# Gravitational waves and massive gravitons

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## Engineering Physics

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[1] actually this whole course, but that's a story for another day

## Resumo

Teorias de gravitões massivos são interessantes como alternativa à Relatividade Geral (RG) devido às suas soluções cosmológicas, que prevêem a aceleração da expansão do universo. Estas teorias são ainda conceptualmente interessantes no que diz respeito a outros fenómenos a que dá origem. Nesta tese estudamos um desses fenómenos de gravidade massiva, a emissão de ondas gravitacionais. Analisamos este fenómeno no contexto da perturbação de um buraco negro de Schwarzschild por uma partícula pontual para duas trajectórias: uma queda radial ultra-relativista e uma órbita circular. Simplificando as equações de campo através da sua decomposição em harmnónicas esféricas tensoriais, mostramos que as perturbações monopolar e dipolar levam à emissão de ondas gravitacionais, ao contrário do que acontece em RG. Esta emissão deve-se à excitação de novos modos do gravitão, com origem na sua massa. Finalmente, calculamos as soluções para o modo monopolar, descobrindo que, embora as oscilações resultantes tenham uma amplitude mensurável, a sua frequência muito baixa impede a detecção pelas experiências actuais e do futuro próximo. Este e outros resultados, nomeadamente relacionados com os modos dipolares, são tratados na referência [1].



**Palavras-chave:** teoria de Proca, gravidade massiva dRGT, buracos negros, teoria de perturbações, ondas gravitacionais

# Abstract

Massive graviton theories are of interest as an alternative to General Relativity (GR) for their self-accelerating cosmological solutions. They are also conceptually interesting for several other aspects and phenomena they provide. In this thesis we study one such aspect of massive gravity, that of the emission of gravitational waves. We analyse this phenomenon for the case of a Schwarzschild black hole perturbed by a point particle in one of two geodesic trajectories: highly relativistic radial infall and circular orbit. Simplifying the field equations of this system through the decomposition in tensor spherical harmonics, we show that the monopolar and dipolar perturbations both lead to the emission of gravitational waves, unlike in GR. This emission corresponds to the new excitation modes introduced by the mass of the graviton. Finally, we explicitly compute the solutions for the monopolar mode, finding that, although it causes oscillations of a measurable amplitude, their frequency precludes detection by current and near future experiments. This and results for the dipolar modes are further discussed in reference [1].

**Keywords:** Proca theory, dRGT massive gravity, black holes, perturbation theory, gravitational waves

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# Acronyms

**BH** Black Hole.

**dRGT** de Rham, Gabadadze and Tolley.

**FP** Fierz-Pauli.

**GR** General Relativity.

**GW** Gravitational Wave.

**LIGO** Laser Interferometer Gravitational-wave Observatory.

**LISA** Laser Interferometer Space Antenna.

# Chapter 1

# Introduction

The theory of General Relativity has, since its birth in 1915, given us the most accurate description of gravitation we have ever accomplished. Simply put, it states that the motion of matter is ruled by the ”shape” of spacetime, represented by its metric $g_{\mu\nu}$, which, in turn, is determined by the matter content of the universe. This interplay is given quantitative meaning through the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu}\,, \tag{1.1}$$

where the Ricci tensor $R_{\mu\nu}$ is a function of first and second derivatives of the metric and $T_{\mu\nu}$ is the stress-energy tensor of all matter present.

Besides reducing correctly to Newtonian gravity in the proper limit, this theory has withstood several experimental tests until now, being able to accurately predict many minute phenomena of astrophysical scale. For example, it predicts that light passing close to a massive object is deviated by an angle of $\frac{4GM}{Rc^2}$ ($M$ and $R$ the mass and radius of the object), a value that was first successfully compared with experiment by Arthur Eddington almost a century ago. Besides, the theory can also be used to model the universe as a whole and its dynamics. Here, however, things get subtler.

## 1.1 Motivation

Observationally, it is known that the universe is not only expanding but also that its expansion is accelerating with time. On the other hand, if we model the universe purely with equations (1.1), we would conclude that the expansion was not accelerating but rather slowing down, as a consequence of gravitational attraction. One way to obtain the correct result is to add another term to the equation, called the cosmological constant, which needs to be $\Lambda = 1.11 \times 10^{-52} m^{-2}$ [2] so the predicted rate of expansion matches the measured one. Still, this should not satisfy us completely, as it suggests no explanation as to why GR needs this cosmological constant to predict the evolution of the universe.

It would be interesting, then, to find a deeper reason behind all this. One possibility is that the assumptions we are making are somewhat wrong. If, for instance, there was something else in the universe besides the matter we know of today, it might explain the present rate of expansion and, through it, the apparently arbitrary value of $\Lambda$. This would only change the rhs of equation (1.1), assuming GR to be correct. This is the basis of the idea behind dark energy, and it is not the approach we took.

Another hypothesis is that while the contents of the universe are sufficiently correct, GR itself is not, failing to describe our universe at certain scales. The path, then, would be to take GR as a starting point and extend it somehow, modifying it at these scales. However, there is a great variety of extensions one can make. How, then, to pick one?

To do it, we can take inspiration from another (apparently unrelated) issue with gravity as described by general relativity, which is its incompatibility with quantum mechanics and, consequently, with the description of the other three fundamental interactions. While gravity in general relativity is simply an effect of 4-dimensional geometry, each of these other interactions is associated to a gauge boson, a particle with integer spin such as the photon and the W and Z bosons. Curiously, however, it was found that the equations of motion given by GR are replicated if we consider a theory of a massless spin-2 gauge boson, tentatively called a graviton. So, apparently, there is a possibility that gravitation can be described like the other forces. However, the chances of finding evidence for isolated gravitons as was done for the other bosons are slim, as the gravitational interaction is much weaker than any other, leading to a too small interaction cross-section for it to be measured.

