

$$y(t) = Cx(t) = CT^{-1}z(t)$$



## Conclusion: Transformation of coordinates in the state model

Given the state model with equations

$$\dot{x}(t) = Ax(t) + bu(t) \quad y(t) = Cx(t)$$

Perform the linear coordinate change

$$z(t) = Tx(t)$$

where  $T$  is square and invertible (preserve the order).

In the new coordinates, the state equations are

$$\dot{z}(t) = Ez(t) + \Gamma u(t) \quad y(t) = Hz(t)$$

$$E = TAT^{-1} \quad H = CT^{-1}$$

Issue to think: Prove that two different state realizations connected by a linear coordinate change have the same transfer function.

## 4.The homogeneous equation

### **Objective:**

*Present the structure of the solution of the homogeneous equation.*

[JML-CEE2019] pp. 68-81



## The homogeneous equation

The equation

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0$$

Is called **homogeneous equation**.

The solution of the homogeneous equation plays a fundamental role.

The structure of its solution depends on the eigenvalues and the eigenvectors of  $A$ .

## Orbits in state space

Homogeneous equation

$$\dot{x}(t) = Ax(t)$$

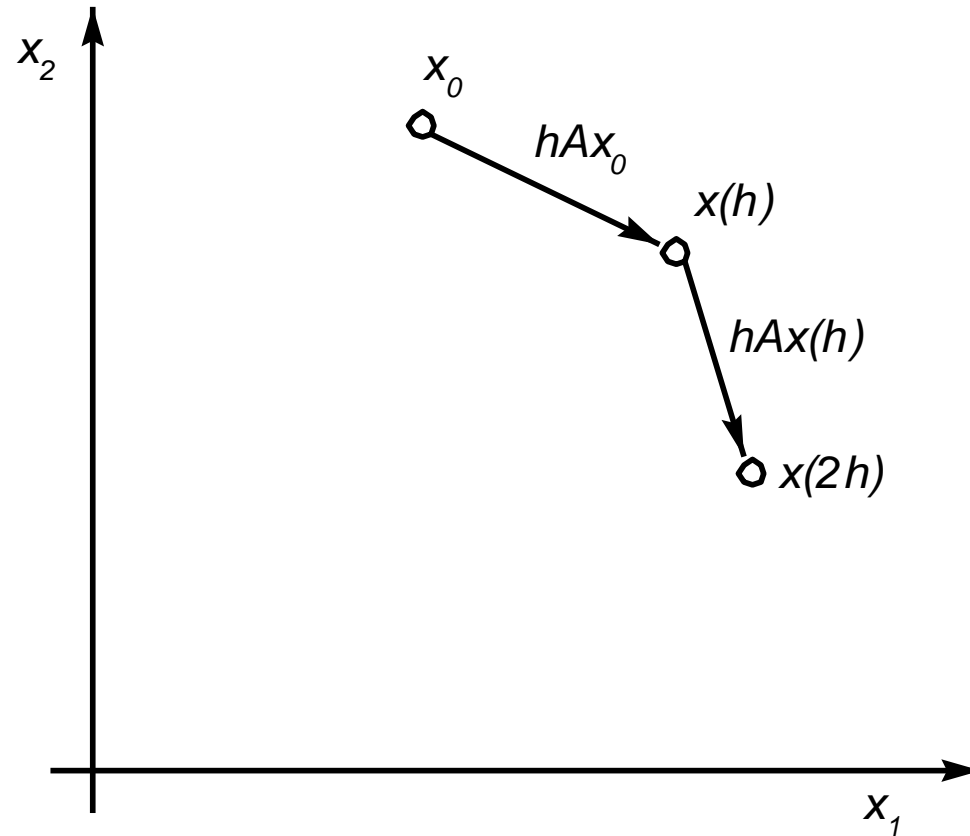
Approximate the derivative by finite differences

