Blade Element Theory

• Momentum theory gives rapid and simple method to estimate of necessary Power.

• This approach is sufficient to size a rotor (i.e. select the disk area) for a given power plant (engine), and a given gross weight.

• This approach is not adequate for designing the rotor.
Blade Element Theory

• The momentum theory does not take into account
  – Number of blades
  – Airfoil characteristics (lift, drag, angle of zero lift)
  – Blade planform (taper, sweep, root cut-out)
  – Blade twist distribution
  – Compressibility effects
Blade Element Theory

- Blade Element Theory (BET) was first proposed by Drzewiecki in 1892 for the analysis of airplane propeller.
- BET assumes that each blade section acts as a two-dimensional airfoil to produce aerodynamic forces.
- The blade is then divided into non-interacting sections where all the computations are performed using 2-D aerodynamics.
- An integration over the blade length gives the total thrust and total power.
BET Model
BET Model

- The in plane Velocity $U_T = \Omega y$
- The out of plane Velocity $U_P = V_C + v_i$
- Therefore the total velocity is $U = \sqrt{U_T^2 + U_P^2}$
BET model

• The relative inflow angle:

\[ \phi = \tan^{-1}\left( \frac{U_P}{U_T} \right) \]

• If the blade element has a pitch angle of \( \theta \), the effective angle of attack is:

\[ \alpha = \theta - \phi = \theta - \tan^{-1}\left( \frac{U_P}{U_T} \right) \]
BET model

• The incremental lift per unit span:

\[ dL = \frac{1}{2} \rho U^2 c C_l dy \]

• The incremental drag per unit span:

\[ dD = \frac{1}{2} \rho U^2 c C_d dy \]

• Or in quantities parallel and perpendicular to the rotor disk plane:

\[
\begin{align*}
   dF_z &= dL \cos \phi - dD \sin \phi \\
   dF_x &= dL \sin \phi + dD \cos \phi
\end{align*}
\]
BET model

- We can then calculate the Thrust:
  \[ dT = N_b dF_z \]
- The Torque
  \[ dQ = N_b dF_x y \]
- The Power
  \[ dP = N_b dF_x \Omega y \]
- Remember \( N_b \) is the number of blades
BET model

- And we can relate all three with $C_l$ and $C_d$

\[
\begin{align*}
 dT &= N_b (dL \cos \phi - dD \sin \phi) \\
 dQ &= N_b (dL \sin \phi + dD \cos \phi) y \\
 dP &= N_b (dL \sin \phi + dD \cos \phi) \Omega y
\end{align*}
\]
BET model assumptions

- The following assumptions are valid within the helicopter aerodynamics

\[ U_T \gg U_P \implies U = \sqrt{U_P^2 + U_T^2} \approx U_T \]

\[ \phi = \tan^{-1}\left(\frac{U_P}{U_T}\right) \approx \frac{U_P}{U_T} \]

\[ \phi \approx 0 \implies \sin \phi = \phi \]

\[ \cos \phi = 1 \]

\[ dD \ll dL \implies dD \sin \phi \approx dD \phi \approx 0 \]
Basic Equations

- The expression for Thrust, Torque and Power are:

\[
\begin{align*}
\frac{dT}{dB} &= N_b \left( dL \cos \phi - dD \sin \phi \right) = N_b (dL) \\
\frac{dQ}{dB} &= N_b \left( dL \sin \phi + dD \cos \phi \right) = N_b (dL \phi + dD) \\
\frac{dP}{dB} &= N_b \left( dL \sin \phi + dD \cos \phi \right) \Omega y = N_b (dL \phi + dD) \Omega y
\end{align*}
\]

- Let’s now nondimensionalize using for length \( R \) and for speed \( V_{tip} = \Omega R \)
Nondimensional form

- \( r = \frac{y}{R} \)
- \( \frac{U_T}{\Omega R} = \frac{\Omega y}{\Omega R} = \frac{y}{R} = r \)
- And the thrust, torque and power coefficients already defined:

\[
\frac{dC_T}{dT} = \frac{dQ}{\rho A (\Omega R)^2}, \quad \frac{dC_Q}{dQ} = \frac{dP}{\rho A (\Omega R)^3}
\]