Having this description of gravity through a massless graviton, we can see a natural way to modify it: give the graviton a mass. And, in fact, there are other compelling reasons to do so. The mass (or absence thereof) of a gauge boson defines if the effective potential has a limitless range (massless case) or decays exponentially (massive case):

$$V_{m_g=0} \sim \frac{1}{r} \quad , \quad V_{m_g\neq 0} \sim \frac{1}{r}e^{-m_g r}\,, \tag{1.2}$$

the latter being what is called a Yukawa-like potential. This means that while the massless case at low energies does correspond to the Newtonian potential with which we are acquainted, the massive case does not, decaying more rapidly at distances at which $m_g r \sim 1$, $m_g$ being the mass of the graviton. However, it is feasible that a sufficiently small graviton mass leads to both a seemingly Newtonian potential at the scales of our regular measurements and an attenuated potential at very long ranges. These long ranges are, on the other hand, the ones where we had our original expansion problem. Therefore, a massive graviton may give a reason for the cosmological constant [3, 4], modifying GR at the correct scales.

Thus, this theory of massive gravity, as it is called, is of interest both for giving General Relativity a description closer to that of the other forces and for solving its initial problem of the expansion of the universe. However, to verify the validity of this theory we would like to measure the mass of this graviton or, at least, find some evidence of it being non-zero. Since nowadays the detection of gravitational waves is already a reality, investigating in what manner a graviton mass changes the behaviour of this phenomenon and compare these predictions with what is obtained experimentally seems a promising avenue.

## 1.2 Topic Overview

### 1.2.1 Massive gravity

The idea of considering a massive graviton, or, in other words, a spin-2 field with mass, started quite early in 1939, with Markus Fierz and Wolfgang Pauli [5], although not much was done until some decades later. The approach that was taken was similar to what is usually done for the other interactions. We consider a linear field, in this case of spin 2, over a background Minkowski metric and add to it a mass term. This addition causes the loss of some of the gauge freedom the massless graviton had, adding, in general, four degrees of freedom to the two that already existed. However, picking correctly, as Fierz and Pauli did, the mass term, one can eliminate one of these new degrees of freedom. Summing this all up, while the massless graviton has two degrees of freedom, corresponding to its helicity (as for any massless particle with spin), the massive graviton has now 5 degrees of freedom , which is in agreement with the formula for the states of a massive particle ($2s+1$, with $s$ the spin).

Much later, in 1970, an unexpected feature of the Fierz-Pauli theory was discovered [6, 7]. Considering the theory in the limit of zero graviton mass (also called the decoupling limit) it gave predictions that differed significantly from GR. The classic example is that of the deflection of light by the Sun, which the Fierz-Pauli theory predicts to be $\frac{3}{4}$ of the measured value, this result being called the vDVZ discontinuity. The reason for it was soon found: out of the five modes the massive graviton has, the ones corresponding to spin 1 decoupled from matter in the limit, as expected, while the scalar, spin-0 mode, did not. This means that the decoupling limit of the linear theory of a massive spin-2 field is not, as expected, equivalent to a massless spin-2 field but is equivalent, in fact, to a massless spin-2 plus a scalar field. Recently, this scalar field, renamed galileon, has been studied as a modification of GR by itself [8, 9].

Soon after, it was found that this discontinuity appeared during the linearisation itself, and that inside some region of spacetime nonlinear terms on the fields had to be taken into account and would cure the discontinuity, this effect being called the Vainshtein screening [10]. However, constructing a nonlinear theory raised yet another issue [11]. The generalization of the Fierz-Pauli choice of mass term was not trivial, giving rise to the previously avoided sixth degree of freedom. Not only that, this degree of freedom, called the Boulware-Deser ghost, permitted modes with negative kinetic energy, which is not physically allowed. At the time, and until recently, this seemed to be the end of the road for the idea of a massive graviton.

Then, in 2011, de Rham et al. [12, 13] described how to pick a nonlinear mass term without causing the appearance of the Boulware-Deser ghost, by choosing wisely the coefficients of the nonlinear mass terms, this being called the dRGT massive gravity theory. Such a development naturally opened the way for studying other implications of a graviton mass, such as, for instance, its effect on black holes and in gravitational waves. It is on this type of phenomena that we focus our work.

### 1.2.2 Gravitational waves and black hole perturbations

One of the most well known solutions to the Einstein equations, and in fact the first non-trivial solution to be found, was the Schwarzschild black hole metric. It corresponds to a static and spherically symmetric vacuum spacetime, as might occur outside a mass distribution in the same conditions. This solution is characterised by its event horizon, which is such that if the trajectory of a particle starts inside it, it will inevitably stay inside it. While there are more realistic models for black holes, such as the Kerr (rotating) solution, the Schwarzschild metric is still useful as a simple model, for a perhaps more qualitative understanding of black hole-related phenomena.

One such phenomena is the emission of gravitational waves in black hole spacetimes. Perturbation theory is a method of obtaining approximate solutions to the Einstein equations by starting from simpler ones. Analogously to many other areas of physics, we consider that the metric can, in fact, be separated into two tensors, a background, fixed metric $\bar{\mathbf{g}}$ and a perturbation $\mathbf{h}$ over it, which is the one to be determined: $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$. From here we can calculate the Ricci tensor to any order we want in what concerns $h_{\mu\nu}$. One can obtain a particularly interesting solution when considering a Minkowski background and working to first order. This metric corresponds to gravitational waves. In essence, it contains oscillating terms which, physically speaking, represent oscillations of spacetime itself (and, therefore, of the distance at which two objects are) that propagate as time goes by. Although minuscule at our distance from their source, these oscillations can be measured by extremely large interferometers, such as the ones in the LIGO and VIRGO experiments, and these measurements, in turn, are yet another way to test the validity of general relativity. For a brief introduction to the topic of GWs we suggest [14].