$$\dot{x}(t) \approx \frac{x((k+1)h) - x(kh)}{h}$$

Approximate the homogeneous equation by the difference equation

$$x((k+1)h) = x(kh) + hAx(kh)$$

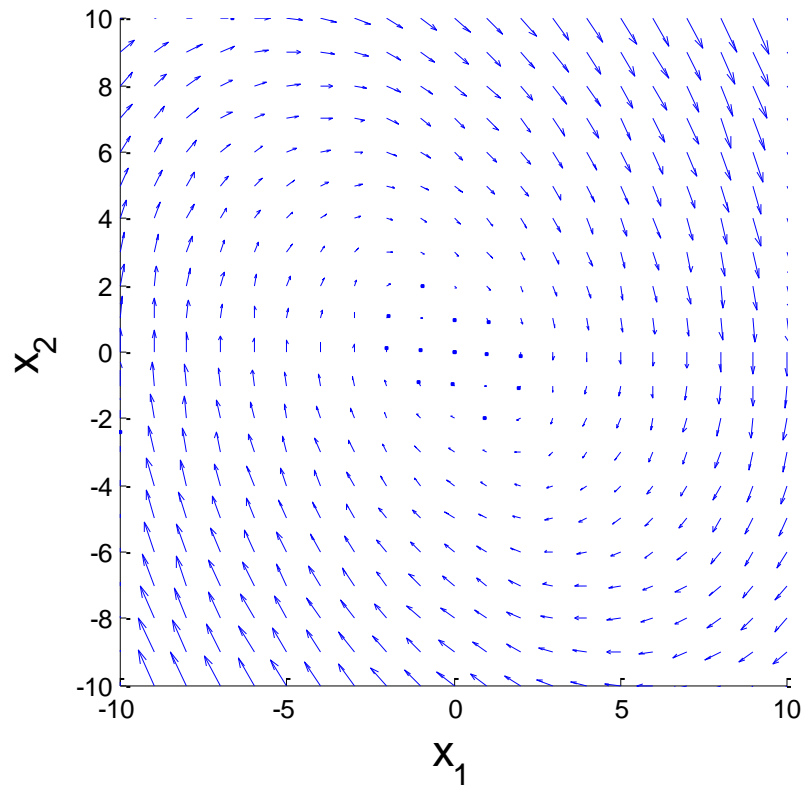
$$x((k + 1)h) = x(kh) + h Ax(kh)$$



$$x((k + 1)h) = x(kh) + h Ax(kh)$$

The orbits in state space may be approximated as follows:

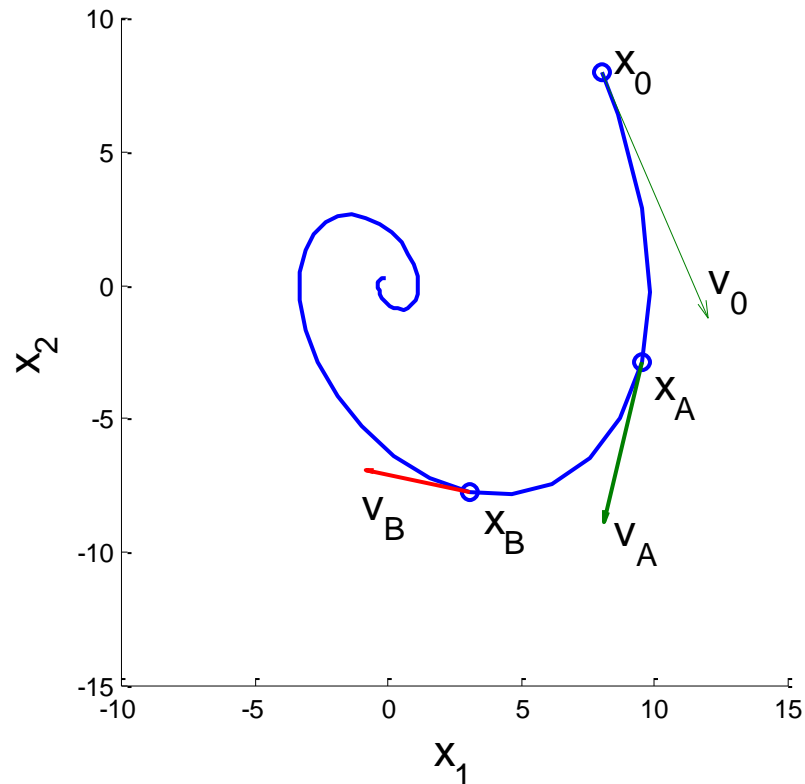
- Start with an initial condition  $x_0$  at the instant  $k = 0$ .
- To get a new point in the instant  $k = h$  add to  $x_0$  a vector proportional to  $Ax_0$  (more precisely,  $hAx_0$ ). We obtain  $x(h) = hAx_0$ .
- Iterate.