- Now the inflow ratio is

\[
\lambda = \frac{V_c + v_i}{\Omega R} = \frac{V_c + v_i}{\Omega y} \left( \frac{\Omega y}{\Omega R} \right) = \frac{U_p}{U_T} \left( \frac{y}{R} \right) = \phi r
\]
• Substituting the previous equations in the Thrust coefficient equation:

\[
dC_T = \frac{N_b dL}{\rho A (\Omega R)^2} = \frac{N_b \left( \frac{1}{2} \rho U_T^2 c C_l \, dy \right)}{\rho A (\Omega R)^2}
\]

\[
= \frac{1}{2} \left( \frac{N_b c}{\pi R} \right) C_l \left( \frac{y}{R} \right)^2 d\left( \frac{y}{R} \right) = \frac{1}{2} \sigma C_l r^2 \, dr
\]
Power coefficient (incremental)

- Using the same analysis for the Power coefficient

\[
dC_P = dC_Q = \frac{dQ}{\rho A(\Omega R)^2 R} = \frac{N_b (\phi dL + dD) y}{\rho A(\Omega R)^2 R} \\
= \frac{1}{2} \sigma (\phi C_l + C_D) \left( \frac{y}{R} \right)^3 d\left( \frac{y}{R} \right) \\
= \frac{1}{2} \sigma (\phi C_l + C_D) r^3 dr
\]
Total Thrust and Power

- To find the total blade contribution for Thrust and power we have on integrate between the root and tip of the blade

\[
C_T = \frac{1}{2} \int_0^1 \sigma C_l r^2 \, dr = \frac{1}{2} \sigma \int_0^1 C_l \, dr
\]

- If the blade is rectangular \( c = \text{const} \)
- For the torque and power coefficient

\[
C_Q = C_P = \frac{1}{2} \sigma \int_0^1 (\phi C_l + C_d) r^3 \, dr = \frac{1}{2} \sigma \int_0^1 (\lambda C_l r^2 + C_d r^3) \, dr
\]
Total Thrust and Power

• To evaluate the previous expressions we need:
  • Inflow ratio $\lambda = \lambda(r)$
  • Lift coefficient $C_l = C_l(\alpha, Re, M)$
  • Drag coefficient $C_d = C_d(\alpha, Re, M)$
  • AOA $\alpha = \alpha(V_C, \theta, v_i)$
  • Induced Velocity $v_i = v_i(r)$

Numerical Solution needed
Approximations

• With certain assumptions and approximations it is possible to find closed form analytical solutions.

• The solutions are important because they serve to illustrate the fundamental form of the results in term of operational and geometric parameters of the rotor

• Let’s the assume a rectangular blade $c=const$. From the definition $\sigma=const.$ too.
Thrust approximation

- From the Steady linearized aerodynamics:
  \[ C_l = C_{l\alpha} (\alpha - \alpha_0) = C_{l\alpha} (\theta - \phi - \alpha_0) \]

- We can consider \( C_{l\alpha} \) constant without serious loss of accuracy

- Let’s also assume symmetric airfoils \( \alpha_0 = 0 \)

- We can then write:
  \[
  C_T = \frac{1}{2} \int_0^1 \sigma C_l r^2 \, dr = \frac{1}{2} \sigma C_{l\alpha} \int_0^1 (\theta - \phi) r^2 \, dr
  \]
  \[
  C_T = \frac{1}{2} \sigma C_{l\alpha} \int_0^1 (\theta r^2 - \lambda r) \, dr
  \]
Untwisted Blades

• For a blade with zero twist $\theta=\text{const.}=\theta_0$.
• Let’s also assume uniform inflow velocity, as assumed in the momentum theory $\lambda=\text{const.}$.
• The Thrust coefficient can be written as:

$$C_T = \frac{1}{2} \sigma C_l \alpha \int_0^1 \left( \theta_0 r^2 - \lambda r \right) dr = \frac{1}{2} \sigma C_l \alpha \left[ \theta_0 \frac{r^3}{3} - \lambda \frac{r^2}{2} \right]_0$$

$$C_T = \frac{1}{2} \sigma C_l \alpha \left[ \frac{\theta_0}{3} - \frac{\lambda}{2} \right]$$
Uniform inflow

