Resonator optics

Fabry-Perot resonator
  • no losses / with losses
  • resonator modes
  • spectral width and finesse
  • resonator lifetime and quality factor

Spherical mirror resonators
  • ray confinement
  • Gaussian modes

*Fundamentals of Photonics*, Ch. 10
An optical resonator confines and stores light

... but not any kind of light. The configuration of the resonator determines its resonance frequencies.
Applications of optical resonators

Can be used as a frequency analyzers or optical filters.
One of the main uses is in laser resonators.

Fabry-Perot interferometer

HeNe laser cavity
The simplest type of resonator is the Fabry-Perot

It is composed of two parallel, highly reflective, flat mirrors separated by a distance $d$.

First we are going to study the case where there are no losses.

Charles Fabry (1867-1945)

Alfred Perot (1863-1925)
A resonator only allows certain modes

Consider a monochromatic wave $U(r)$ that satisfies the Helmholtz eq.:

$$u(r,t) = \text{Re}\{U(r) \exp(i2\pi \nu t)\}$$

The resonator modes (allowed $U(r)$) are the solutions under the boundary conditions defined by the geometry of the resonator.

Lossless mirrors $\Rightarrow U(r)=0$ at the mirrors

$$U(r) = A \sin(kz), \quad kd = q\pi \quad (q = 1, 2, \ldots)$$

$$k_q = q \frac{\pi}{d}$$

$$U_q(r) = A_q \sin(k_q z)$$

The modes are standing waves and $q=1,2,\ldots$ is the mode number.
Resonance frequencies and frequency spacing

Just as the wavenumber $k$ is restricted to discrete values $k_q$, so is the frequency:

$$\nu = \frac{c}{\lambda} = \frac{k_c}{2\pi}$$

$$k_q = q \frac{\pi}{d}$$

$$\Rightarrow \nu \equiv \nu_q = q \frac{c}{2d}$$

$c = c_0/n$

speed of light in the medium!

resonance frequencies

frequency spacing = free spectral range (FSR)
Resonance: examples

Consider a 30 cm long resonator filled with air \((n=1)\):

\[
d = 30 \text{ cm} \quad \nu_F = \frac{c}{2d} = 500 \text{ MHz}
\]

\[
\lambda_q = \frac{2d}{q} = 60 \text{ cm}, 30 \text{ cm}, 20 \text{ cm}, 15 \text{ cm} \ldots
\]

For a much smaller resonator (30 µm long) we have

\[
d = 3 \text{ µm}, \quad \nu_F = \frac{c}{2d} = 50 \text{ THz}
\]

\[
\lambda_q = \frac{2d}{q} = 6 \text{ µm}, 3 \text{ µm}, 2 \text{ µm}, 1.5 \text{ µm} \ldots
\]
Resonator modes must repeat themselves at the same place

If a given wave is a mode, it must repeat itself after a roundtrip.

In terms of wave optics:

\[ \varphi = 2dk \quad (= q2\pi) \]

\[ \Rightarrow k = k_q = q \frac{\pi}{d} \]

This can be understood as a feedback mechanism: only similar waves will add up and build power (i.e. resonate) in the resonator.
Real resonator have losses

1. Imperfect reflection at the end mirrors
   • reflection <100% (e.g. partially reflecting mirror)
   • finite size of the mirror aperture
2. Absorption and scattering in the medium in-between

One case when we want to introduce losses is the laser:
Calculating the effect of losses: amplitude

Losses may affect the amplitude and the phase: they are accounted for by a complex roundtrip attenuation factor:

\[ h = r e^{-i\phi} \]

We have for the consecutive amplitudes \( U_1, U_2 \ldots \)

\[ U_n = hU_{n-1} = h^n U_0 \]

\[ U = U_0 + U_1 + U_2 + \cdots \]

\[ = U_0 (1 + h + h^2 + \cdots) = U_0 / (1 - h) \]

The phase shift \( \phi \) corresponding to a full roundtrip is:

\[ \phi = k(2d) = (2\pi\nu / c)(2d) = 4\pi\nu d / c \]
Resonator with losses: intensity and finesse

Let’s now calculate the corresponding intensity:

\[ I = |U|^2 = \frac{|U_0|^2}{1 - |r e^{-i\phi}|^2} = \frac{I_0}{1 + |r|^2 - 2|r| \cos \phi} \]

We can write this result in a more interesting way:

\[ I = \frac{I_{\text{max}}}{1 + (2F/\pi)^2 \sin^2(\phi/2)} \]

\[ I_{\text{max}} = \frac{I_0}{(1 - |r|)^2} \]

\[ F = \frac{\pi \sqrt{|r|}}{1 - |r|} \]

\[ \sin^2(\phi/2) = \frac{1}{2} (1 - \cos \phi) \]