At each point  $x$  of the state space  $Ax$  defines a vector (vector field) that indicates the direction followed by the solution at that point.

The **vector field** can be plotted with MATLAB using the function **quiver**.

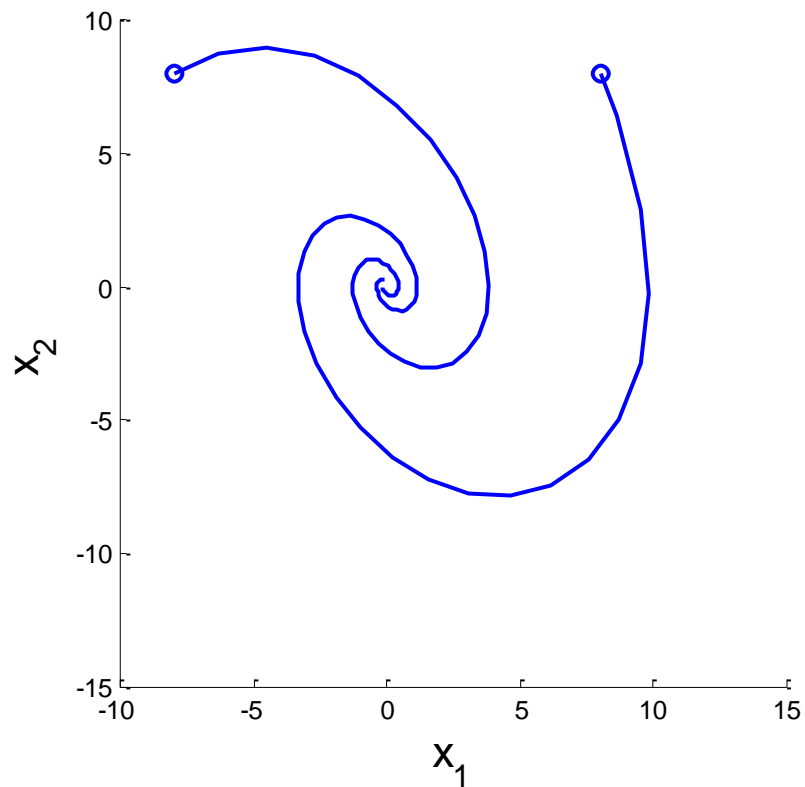




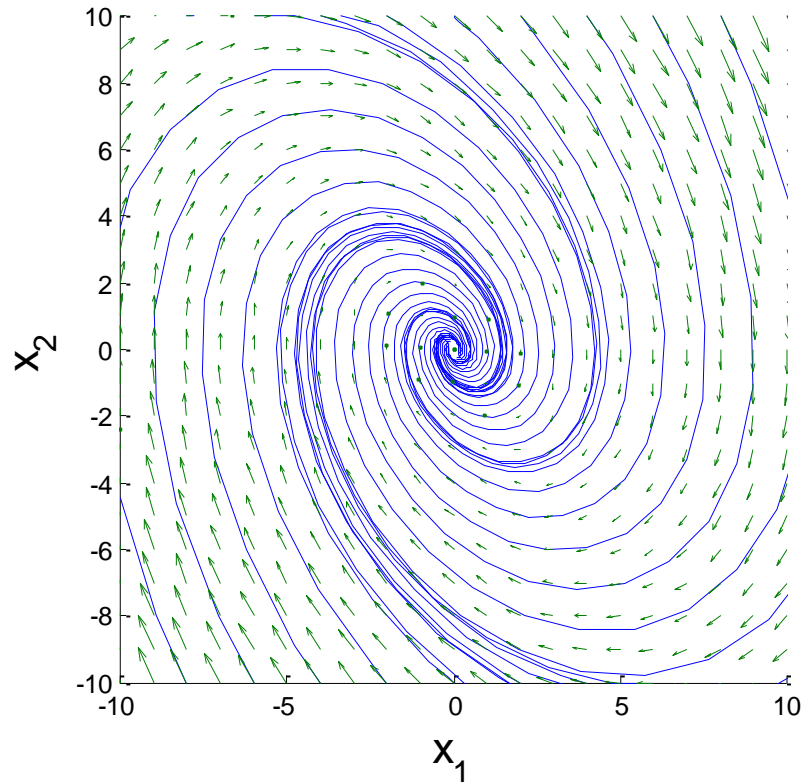
Starting from  $x_0$ , the solution progresses (locally) along the direction