- Let’s use the result from the momentum theory
  \[ \lambda_i = \lambda_h = \sqrt{\frac{C_T}{2}} \]
- So the thrust coefficient is:
  \[ C_T = \frac{1}{2} \sigma C_{l\alpha} \left( \frac{\theta_0}{3} - \frac{1}{2} \sqrt{\frac{C_T}{2}} \right) \]
- And we can calculate the pitch angle
  \[ \theta_0 = \frac{6C_T}{\sigma C_{l\alpha}} + \frac{3}{2} \sqrt{\frac{C_T}{2}} \]
Untwisted Blades, Uniform inflow
Linearly Twisted Blades, Uniform Inflow

- Let’s now assume that we have a linear twist, common practice in helicopter blades:
  \[ \theta(r) = \theta_0 + r \theta_{tw} \]

- Substituting in the \( C_T \) equation:

\[
C_T = \frac{1}{2} \sigma C_{l_\alpha} \int_0^1 \left( (\theta_0 + r \theta_{tw}) r^2 - \lambda r \right) dr = \\
\frac{1}{2} \sigma C_{l_\alpha} \left[ \frac{\theta_0}{3} + \frac{\theta_{tw}}{4} - \frac{\lambda}{2} \right]
\]
Linearly Twisted Blades, Uniform inflow

- If the reference blade pitch angle is taken a 3/4-radius ($\theta_{0.75}$) then

\[ \theta(r) = \theta_{0.75} + (r - 0.75)\theta_{tw} \]

\[ C_T = \frac{1}{2} \sigma C_{l\alpha} \int_0^1 \left( (\theta_{0.75} + (r - 0.75)\theta_{tw})r^2 - \lambda r \right) dr = \]

\[ = \frac{1}{2} \sigma C_{l\alpha} \int_0^1 \left( \theta_{0.75}r^2 + \theta_{tw}r^3 - 0.75\theta_{tw}r^2 - \lambda r \right) dr = \]

\[ = \frac{1}{2} \sigma C_{l\alpha} \left[ \frac{\theta_{0.75}}{3} + \frac{\theta_{tw}}{4} - \frac{\theta_{tw}}{4} - \frac{\lambda}{2} \right] = \frac{1}{2} \sigma C_{l\alpha} \left[ \frac{\theta_{0.75}}{3} - \frac{\lambda}{2} \right] \]

- Same result as for the constant pitch blade
Power approximations

- We have seen that the incremental power coefficient (that is equal to the torque coefficient):

\[ dC_P = \frac{1}{2} \sigma (\phi C_l + C_d) r^3 dr = \frac{1}{2} \sigma (\lambda C_l r^2 + C_d r^3) dr = \]

\[ = \frac{1}{2} \sigma \lambda C_l r^2 dr + \frac{1}{2} \sigma C_d r^3 dr = \]

\[ = dC_{P_i} + dC_{P_0} \]

- Remembering that

\[ dC_{P_i} = \lambda dC_T \implies dC_P = \lambda dC_T + dC_{P_0} \]
Power approximations

• Therefore the total power:
\[ C_P = \int_{r=0}^{r=1} \lambda dC_T + \int_0^1 \frac{1}{2} \sigma C_d r^3 \, dr = \lambda C_T + \frac{1}{8} \sigma C_{d0} \]

• Assuming uniform inflow and \( C_d = C_{d0} = \text{const.} \)
• Using once more the inflow expression obtained in hover:
\[ C_P = \frac{C_T^{3/2}}{\sqrt{2}} + \frac{1}{8} \sigma C_{d0} \]
• Expression already obtained in the momentum theory
FM for BET

\[
FM = \frac{C_{P_{\text{ideal}}}}{C_{P_{\text{real}}}} = \frac{\lambda C_T}{\lambda C_T + \sigma C_{d0} / 8}
\]

- High solidity \( \sigma \) (lot of blades, wide-chord, large blade area) leads to higher Power consumption, and lower FM.
- Low drag Airfoils leads to higher FM
Average Lift coefficient