However, none of this tells us how gravitational waves are generated, only that they can exist and propagate. Several generating phenomena have been studied, among them the inspiral orbit of two massive bodies, whose signal was the one detected in the above mentioned experiments. Another possibility for this generation is the free fall of a body in a black hole, be it Schwarzschild or Kerr. This system can also be studied via perturbation theory, as long as the body's mass is small when compared to the black hole mass (being, essentially, pointlike). The metric is now divided into a black hole metric background and a perturbation over it, induced by the particle infall. Also, the final solution is not in the vacuum, as there is a first order term in the stress-energy tensor, corresponding to the pointlike particle's movement. This system has already been studied in the context of GR and solved through the decomposition of the equations into spherical tensor harmonics (see [15, 16]). These are a generalisation of the spherical harmonics method used, for example, in the Schrödinger equation for the hydrogen atom, and help us simplifying the equations to be solved. These works found not only that this kind of perturbations do generate gravitational waves but also that the original black hole solutions are stable after imposing such perturbations.

### 1.2.3 State of the art

Despite the problems raised by the Boulware-Deser ghost, there have been some attempts, pre-dRGT, to impose bounds on the mass of the graviton. Many of these have been based in alterations this mass implies in Solar System phenomena, such as the precession of the perihelion of the various planets in it. An important bound related to this approach was obtained by Talmadge et al. in [17], having recently received a major update by Clifford Will [18]. Other approach that was followed in [19, 20] was to study the power emitted by binary pulsars in a linearized massive graviton theory with a mass term different than the one from Fierz and Pauli and compare it with observations of its orbital decay. The most relevant of these results are presented in section 3.1.

In more recent developments, theories consisting of a massless graviton plus a galileon were studied in [21], in the context of the emission of gravitational radiation by binary systems. This was done by considering a Minkowski background along with a spherically symmetric background for the galileon field and then perturbing the galileon to first order. The results obtained, while representing a small correction for systems with high mass ratio, diverged when the masses were similar. This led to the conclusion that the perturbation used was not valid, being necessary to consider higher orders of the galileon field, to cause a suppression like that of the Vainshtein mechanism in the Fierz-Pauli theory.

As for dRGT massive gravity theory, there have been several works on phenomena such as black holes, ranging from new solutions besides the ones found in GR to the study of the stability of the well known solutions, such as the Schwarzschild and the Kerr. From the latter it has also been extracted a bound on the mass of the graviton, discussed in the above mentioned section 3.1. However, to our knowledge, there is no previous work attempting to compute the generation and emission of gravitational waves through the perturbation of black holes in dRGT massive gravity.

## 1.3 Objectives

Having presented the dRGT massive gravity theory, the first successful nonlinear theory of a massive graviton, and the phenomena of generation of gravitational waves, the purpose of this work was to join the two. We studied the perturbation of a Schwarzschild black hole in the bimetric formalism of dRGT massive gravity, which will be presented further along, by a pointlike particle in free fall, that is, following some (background) geodesic motion. In particular, we studied the equation:

$$\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma} + \mu^2(h_{\mu\nu} - \overline{g}_{\mu\nu}h) = 8\pi T^{(1)}_{\mu\nu}\,, \tag{1.3}$$

where the lhs is the usual linearised Einstein equations plus a term related to the mass of the graviton and the rhs is the 1st order stress-energy tensor of the pointlike particle.

The solution to these equations was found by applying methods similar to those used by Zerilli in [15] for the same system in GR, which consists in using tensor spherical harmonics and Fourier transforms to simplify equations (1.3) into a system of ODEs. This system was then solved numerically and the solutions transformed back into the time domain.

We obtained the simplified equations of this setup for two different behaviours of the matter perturbation, that of a radial infall and that of a circular orbit. In this thesis we only present the full, numerical solution for one of these cases, the radial infall, in the lowest multipolar order of $l = 0$, which is excited in dRGT massive gravity, unlike in GR. For this case, we obtained not only the perturbation metric elements but also the spectrum of the energy lost by the system.

# Chapter 2

# Proca Theory

A Proca field, first conceived by Alexandru Proca [22], is a massive spin-1 (or vector) field, corresponding, for a common analogue, to an electromagnetic field due to a photon with a nonzero mass. Although the massive photon case has been studied extensively, the Proca field has found several other applications.

Our interest in this subject, however, is not in these applications but in the case study it provides us. Perturbing a massive tensor field, as we have proposed to do in this thesis, is not as trivial and direct as for scalar fields. As such, before delving directly into the spin-2 field we will analyze here the simpler case of studying a Proca field over a Schwarzschild background. This will make it easier to present the concepts and mechanisms used throughout this work, having to deal with less technical complexity than in its main topic.

## 2.1 Massive spin-1 field

As said, to study a Proca field is equivalent to study a theory of a massive photon. It is described, therefore, by the lagrangian of the electromagnetic field plus a mass term proportional to $m_\gamma^2$, the square of the putative photon mass in our analogy:

$$\mathcal{L} = -\frac{1}{2}(\nabla_\mu A_\nu - \nabla_\nu A_\mu)(\nabla^\mu A^\nu - \nabla^\nu A^\mu) + m_\gamma^2 A_\mu A^\mu + \kappa A_\mu J^\mu \,, \tag{2.1}$$

where $J^\mu$ is the source of the field, $\kappa$ is its coupling constant and $\nabla_\mu$ is the covariant derivative, containing the information pertaining to the backgorund over which we are studying this field. This leads to the Proca equations:

$$\Box A^\nu - \nabla_\mu \nabla^\nu A^\mu - m_\gamma^2 A^\nu = \kappa J^\nu \,. \tag{2.2}$$

Taking the covariant derivative $\nabla_\nu$ of this equation we obtain the equivalent of the Lorentz gauge condition

$$\nabla_\nu A^\nu = -\frac{\kappa}{m_\gamma^2}\nabla_\nu J^\nu \,. \tag{2.3}$$

It is noteworthy that the presence of the mass term leads, in general, to the possible non-conservation of the source term (or 4-current), unlike what occurs for the massless theory (although conserved currents will still be useful for our analysis). Another consequence is that there isn't an extra gauge transformation that can be made after relation (2.3). In other words, we can only reduce the 4 original degrees of freedom of the field $A^\mu$ to 3 degrees of freedom, instead of the 2 degrees existent in the massless theory. This makes sense when thought of in context of the degrees of freedom of the particle represented by the field $A^\mu$ in each case: a massless particle always has two possible states, corresponding to its helicity, while a massive particle has $2s+1$ states (3 for $s=1$).