$$v_0 = Ax_0.$$

At each point, the solution (orbit) is tangent to the vector field at that point.



If we start at another initial condition, we get a different trajectory. The graph shows two orbits (solution trajectories) generated from two different initial conditions.



Starting with different initial conditions (there are infinite!) we get the **phase portrait** (*retrato de fase*), shown here together with the vector field.

## Note on linear algebra: Eigenvalues and eigenvectors

Given a square matrix  $A$   $[n \times n]$ , their eigenvectors  $v_i$  satisfy

$$Av_i = \lambda_i v_i$$

where  $\lambda_i$  is the corresponding eigenvalue.

At most there are  $n$  linearly independent eigenvectors (but they may be less).

The eigenvectors are also called *modal vectors* (*vectores modo*).

## Computing eigenvectors and eigenvalues

Since

$$Av_i = \lambda_i v_i$$

The eigenvectors satisfy the algebraic system of equations

$$(A - \lambda_i I)v_i = 0$$

In order for this system to have non-trivial solutions  $v_i \neq 0$ , it must be undetermined, and:

$$\det(A - \lambda_i I) = 0$$

## Computing the eigenvalues and eigenvectors of a square matrix $A$

a) Compute the eigenvalues by solving the polynomial equation

$$\det(A - \lambda_i I) = 0$$

b) For each eigenvalue  $\lambda_i$  compute the corresponding eigenvector from

$$(A - \lambda_i I)v_i = 0$$

Since this system is undetermined, its solution is given up to a normalization constant that may be selected in different ways.

## Computing the eigenvalues and eigenvectors – Example

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \quad A - \lambda I = \begin{bmatrix} 4 - \lambda & -5 \\ 2 & -3 - \lambda \end{bmatrix}$$

Matrix characteristic polynomial

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 4 - \lambda & -5 \\ 2 & -3 - \lambda \end{vmatrix} = (4 - \lambda)(-3 - \lambda) + 10 = \lambda^2 - \lambda - 2 = (\lambda + 1)(\lambda - 2)$$

The eigenvalues are the roots of this polynomial

$$\lambda_1 = -1 \quad \lambda_2 = 2$$

Eigenvectors:

$$\lambda_1 = -1 \quad (A - \lambda_1 I)v_1 = \begin{bmatrix} 5 & -5 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is any multiple of  $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\lambda_2 = 2 \quad (A - \lambda_2 I)v_2 = \begin{bmatrix} 2 & -5 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solution is any multiple of  $v_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$



## Matrix diagonalization

Assumption:  $A$  has  $n$  linearly independent eigenvectors.

Modal matrix (columns are the eigenvectors):

$$M = \begin{bmatrix} v^1 & \dots & v^n \end{bmatrix}$$

Diagonal matrix of eigenvalues

$$\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$$

Warning: Not all matrices verify this assumption.