This is the definition of the finesse. We will see that this parameter is fundamental in defining the characteristics of a resonator.
Intensity vs. phase / Finesse

\[ I_{\text{min}} = \frac{I_{\text{max}}}{1 + \left(2F / \pi \right)^2} \]
Inteensity vs. frequency of a monochromatic wave

\[ I(\nu) = \frac{I_{\text{max}}}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2\left(\frac{\pi \nu}{\nu_F}\right)} \]

\[ \nu_F = \frac{c}{2d} \]

\[ \phi = \frac{2\pi \nu d}{c} = \frac{\pi \nu}{\nu_F} \]

\[ \delta \nu \approx \frac{\nu_F}{F} \]
Finding a simpler expression for the finesse

We can write the **losses in the medium** as a distributed loss proportional to \( d \)

The **losses in the mirrors** are considered located losses at the mirrors 1 and 2

The **total losses** are then:

\[
|r|^2 = R_1 R_2 \exp(-2\alpha_s d)
\]

\[
= \exp(-2\alpha_r d)
\]

\[
\rightarrow \alpha_r = \alpha_s + \frac{1}{2d} \ln \left( \frac{1}{R_1 R_2} \right)
\]

\( \alpha_r = \text{loss coefficient} \)

If we replace \( |r|^2 \) in the expression for the finesse we obtain (approx. valid for \( \alpha_r d \ll 1 \)):

\[
\mathcal{F} = \frac{\pi \exp(-\alpha_r d / 2)}{1 - \exp(-\alpha_r d)} \approx \frac{\pi}{\alpha_r d}
\]
Finesse vs. $r$

\[
|r|^2 = \exp(-2\alpha_r d)
\]

![Graph showing relationship between Finesse \(F\) and Loss factor \(\alpha_r d\).]
Exercise - Resonator Modes and Spectral Width

Calculate:

- frequency spacing $\nu_F$
- spectral width $\delta \nu$

of the modes of a Fabry-Perot resonator whose mirrors have reflectances $R_1=0.98$ and $R_1=0.99$ and are separated by a distance $d = 100$ cm.

Assume that the medium has refractive index $n = 1$ and negligible losses.

Is the derived approximation appropriate in this case?
Why do resonator losses cause spectral line broadening?

Consider the expression for the spectral width:

\[ \delta \nu \approx \frac{\nu_F}{\mathcal{F}} \approx \frac{c}{2d} \left( \frac{\pi}{\alpha_r d} \right) = \frac{c \alpha_r}{2\pi} \]

Note that \( c \alpha_r \) has dimensions of (time)\(^{-1}\). We define the characteristic decay time as:

\[ \tau_p = \frac{1}{c \alpha_r} \]

The relation between time width and spectral width has the form of an uncertainty product:

\[ \delta \nu \cdot \tau_p = \frac{1}{2\pi} \]

\[ \propto \exp\left( -\frac{t}{2\tau_p} \right) \]

\[ \propto \frac{1}{1 + (4\pi \nu \tau_p)^2} \]
The quality summarizes the resonator characteristics

$$Q = 2\pi \frac{\text{stored energy}}{\text{energy lost per cycle}}$$

The energy decays exponentially with \(\tau_p\) (check expression for E-field, prev. slide):

$$E(t) \propto E_0 \exp\left(-\frac{t}{\tau_p}\right)$$

The energy lost in a cycle (=period \(T\)) is:

$$\Delta E = (dE / dt) \cdot T$$

So we obtain:

$$Q = \frac{2\pi}{(1/\tau_p) \cdot T} = 2\pi \tau_p \nu_0 = \nu_0 / \delta\nu$$

Finally, using the previous relations, we can arrive to an expression relating

- quality factor
- frequency spacing
- finesse

$$Q = \frac{\nu_0}{\delta\nu} = \frac{\nu_0}{\nu_F} \mathcal{F}$$
# Fabry-Perot resonators: summary

<table>
<thead>
<tr>
<th><strong>Lossless resonator</strong></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>only selected frequencies are allowed</td>
<td>$v_q = q \frac{c}{2d}$</td>
</tr>
<tr>
<td>Spacing between modes (FSR)</td>
<td>$v_F = \frac{c}{2d}$</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Lossy resonator</strong></th>
<th></th>
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<tbody>
<tr>
<td>the resonance line is broadened – the spectral width is</td>
<td>$\delta v \approx \frac{v_F}{F}$</td>
</tr>
<tr>
<td>The losses may be described by two parameters</td>
<td>$\alpha_r , (\text{cm}^{-1})$</td>
</tr>
<tr>
<td>Resonator quality is characterized by two dimensionless parameters</td>
<td>$\tau_p = \frac{1}{c \alpha_r} , (s)$</td>
</tr>
<tr>
<td></td>
<td>$F = \frac{\pi}{\alpha_r d}$</td>
</tr>
<tr>
<td></td>
<td>$Q = \left(\frac{v_0}{v_F}\right)F$</td>
</tr>
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</table>
Extra: Fabry-Perot interferometer

In this case a plane wave comes from outside and is transmitted through a mirror with amplitude transmittance $t$ and amplitude reflectance $r$:

\[
U_0 t^2 r^4 e^{-i5\varphi} \\
U_0 t^2 r^2 e^{-i3\varphi} \\
U_0 t^2 e^{-i\varphi}
\]
By making calculations similar to the FP resonator, we obtain a similar expression for the transmitted intensity

\[ I_t = \frac{I_0}{1 + \left(\frac{2F}{\pi}\right)^2 \sin^2(\varphi / 2)} \]

This depends on the separation \( d \) between the mirrors
In particular, this means that the reflectance coefficient of the mirrors is not absolute!
Spherical mirror resonators

Pay attention!
In this case we have: $R_1 < 0, R_2 < 0$
Conditions for ray confinement

In ray matrix theory we studied the conditions for a bounded solution in a periodic system.