We would expect, however, that the third degree of freedom disappeared when considering the limit where the photon's mass is zero, which would be equivalent to the usual, zero mass photon theory. To ascertain this, we follow the approach in [14] and start by considering a free (sourceless) massive photon over a Minkowski background. The Proca and gauge equations reduce, respectively, to

$$(\Box - m_\gamma^2)A^\mu = 0 \quad , \quad \partial_\mu A^\mu = 0\,. \tag{2.4}$$

The solutions to these equations are of the form $A^\mu = \epsilon^\mu(k)e^{ik\cdot x}$, where $k^\mu$ is the momentum 4-vector, having norm $k_\mu k^\mu = -m_\gamma^2$. We pick a frame where $k = (\omega, 0, 0, k_3)$, such that $\omega^2 = k_3^2 + m_\gamma^2$ and proceed to write possible orthonormal solutions for $\epsilon^\mu(k)$, keeping in mind that the Lorentz gauge condition demands that $\epsilon_\mu(k)k^\mu = -\omega J^0 + k_3 J^3 = 0$. Two of them correspond to the transverse modes already known from the massless photon theory (and electromagnetic theory):

$$\epsilon^{(1)}(k) = (0,1,0,0) \quad , \quad \epsilon^{(2)}(k) = (0,0,1,0)\,, \tag{2.5}$$

while the third possible solution is, unlike the others, a longitudinal mode:

$$\epsilon^{(3)}(k) = \frac{1}{m_\gamma}(-k_3, 0, 0, \omega)\,. \tag{2.6}$$

Having this, we can compute the coupling of each of these solutions to a conserved source (as one would be required to have in the massless theory). While for $i = 1,2$ we have $\epsilon_\mu^{(i)}(k)J^\mu = J^i$, as for a massless photon, for the third mode we have:

$$\epsilon_\mu^{(3)}(k)J^\mu = \frac{1}{m_\gamma}(-k_3 J^0 + \omega J^3) = \frac{k_3}{\omega m_\gamma}(-\omega J^0 + \frac{\omega^2}{k_3}J^3) = \frac{m_\gamma}{\omega}J^3\,, \tag{2.7}$$

which goes to zero with the photon's mass. This shows that the zero mass limit of a massive photon is indeed equivalent to a massless photon, as we would have hoped.

## 2.2 Solution of Proca's equations

Having (2.2) and (2.3), we now want to specify the covariant derivatives to a Schwarzschild black hole background and solve the resulting equations. We will start by making a separation of variables between the angular ones ($\theta$ and $\phi$) and $t$ and $r$.

Spherical harmonics are commonly used for the separation of variables in equations pertaining to scalar fields in systems with spherical symmetry (as is our case). These functions, dependent on the variables that define a sphere, $\theta$ and $\phi$, form an orthonormal basis of square-integrable functions. This means that, supposing we have an equation for an unknown scalar function $\Psi(t,r,\theta,\varphi)$, $\Psi$ can be written as a linear combination of spherical harmonics:

$$\Psi(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} H_{lm}(t,r)Y_{lm}(\theta,\varphi)\,, \tag{2.8}$$

where $l$ and $m$ are two parameters that define spherical harmonics. The previously 4-variable PDE on $\Psi$ can then be simplified to merely a 2-variable equation on $H_{lm}$, simplifying the search for a solution.

When dealing with vector instead of scalar fields, the method is similar. As done in [23], we must now decompose $A^\mu$ in the following fashion:

$$\mathbf{A} = \sum_{lm}\sum_{i=0}^{3} h_{lm}^{(i)}(t,r)\mathbf{Z}_{lm}^{(i)}\,, \tag{2.9}$$

where $h_{lm}^{(i)}$ are four new unknown functions and $\mathbf{Z}_{lm}^{(i)}$ are the vector spherical harmonics, given by:

$$\mathbf{Z}_{lm}^{(0)} = \begin{pmatrix} Y_{lm} \\ 0 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{Z}_{lm}^{(1)} = \begin{pmatrix} 0 \\ Y_{lm}/f(r) \\ 0 \\ 0 \end{pmatrix},\ \mathbf{Z}_{lm}^{(2)} = \frac{r}{l(l+1)}\begin{pmatrix} 0 \\ 0 \\ \frac{\partial Y_{lm}}{\partial\theta} \\ \frac{\partial Y_{lm}}{\partial\phi} \end{pmatrix},\ \mathbf{Z}_{lm}^{(3)} = \frac{r}{l(l+1)}\begin{pmatrix} 0 \\ 0 \\ \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\phi} \\ -\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \end{pmatrix}, \tag{2.10}$$

where $f(r) = \left(1 - \frac{2M}{r}\right)$. These vectors can be shown to be orthonormal to each other, with respect to the metric $\eta = \text{diag}(1, f^{-2}(r), r^2, r^2\sin^2\theta)$ and the inner product:

$$\langle \mathbf{T}, \mathbf{S}\rangle = \int d\Omega\, T_\alpha^* \eta^{\alpha\beta} S_\beta\,. \tag{2.11}$$

In the same way, we can decompose the source term in these spherical harmonics:

$$\mathbf{J} = \sum_{lm}\sum_{i=0}^{3} S_{lm}^{(i)}(t,r)\mathbf{Z}_{lm}^{(i)}\,, \tag{2.12}$$

where the $S_{lm}^{(i)}$ functions will be specified in section 2.3.