For any eigenvalue/eigenvector couple

$$Av_i = \lambda_i v_i$$

therefore

$$AM = M\Lambda$$

This means that  $A$  can be decomposed as

$$A = M\Lambda M^{-1}$$

Furthermore, multiplying at the right by  $M$  and at the left by  $M^{-1}$

$$\Lambda = M^{-1}AM$$

## Solution of the homogeneous equation by diagonalization

Valid when  $A$  has  $n$  linearly independent eigenvectors.

$$\dot{x}(t) = Ax(t) \quad x(0) = x_0$$

Variable transformation associated to the modal matrix:

$$z = M^{-1}x \quad \text{ou} \quad x = Mz$$

In the  $z$  coordinates the dynamics becomes

$$\dot{z} = M^{-1}\dot{x} = M^{-1}Ax = M^{-1}AMz = \Lambda z$$

This means that the components of  $z$  are **decoupled** and the equations can be solved separately!

$$\dot{z} = \Lambda z$$

This equation corresponds to

$$\begin{cases} \dot{z}_1 = \lambda_1 z_1 \\ \vdots \\ \dot{z}_n = \lambda_n z_n \end{cases}$$

Since these equations are decoupled, they can be solved separately:

$$z_1(t) = k_1 e^{\lambda_1 t}$$

...

$$z_n(t) = k_n e^{\lambda_n t}$$

Os  $k_i$  são constantes que dependem das condições iniciais

Structure of the response in the coordinates  $x$  :

$$x = Mz = \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} k_1 e^{\lambda_1 t} \\ \vdots \\ k_n e^{\lambda_n t} \end{bmatrix}$$

or

$$x = k_1 v_1 e^{\lambda_1 t} + \dots + k_n v_n e^{\lambda_n t}$$

Each term

$$v_i e^{\lambda_i t}$$

Is called a *system mode*. The response is a linear combination of modes. The coefficients depend on the initial conditions.

### Exemplo

$$\dot{x}(t) = Ax(t)$$

$$A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix} \quad x(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

Time response:

$$x(t) = k_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-1t} + k_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{2t}$$

where  $k_1$  and  $k_2$  depend on the initial conditions.

Computing  $k_1$  and  $k_2$  from the initial conditions:

For  $t = 0$  :

$$\begin{bmatrix} 8 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} k_1 + \begin{bmatrix} 5 \\ 2 \end{bmatrix} k_2$$

This algebraic system can be written as

$$\begin{bmatrix} 1 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \quad k_1 = 3, \quad k_2 = 1$$

## Natural frequencies in a coordinate change

Consider

$$\dot{x}(t) = Ax(t) \quad x(t_0) = x_0$$

Perform the coordinate change

$$z(t) = Tx(t)$$

In which  $T$  is constant and invertible.

Show that the new state realization has the same natural frequencies (eigenvalues of the dynamic matrix).

**Suggestion:** Compute the characteristic polynomial of the dynamics matrix in the  $z$  coordinates.



In the new coordinates the dynamics is

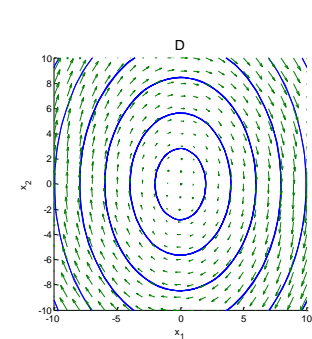
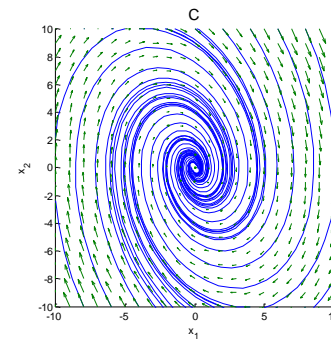
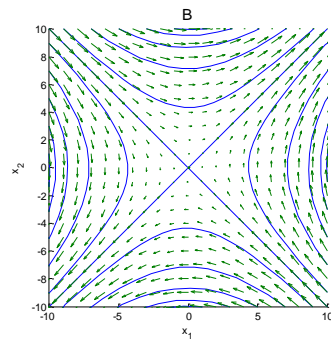
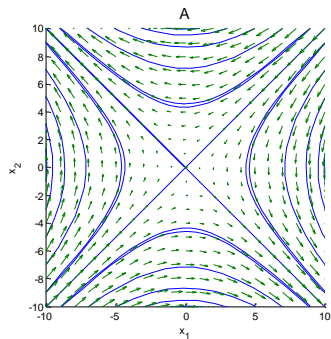
$$\dot{z}(t) = TAT^{-1}z(t)$$