- The average Lift coefficient is defined to give the same thrust coefficient when the blade is operating at the same local lift coefficient (optimum rotor):

\[ C_T = \frac{1}{2} \int_0^1 \sigma r^2 C_l dr = \frac{1}{2} \int_0^1 \sigma r^2 \overline{C_l} dr = \frac{1}{6} \sigma \overline{C_l} \]

- Or \( \overline{C_l} = 6 \frac{C_T}{\sigma} \)

- Typically \( \overline{C_l} \) is found to be on the range of 0.5 to 0.8.
FM for Average Lift Coefficient

\[ FM = \frac{\lambda C_T}{\lambda C_T + \sigma C_{d_0}/8} = \frac{1}{1 + \sigma C_{d_0}/(8 C_T \lambda)} = \frac{1}{1 + \sigma C_{d_0}/(\frac{8}{6} \sigma \bar{C}_L \lambda)} = \frac{1}{1 + \frac{3}{4} \left(\frac{C_{d_0}}{\bar{C}_1}\right)/\lambda} \]

• FM is maximized if \( \left(\frac{C_{d_0}}{\bar{C}_1}\right) \) is minimized
Tip Loss factor

- We can assume that the outer portion of the blade \((R-R_e=R-BR)\) does not produce lift. Therefore the thrust coefficient is:

\[
C_T = \frac{1}{2} \sigma C_{\ell \alpha} \int_0^B \left( \theta r^2 - \lambda r \right) dr
\]

- For a untwisted blade \((\theta=\theta_0)\):

\[
C_T = \frac{1}{2} \sigma C_{\ell \alpha} B^2 \left[ \frac{\theta_0 B}{3} - \frac{\lambda}{2} \right]
\]
Tip Loss factor

- For a twisted blade ($\theta = \theta_{tip}/r$):

\[
C_T = \frac{1}{2} \sigma C_{l\alpha} \int_0^B (\theta_{tip} r - \lambda r) \, dr = \frac{1}{4} \sigma C_{l\alpha} B^2 [\theta_{tip} - \lambda]
\]

- For $B$ between 0.95 and 0.98 we can calculate a 6 to 10% reduction in rotor thrust.

- Let’s now assume that instead of having the blade tip not carrying any lift, let’s see it’s effect of the induced inflow velocity:

\[
\nu_h = \sqrt{\frac{T}{2 \rho A_e}} = \sqrt{\frac{T}{2 \rho (AB^2)}} = \frac{1}{B} \sqrt{\frac{T}{2 \rho A}}
\]
Tip Loss factor

- Since the influence is a increase of $\lambda$ by $B^{-1}$ we can substitute in the equations obtained for no tip losses:
  - Untwisted blades and uniform inflow
    \[
    C_T = \frac{1}{2} \sigma C_{l\alpha} \left[ \frac{\theta_0}{3} - \frac{\lambda}{2B} \right]
    \]
  - Twisted blades and uniform inflow
    \[
    C_T = \frac{1}{4} \sigma C_{l\alpha} \left[ \theta_{tip} - \frac{\lambda}{B} \right]
    \]
Tip Loss factor

• Comparing with the results obtained earlier we see that these overpredict the effect of tip losses.

• Performing the same calculation for the power coefficient

\[
\begin{align*}
\theta = \theta_0 & \implies C_P = \frac{\sigma C_{l\alpha}}{2} \frac{\lambda}{B} \left[ \frac{\theta_0}{3} - \frac{\lambda}{2B} \right] + \frac{1}{8} \sigma C_{d_0} \\
\theta = \frac{\theta_{tip}}{r} & \implies C_P = \frac{\sigma C_{l\alpha}}{4} \left[ \frac{\lambda}{B} \left( \theta_{tip} - \frac{\lambda}{B} \right) \right] + \frac{1}{8} \sigma C_{d_0}
\end{align*}
\]
Tip Loss factor

![Graph showing the tip loss factor vs. blade-pitch angle for different values of \( \sigma \). The graph illustrates the power coefficient, \( C_p \), as a function of the blade-pitch angle, \( \theta_0 \), with theoretical curves and data points for \( \sigma = 0.042, 0.064, 0.085, \) and \( 0.106 \).]