For a spherical mirror we have the confinement condition

$$0 \leq g_1 g_2 \leq 1, \quad g_{1,2} = \left( 1 + \frac{d}{R_{1,2}} \right)$$

The stability depends on the product of the $g$-parameters.

- $0 \leq g_1 g_2 \leq 1$ 
  - stable resonator
- $g_1 g_2 < 0$ 
  - unstable resonator
- $g_1 g_2 = 0$ 
  - conditionally stable resonator
- $g_1 g_2 = 1$ 
  - Fabry-Perot: lossless
- $g_1 g_2 > 1$ 
  - Fabry-Perot: lossy
Diagram of resonator stability

The dotted red line marks the position of **symmetric resonators** for which $g_1 = g_2$.

The equations for the stability factor $g_{1,2}$ are:

$$g_{1,2} = \left(1 + \frac{d}{R_{1,2}}\right)$$

<table>
<thead>
<tr>
<th>$R$</th>
<th>$g$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\infty$</td>
<td>1</td>
</tr>
<tr>
<td>$-d$</td>
<td>0</td>
</tr>
<tr>
<td>$-d/2$</td>
<td>-1</td>
</tr>
</tbody>
</table>

The diagram shows different types of resonators:
- Hemispherical $(0, 1)$
- Plane-parallel $(1, 1)$
- Concentric $(-1, -1)$
- Confocal $(0, 0)$
- Concave-convex $(2, 1/3)$

Fabry-Perot: lossless
Fabry-Perot: lossy
Spherical mirror
Symmetric resonators: confinement condition

In this case $R_1 = R_2 = R$ and $g_1 = g_2 = g$:

$$0 \leq g^2 \leq 1 \iff 0 \leq \frac{d}{(-R)} \leq 2$$

A stable symmetric resonator must use mirrors with a radius of curvature greater than (length/2).

Example: symmetric confocal ($R = -d$)

(a) Planar ($R_1 = R_2 = \infty$)
(b) Symmetric confocal ($R_1 = R_2 = -d$)
(c) Symmetric concentric ($R_1 = R_2 = -d/2$)

Fabry-Perot: lossless  Fabry-Perot: lossy  Spherical mirror
Gaussian beams: quick reminder

\[ I(\rho, z) = I_0 \frac{W_0}{W(z)} \exp \left[ -\frac{\rho^2}{W^2(z)} \right] \]
\[ \times \exp \left[ -i(kz - \zeta(z)) \right] \]
\[ \times \exp \left[ -ik \frac{\rho^2}{2R(z)} \right] \]

\[ z_0 = \frac{\pi W_0^2}{\lambda} \]
\[ W(z) = W_0 \sqrt{1 + \left( \frac{z}{z_0} \right)^2} \]
\[ R(z) = z \left[ 1 + \left( \frac{z_0}{z} \right)^2 \right] \]
\[ \zeta(z) = \tan^{-1} \frac{z}{z_0} \]
\[ W_0 = \sqrt{\frac{\lambda z_0}{\pi}} \]
The Gaussian beam is a mode of the spherical mirror resonator

A Gaussian beam will be reflected at a spherical mirror and retrace its way back exactly if:

$$R_{\text{wavefront}} = R_{\text{mirror}}$$
Conditions for Gaussian beam confinement

We have the following three conditions linking the positions \( z \) and radius \( R_1, R_2 \) of the concave mirrors, and the \( R(z) \) of the beam:

\[
\begin{align*}
    z_2 &= z_1 + d \\
    R_1 &= z_1 + \frac{z_0^2}{z_1} \\
    (-R_2) &= z_2 + \frac{z_0^2}{z_2}
\end{align*}
\]

This gives us the position of the beam center and Rayleigh length:

\[
\begin{align*}
    z_1 &= \frac{-d(R_2+d)}{R_2+R_1+2d} \\
    z_0^2 &= \frac{-d(R_1+d)(R_2+d)(R_1+R_2+d)}{(R_2+R_1+2d)^2}
\end{align*}
\]

Parameters of the Gaussian beam that obeys the boundary conditions:

\[
\begin{align*}
    W_0 &= \sqrt{\frac{\lambda z_0}{\pi}} \\
    W_i &= W_0 \sqrt{1 + \left( \frac{z_i}{z_0} \right)^2}, \quad i = 1, 2
\end{align*}
\]
Exercise – plano-concave resonator

When mirror 1 is planar ($R_1 = \infty$), determine as a function of $d / |R_2|$:  

- the confinement condition  
- the depth of focus ($= 2z_0$)  
- beam width at the waist and at each of the mirrors