The characteristic polynomial of  $TAT^{-1}$  is

$$\begin{aligned}\det(sI - TAT^{-1}) &= \det\left[T(sI - A)T^{-1}\right] = \\ \det(T) \det(sI - A) \det(T^{-1}) &= \det(T) \det(sI - A) \frac{1}{\det(T)} = \\ &= \det(sI - A)\end{aligned}$$

## Relationship between phase portrait and dynamics matrices

$$A_1 = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ -2 & -0,7 \end{bmatrix}, A_4 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$



Which matrix corresponds to each phase portrait?

*Suggestion:* Compute the eigenvalues and, if needed, the eigenvectors

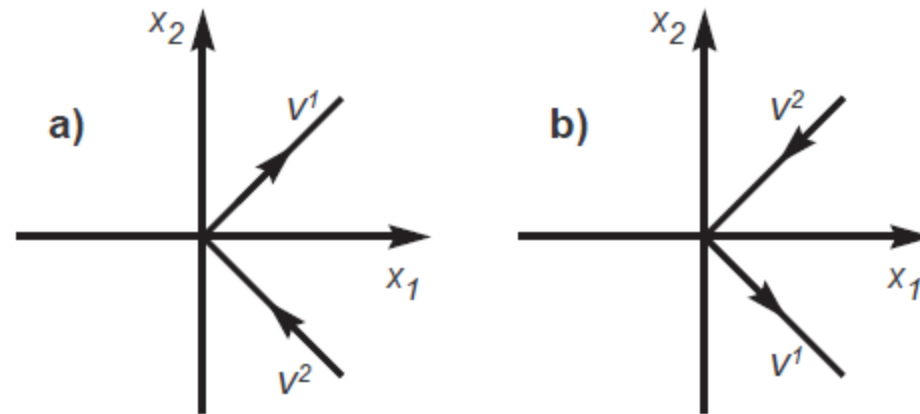
A, B → saddle points (*pontos de sela*)

C → stable focus

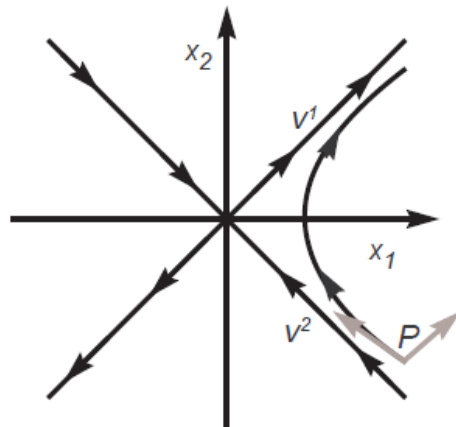
D → centre

Real eigenvalues, both negative → stable node

$$x(t) = k_1 v_1 e^{\lambda_1 t} + k_2 v_2 e^{\lambda_2 t}$$



Directions defined by the eigenvectors of matrices  $A_2$  and  $A_4$



Orbit that starts in an initial condition not aligned with one eigenvector in a saddle point.

## 5.The state transition matrix

### ***Objective:***

*The solution of the homogeneous equation as a transformation of the state given by the transition matrix. Main properties of the transition matrix.*

[JML-CEE2019] pp. 82 – 88, 94

## The Peano-Baker series and the transition matrix

$$\dot{x} = Ax \quad x(t_0) = x_0$$

The solution can be written as

$$x(t) = \Phi(t, t_0)x(t_0)$$

Where the matrix  $\Phi(t, t_0)$ , is called **transition matrix**, being given by the series that converges uniformly and defines the **exponential matrix**

$$\Phi(t, t_0) = e^{A(t-t_0)} = I + A(t-t_0) + \frac{1}{2!} A^2(t-t_0)^2 + \frac{1}{3!} A^3(t-t_0)^3 + \dots$$