Restricting ourselves to the simplest case of $l = 0$, in which $h^{(2)}(t,r) = h^{(3)}(t,r) = 0$, the only remaining equations are the $t$, $r$ and gauge, respectively:

$$\frac{(1-\frac{2M}{r})}{r}\frac{\partial^2 h_0}{\partial r^2}-\frac{m_\gamma^2}{r}h_0(t,r)-\frac{(1-\frac{4M}{r})}{r^2(1-\frac{2M}{r})}\frac{\partial h_1}{\partial t}-\frac{1}{r}\frac{\partial^2 h_1}{\partial t\partial r}=\kappa S^{(0)}(t,r)\,, \tag{2.13a}$$

$$-\frac{1}{r^2}\frac{\partial h_0}{\partial t}+\frac{1}{r}\frac{\partial^2 h_0}{\partial t\partial r}-\frac{1}{r\left(1-\frac{2M}{r}\right)}\frac{\partial^2 h_1}{\partial t^2}-\frac{m_\gamma^2}{r}h_1(t,r)=\kappa S^{(1)}(t,r)\,, \tag{2.13b}$$

$$-\frac{1}{r\left(1-\frac{2M}{r}\right)}\frac{\partial h_0}{\partial t}+\frac{1}{r}\frac{\partial h_1}{\partial r}+\frac{1}{r^2}h_1(t,r)=-\frac{\kappa}{m_\gamma^2}\left(\frac{\partial S^{(1)}}{\partial r}+\frac{2}{r}S^{(1)}(t,r)\right), \tag{2.13c}$$

where we are defining $h_i(t,r)\equiv h_{lm}^{(i)}(t,r)$ from here on, for the sake of simplicity. The next step for solving our system of differential equations is to take its Fourier transform with respect to the $t$ variable. This makes it so we now have a system of coupled ODEs with respect to $r$:

$$\frac{(1-\frac{2M}{r})}{r}\frac{d^2 h_0}{dr^2}-\frac{m_\gamma^2}{r}h_0(\omega,r)+\frac{i\omega\left(1-\frac{4M}{r}\right)}{r^2(1-\frac{2M}{r})}h_1(\omega,r)+\frac{i\omega}{r}\frac{dh_1}{dr}=\kappa S^{(0)}(\omega,r)\,, \tag{2.14a}$$

$$\frac{i\omega}{r^2}h_0(\omega,r)-\frac{i\omega}{r}\frac{dh_0}{dr}+\frac{\omega^2}{r\left(1-\frac{2M}{r}\right)}h_1(\omega,r)-\frac{m_\gamma^2}{r}h_1(\omega,r)=\kappa S^{(1)}(\omega,r)\,, \tag{2.14b}$$

$$\frac{i\omega}{r\left(1-\frac{2M}{r}\right)}h_0(\omega,r)+\frac{1}{r}\frac{dh_1}{dr}+\frac{1}{r^2}h_1(\omega,r)=-\frac{\kappa}{m_\gamma^2}\left(\frac{dS^{(1)}}{dr}+\frac{2}{r}S^{(1)}(\omega,r)\right), \tag{2.14c}$$

where, effectively, we redefined $h_i(t,r)=h_i(\omega,r)e^{-i\omega t}$.

Finally, we can solve the gauge equation for $h_0(r)$ in terms of $h_1(r)$ and its derivatives (2.15) and, substituting this new expression in the radial equation, obtain a 2nd order ODE for $h_1(r)$ (2.16).

$$h_0(\omega,r)=\frac{i\left(1-\frac{2M}{r}\right)}{\omega}\frac{dh_1}{dr}+\frac{i\left(1-\frac{2M}{r}\right)}{r\omega}h_1(\omega,r)+\kappa e^{i\omega t}\frac{ir\left(1-\frac{2M}{r}\right)}{m_\gamma^2\omega}\frac{dS^{(1)}}{dr}+\kappa e^{i\omega t}\frac{2i\left(1-\frac{2M}{r}\right)}{m_\gamma^2\omega}S^{(1)}(r)\,, \tag{2.15}$$

$$\begin{aligned}&\frac{d^2h_1}{dr_*{}^2}+\left(\omega^2-\frac{\left(1-\frac{2M}{r}\right)\left(m_\gamma^2r^3-6M+2r\right)}{r^3}\right)h_1(\omega,r_*)=\\&=\frac{\kappa\left(1-\frac{2M}{r}\right)}{m_\gamma^2r^2}\left((-8M+m_\gamma^2r^3+2r)S^{(1)}(\omega,r)-2r^2\left(1-\frac{M}{r}\right)\frac{dS^{(1)}}{dr}-r^3\left(1-\frac{2M}{r}\right)\frac{d^2S^{(1)}}{dr^2}\right).\end{aligned} \tag{2.16}$$

Note that, in equation (2.16) we have changed the variable of the function $h_1$ to the tortoise coordinate, defined by $\frac{dr_*}{dr}=\frac{1}{1-\frac{2M}{r}}$. This is done so that our equation can have the format of an inhomogeneous wave equation (2.17), with some potential $V$,

$$\frac{d^2\Psi}{dr_*^2}+(\omega^2-V(r))\Psi(\omega,r)=S(\omega,r)\,, \tag{2.17}$$

which will be useful when calculating exact or asymptotic solutions.

## 2.3 Radial infall source term

Having obtained (2.16), we only have to specify its rhs to be able to solve it. The $S^{(i)}$ functions we want to find are, as defined in (2.12), the result of the decomposition of the 4-current into vector spherical harmonics. Therefore, we can obtain them by taking the inner product between $\mathbf{J}$ and each of the spherical harmonics:

$$S^{(i)}(t,r) = \langle \mathbf{Z}^{(i)}, \mathbf{J} \rangle \,. \tag{2.18}$$

The 4-current $\mathbf{J}$, in its turn, is assumed to be that of a particle of charge $q$ following a geodesic of the background metric, in our case Schwarzschild, that is:

$$\begin{aligned} J^\mu &= q \int d\tau \, \frac{dz^\mu}{d\tau} \delta^{(4)}(x - z(\tau)) = q \frac{dT}{d\tau} \frac{dz^\mu}{dt} \frac{\delta(r - R(t))}{r^2} \frac{\delta(\theta - \Theta(t))}{\sin\theta} \delta(\phi - \Phi(t)) \\ &= q \frac{dT}{d\tau} \frac{dz^\mu}{dt} \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \,, \end{aligned} \tag{2.19}$$

where $z(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ is the parametrization of the worldline of the charged particle.

With this, we can compute the source functions:

$$\begin{aligned} S^{(i)}(t,r) &= q \frac{dT}{d\tau} \int d\Omega \, Z_\alpha^{(i)*}(r,\theta,\phi) \eta^{\alpha\beta}(r,\theta) g_{\beta\rho}(r,\theta) \frac{dz^\rho}{dt} \frac{\delta(r - R(t))}{r^2} \delta^{(2)}(\Omega - \Omega(t)) \\ &= q \frac{dT}{d\tau} Z_\alpha^{(i)*}(r, \Theta(t), \Phi(t)) \eta^{\alpha\beta}(r, \Theta(t)) g_{\beta\rho}(r, \Theta(t)) \frac{dz^\rho}{dt} \frac{\delta(r - R(t))}{r^2} \,. \end{aligned} \tag{2.20}$$

An interesting example to consider for the 4-current is that of a highly relativistic particle falling radially into the black hole. As for this case $\frac{dR}{dt} \neq 0$, we can write the Fourier transform of the source functions as:

$$\begin{aligned} S^{(i)}(\omega,r) &= \int_{-\infty}^{\infty} dt \, e^{-i\omega t} S^{(i)}(t,r) \\ &= q e^{-i\omega T(r)} \frac{dT}{d\tau} Z_\alpha^{(i)*}(r, \Theta(t), \Phi(t)) \eta^{\alpha\beta}(r, \Theta(t)) g_{\beta\rho}(r, \Theta(t)) \frac{dz^\rho}{dt} \frac{1}{\frac{dR}{dt} r^2} \,. \end{aligned} \tag{2.21}$$

Substituting the following definitions for the radial infall:

$$\frac{dT}{d\tau} = \frac{1}{1 - \frac{2M}{r}}, \; \frac{dR}{dt} = -\left(1 - \frac{2M}{r}\right), \; \Theta(t) = 0, \; \Phi(t) = 0 \,, \tag{2.22}$$

we can finally write the explicit expressions for $S^{(i)}(r)$:

$$S^{(0)}(\omega,r) = S^{(1)}(\omega,r) = \frac{q}{2\pi r^2 \left(1 - \frac{2M}{r}\right)} \,, \tag{2.23a}$$

$$S^{(2)}(\omega,r) = S^{(3)}(\omega,r) = 0 \,, \tag{2.23b}$$

and the source term in equation (2.16):

$$S(\omega,r) = -\kappa q e^{i\omega T(r)} \frac{\left(12M^2 - 2Mr\left(m_\gamma^2 r^2 + 4\right) + m_\gamma^2 r^4\right)}{2\sqrt{2}\pi m_\gamma^2 r^5 \left(1 - \frac{2M}{r}\right)}\,, \tag{2.24}$$

where the expression for $T(r)$ is defined via $\frac{dT}{dr} = -\left(1 - \frac{2M}{r}\right)^{-1}$.

# Chapter 3

# dRGT massive gravity

## 3.1 Massive gravity pre-dRGT

The most generic lagrangian for a massive graviton can be written as follows:

$$S = \frac{1}{64\pi}\int d^4x(-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu}+2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho}-2\partial_\nu h^{\mu\nu}\partial_\mu h+\partial_\mu h\partial^\mu h+m_g^2(h^2-\kappa h_{\mu\nu}h^{\mu\nu})+32\pi G h_{\mu\nu}T^{\mu\nu})\,. \tag{3.1}$$

From this we can obtain the equations of motion for $h_{\mu\nu}$:

$$\Box h^{\mu\nu} - (\partial^\nu\partial_\rho h^{\mu\rho} + \partial^\mu\partial_\rho h^{\nu\rho}) + \eta^{\mu\nu}\partial_\rho\partial_\sigma h^{\rho\sigma} + \partial^\mu\partial^\nu h - \eta^{\mu\nu}\Box h = m_g^2(h^{\mu\nu} - \kappa\eta^{\mu\nu}h) - 16\pi G T^{\mu\nu}\,, \tag{3.2}$$

and the following conditions by taking the gradient $\partial^\nu$ and the trace of (3.2), respectively.

$$\partial^\mu h_{\mu\nu} = \kappa\partial_\nu h \quad , \quad 2(1-\kappa)\Box h + (1-4\kappa)m_g^2 h = 16\pi G T\,. \tag{3.3}$$