## Computing the transition matrix with the Laplace transform

$$\dot{x} = Ax \quad x(0) = x_0$$

Take the Laplace transform

$$sX - x_0 = AX$$

$$(sI - A)X = x_0$$

$$X = (sI - A)^{-1} x_0$$

$$x(t) = TL^{-1} \left\{ (sI - A)^{-1} \right\} x_0$$

Conclusion

$$\Phi(t, t_0) = TL^{-1} \left\{ (sI - A)^{-1} \right\}$$

**Example: Computing the transition matrix with the Laplace Transform**

Consider a system with

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$$

Compute the transition matrix using the Laplace transform..

***Solution***

$$\Phi(t, t_0) = TL^{-1} \left\{ (sI - A)^{-1} \right\}$$

$$sI - A = \begin{bmatrix} s-1 & -1 \\ -4 & s-1 \end{bmatrix} \quad \det(sI - A) = (s-3)(s+1)$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s-1}{(s-3)(s+1)} & \frac{1}{(s-3)(s+1)} \\ \frac{4}{(s-3)(s+1)} & \frac{s-1}{(s-3)(s+1)} \end{bmatrix}$$

$$\frac{s-1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{1}{2} \left( \frac{1}{s-3} + \frac{1}{s+1} \right) \quad A = \frac{3-1}{3+1} = \frac{1}{2} \quad B = \frac{-2}{-4} = \frac{1}{2}$$

$$\phi_{11}(t) = \phi_{22}(t) = \frac{1}{2} (e^{3t} + e^{-t})$$

$$\frac{1}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} = \frac{1}{4} \left( \frac{1}{s-3} - \frac{1}{s+1} \right) \quad A = \frac{1}{4} \quad B = -\frac{1}{4}$$

$$\phi_{12}(t) = \frac{1}{4} (e^{3t} - e^{-t}) \quad \phi_{21}(t) = e^{3t} - e^{-t}$$



$$\Phi(t,0) = \begin{bmatrix} \frac{1}{2}(e^{3t} + e^{-t}) & \frac{1}{4}(e^{3t} - e^{-t}) \\ e^{3t} - e^{-t} & \frac{1}{2}(e^{3t} + e^{-t}) \end{bmatrix}$$

*End of the example*

## Computing the transition matrix with the exponential of $A$

$$\Phi(t, t_0) = I + A(t - t_0) + \frac{1}{2!} A^2 (t - t_0)^2 + \frac{1}{3!} A^3 (t - t_0)^3 + \dots$$

This serie is recognised as the exponential of a matrix. Hence:

$$\Phi(t, t_0) = e^{A(t-t_0)}$$

Remark that, since the system is time invariant

$$\Phi(t, t_0) = \Phi(t - t_0)$$

*Warning:* These properties hold true only when  $A$  is constant.

## Differential equation verified by the transition matrix

The transition matrix verifies

$$\frac{d}{dt}\Phi(t, t_0) = A\Phi(t, t_0)$$

$$\Phi(t_0, t_0) = I$$

These properties are a consequence of

$$x(t) = \Phi(t, t_0)x(t_0)$$

And the unicity of solution of the state equation.

## Invertibility of the transition matrix

Abel-Jacobi-Liouville theorem (special case):

$$\det\left[e^{A(t-t_0)}\right] = e^{(t-t_0)\text{tr}A}$$

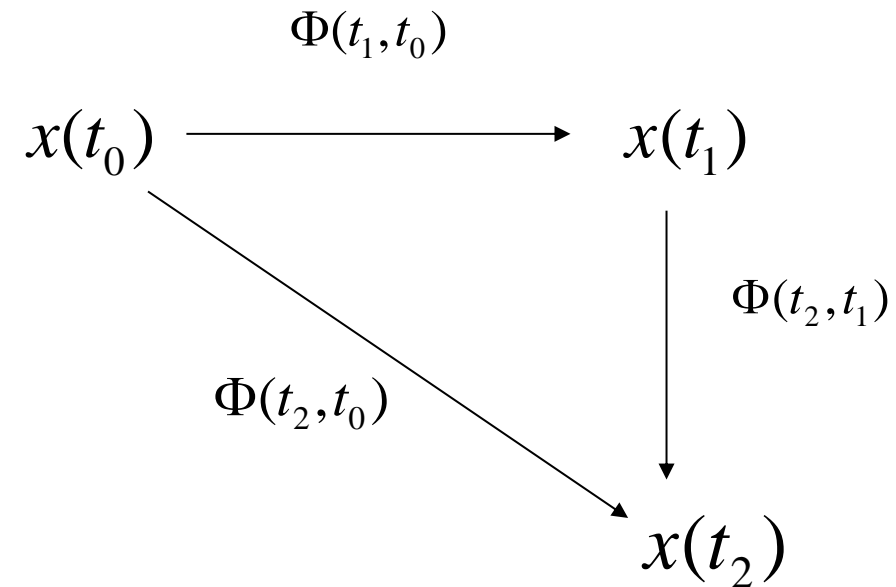
Where the **trace** of  $A$ , represented by  $\text{tr}A$ , is the sum of the diagonal elements.

From here we can conclude that, **in continuous time, the transition matrix is always invertible** because its determinant never vanishes.

This is a major difference with respect to discrete time.

## Semigroup property

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0) \quad \forall t_0, t_1, t_2$$



## Proof of the semigroup property

$$x(t_2) = \Phi(t_2, t_0)x(t_0)$$

On the other way,

$$x(t_2) = \Phi(t_2, t_1)x(t_1) = \Phi(t_2, t_1)\Phi(t_1, t_0)x(t_0)$$

Therefore

$$\Phi(t_2, t_0)x(t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)x(t_0)$$

Since this equality holds  $\forall x(t_0)$ :

$$\Phi(t_2, t_0) = \Phi(t_2, t_1)\Phi(t_1, t_0)$$

## Inverse of the transition inverse

$$\Phi^{-1}(t, t_0) = \Phi(t_0, t) \quad \forall t, t_0 \in R$$

### Proof

The inverse  $\Phi^{-1}(t, t_0)$  always exists (Abel-Jacobi-Liouville theorem).

$$\Phi(t, t_0)\Phi(t_0, t) = \Phi(t, t) = I$$

**Consequence:** Reversibility in time

$$x(t_0) = \Phi(t_0, t)x(t)$$

In continuous time we can recover the initial condition from the present state.

*Not always valid for discrete systems!*

## Continuity

The transition matrix

$$\Phi(t, t_0)$$

Is a continuous function of  $t$  and  $t_0$ .

**Proof:** Not shown. A consequence of the theorem of existence and unicity of solution for ODEs.



## 6.Non-homogeneous systems

### ***Objective:***

*Compute the response of a forced linear time invariant system using the state model*

[JML-CEE2019] pp. 97 – 99

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## Non-homogeneous systems (continuous case)

$$\dot{x}(t) = Ax(t) + bu(t)$$

The response is given by the formula of variation of constants

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^t e^{A(t-\tau)} bu(\tau) d\tau$$

Free regime

Forced regime

### Proof methods

- Superposition principle
- Change of variables
- Show that it verifies the state ODE

## 7.State model for discrete systems

### ***Objective:***

*Very concise introduction of the state model for discrete systems and its response.*

[JML-CEE2019] 99 – 105.

## Homogeneous systems (discrete time)

$$x(k+1) = Ax(k) \quad x(k_0) = x_0$$

As in the continuous time case, the state at time  $k$  is related to the state at time  $k_0 \leq k$  by the state transition matrix:

$$x(k) = A^{k-k_0} x(k_0)$$

## Solution of the non-homogeneous system (discrete time)

$$x(k + 1) = Ax(k) + bu(k)$$

The solution is given by

$$x(k) = A^{k-k_0} x_0 + \sum_{j=k_0}^{k-1} A^{k-j-1} bu(j)$$

Free regime

Forced regime

With zero initial conditions:

$$x(k) = A^{k-1} bu(0) + A^{k-2} bu(1) + \dots + bu(k-1)$$

Proof: Use the superposition principle.