As $h_{\mu\nu}$ is a symmetric, 2-rank tensor, we know from the start that it has 10 degrees of freedom. Going further, the gradient of (3.2) imposes four conditions on the field, reducing the degrees of freedom to 6. Comparing this to the expected 5 degrees of freedom of the massive graviton ($2s+1$, spin $s=2$), we see that we have one too much. To match these two values, we must take $\kappa = 1$, so that the trace equation corresponds to a further constraint on the trace of the metric perturbation. This choice of value for $\kappa$ is exactly the one made by Fierz and Pauli in [5]. This gives us, then, the following equations:

$$h = -\frac{16\pi G}{3m_g^2}T \quad , \quad \partial^\mu h_{\mu\nu} = -\frac{16\pi}{3m_g^2}\partial_\nu h\,, \tag{3.4}$$

$$(\Box - m_g^2)h_{\mu\nu} = -16\pi G\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T + \frac{1}{3m_g^2}\partial_\mu\partial_\nu T\right). \tag{3.5}$$

Determining $h_{\mu\nu}$, we can now compare it with the massless graviton (i.e. General Relativity) case in a well-known example. We shall consider a perturbation caused by a static point particle of mass

$M$, that is, with $T_{\mu\nu} = M\delta^{(3)}(\vec{x})\delta^0_\mu\delta^0_\nu$, as done in [24]. The classic case would result in the linearised Schwarzschild solution:

$$h_{00}(x) = \frac{2GM}{r}\,, \quad h_{rr}(x) = 0\,, \quad h_{ij}(x) = \frac{2GM}{r}\,. \tag{3.6}$$

As for the massive case, substituting the previous stress-energy tensor into (3.5) we obtain

$$(\Box - m_g^2)h_{00}(x) = -16\pi G\frac{2}{3}\delta^{(3)}(\vec{x})\,, \tag{3.7a}$$

$$(\Box - m_g^2)h_{0i}(x) = 0\,, \tag{3.7b}$$

$$(\Box - m_g^2)h_{ij}(x) = -16\pi G\left(\frac{1}{3}\delta^{(3)}(\vec{x}) - \frac{1}{3m_g^2}\partial_i\partial_j\delta^{(3)}(\vec{x})\right). \tag{3.7c}$$

To solve these equations we start by applying a Fourier transform in all four variables, defining $H_{\mu\nu} = \mathcal{F}\left(h_{\mu\nu}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\xi_\alpha x^\alpha} h_{\mu\nu}(x)$, with $\xi$ the Fourier conjugate of the spacetime coordinates $\mathbf{x}$:

$$H_{00}(\xi) = \frac{4}{3}\frac{16\pi GM}{2\pi}\frac{1}{\xi^\alpha\xi_\alpha + m_g^2}\delta(\xi^0)\,, \tag{3.8a}$$

$$H_{0i}(\xi) = 0\,, \tag{3.8b}$$

$$H_{ij}(\xi) = \frac{2}{3}\frac{16\pi GM}{2\pi}\frac{1}{\xi^\alpha\xi_\alpha + m_g^2}\left(\delta_{ij} + \frac{1}{m_g^2}\xi_i\xi_j\right)\delta(\xi^0)\,. \tag{3.8c}$$

Using relations (3.9),

$$\int \frac{d^3\xi}{(2\pi)^3}e^{i\vec{\xi}\cdot\vec{x}}\frac{1}{\vec{\xi}^2 + m^2} = \frac{1}{4\pi}\frac{e^{-mr}}{r}\,, \tag{3.9a}$$

$$\begin{aligned}\int \frac{d^3\xi}{(2\pi)^3}e^{i\vec{\xi}\cdot\vec{x}}\frac{\xi_i\xi_j}{\vec{\xi}^2 + m^2} &= -\partial_i\partial_j\int \frac{d^3\xi}{(2\pi)^3}e^{i\vec{\xi}\cdot\vec{x}}\frac{1}{\vec{\xi}^2 + m^2} \\ &= \frac{1}{4\pi}\frac{e^{-mr}}{r}\left[\frac{1}{r^2}(1+mr)\delta_{ij} - \frac{1}{r^4}(3+3mr+m^2r^2)x_ix_j\right].\end{aligned} \tag{3.9b}$$

we can transform it back, which gives us:

$$h_{00}(x) = \frac{4}{3}\frac{2GM}{r}e^{-mr}\,, \tag{3.10a}$$

$$h_{0i}(x) = 0\,, \tag{3.10b}$$

$$h_{ij}(x) = \frac{2}{3}\frac{2GM}{r}e^{-m_g r}\left(\delta_{ij}\frac{1 + m_g r + m_g^2 r^2}{m_g^2 r^2} - \frac{3 + 3m_g r + m_g^2 r^2}{m_g^2 r^4}x_ix_j\right), \tag{3.10c}$$

which, transforming into spherical coordinates through

$$\left(F(r) + r^2G(r)\right)dr + r^2F(r)d\Omega^2 = \left(F(r)\delta_{ij} + G(r)x_ix_j\right)dx^idx^j\,, \tag{3.11}$$

corresponds to:

$$h_{00}(r) = \frac{4}{3}\frac{2GM}{r}e^{-mr}\,, \tag{3.12a}$$

$$h_{rr}(r) = -\frac{4}{3}\frac{2GM}{r}e^{-mr}\frac{1+mr}{m^2r^2}\,, \tag{3.12b}$$

$$h_{\theta\theta}(r) = r^2\frac{2}{3}\frac{2GM}{r}e^{-mr}\frac{1+mr+m^2r^2}{m^2r^2}\,, \tag{3.12c}$$

$$h_{\phi\phi}(r,\theta) = \sin^2\theta h_{\theta\theta}(r)\,. \tag{3.12d}$$

Considering the zero mass limit $m_g \to 0$, we can easily see that the space metric components diverge, as $\lim_{m_g\to 0} h_{ij} = \frac{2}{3}\frac{2GM}{m_g^2 r^3}e^{-m_g r}$. However, the term in the Fourier transform of the space components that causes this divergence can be shown to not contribute in such limit [24], giving us, instead:

$$h_{00}(r) = \frac{4}{3}\frac{2GM}{r}e^{-mr}\,, \tag{3.13a}$$

$$h_{rr}(r) = \frac{2}{3}\frac{2GM}{r}e^{-mr}\,, \tag{3.13b}$$

$$h_{\theta\theta}(r) = \frac{2}{3}\frac{2GM}{r}e^{-m_g r}\,, \tag{3.13c}$$

$$h_{\phi\phi}(r,\theta) = h_{\theta\theta}(r)\sin^2\theta\,. \tag{3.13d}$$

If we now consider this metric in the Newtonian limit, we get that the corresponding Newtonian potential now is:

$$\phi_{Nm}(r) = -\frac{1}{2}h_{00} = -\frac{4}{3}\frac{GM}{r}e^{-m_g r}\,. \tag{3.14}$$

Comparing this expression with the regular Newtonian potential, $\phi_N(r) = -\frac{GM}{r}$ we notice two differences. First of all, there is now an exponential factor, dependent on the mass of the graviton. This is expected, being a typical feature of the potential of interactions mediated by massive particles. Moreover, taking the limit of zero graviton mass this factor disappears, and we once again obtain the classical dependency on $r$. The second difference is a factor of $\frac{4}{3}$ which does not disappear in the zero mass limit. To correct this, we would need to redefine the coupling constant between the field $h_{\mu\nu}$ and the stress-energy tensor, which we took to be $32\pi G$ in equation (3.1). Choosing it, instead, to be $24\pi G$, the new potential should become exactly the Newtonian one, in the zero mass limit.

However, making such a change affects other results that can be obtained from this theory, namely the value for the deflection of a light ray by a massive body, a classical test of GR. To compute this value we must consider the variation of the $\phi$ and $r$ coordinates with respect to the affine parameter

$$\frac{d\phi}{\lambda} = p^\phi = g^{\phi\phi}p_\phi = g^{\phi\phi}L\,, \tag{3.15}$$

$$0 = E^2g^{tt} + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + L^2g^{\phi\phi} \Leftrightarrow \frac{dr}{d\lambda} = -\sqrt{-E^2g^{tt} - L^2g^{\phi\phi}}\sqrt{g^{rr}}\,, \tag{3.16}$$

where we assumed a generic diagonal metric dependent only on $r$ and $\theta$, and $L$ and $E$ are, respectively, the angular momentum and energy at inifinty of the incoming photon. Fixing, further, $\theta = \frac{\pi}{2}$, we obtain an expression for $\frac{d\phi}{dr}$ dependent only on $r$:

$$\frac{d\phi}{dr} = -\frac{g^{\phi\phi}\sqrt{g_{rr}}}{\sqrt{-b^{-2}g^{tt} - g^{\phi\phi}}}\,, \tag{3.17}$$

where $b = \frac{L}{E}$ is the impact parameter of the photon. Specifying the metric for each case we obtain, using the approximation $\frac{M}{r} \ll 1$:

$$\left.\frac{d\phi}{dr}\right|_{GR} = -\frac{1}{r^2}\frac{\sqrt{1+\frac{2GM}{r}}}{b^{-2}\frac{1}{1-\frac{2GM}{r}} - \frac{1}{r^2}} \Leftrightarrow \left.\frac{d\phi}{du}\right|_{GR} \simeq \left(b^{-2} - u^2 + 2GMu^3\right)^{-\frac{1}{2}}\,, \tag{3.18}$$

$$\left.\frac{d\phi}{dr}\right|_{FP} = -\frac{1}{r^2}\frac{\sqrt{1+\frac{4GM}{3r}}}{\left(1+\frac{4GM}{3r}\right)}b^{-2}\frac{1}{1-\frac{8GM}{3r}} - \frac{1}{r^2}\frac{1}{\left(1+\frac{4GM}{3r}\right)} \Leftrightarrow \left.\frac{d\phi}{du}\right|_{FP} \simeq \left(1-2GMu\right)\left(b^{-2} - u^2 + 4GMu^3\right)^{-\frac{1}{2}}\,, \tag{3.19}$$

where $u = \frac{1}{r}$. To compute the light deflection we wish to integrate the above expressions from an infinite radius, whence the photon comes, until the minimum radius $r_0$ of the trajectory of the photon, which corresponds to the midpoint of its trajectory. From this point on, the trajectory is symmetric, as is the value of the deflection. Therefore, the total light deflection angle is given by:

$$\Delta\phi = 2\int_{\infty}^{r_0} dr\,\frac{d\phi}{dr} = 2\int_0^{1/r_0} du\,\frac{d\phi}{du}\,. \tag{3.20}$$

The minimum radius is given by the root of the denominator of the above expressions (3.18) and (3.19). Computing this integral for $M = 0$, we merely obtain $\Delta\phi = \pi$. This corresponds to a photon passing through empty space, having a change of angle of $\pi$ with respect to the centre of reference. With non-zero $M$ the integration is not as direct. Changing the variables of integration, respectively for equations (3.18) and (3.19), to $y = u\left(1 - GMu\right)$ and $y = u\left(1 - \frac{3}{2}GMu\right)$, we obtain:

$$\Delta\phi|_{M,GR} = \Delta\phi|_{M,FP} = 2\int_0^{1/b} dy\,\frac{1+2GMy}{\sqrt{b^{-2}+y^2}}\,, \tag{3.21}$$

where we define $\Delta\phi|_M$ to be the deflection by the massive body itself. This value has, since the experiment of Eddington in 1919, been measured to a great accuracy to be $\frac{4GM_\odot}{R_\odot}$ for a light ray grazing the surface of the Sun. While the GR value does correspond to it, due to the rescaling of Newton's constant $G \to \frac{3}{4}G$ done previously the value for the Fierz-Pauli theory is now found to be $\Delta\phi|_{M,FP} = \frac{3GM_\odot}{R_\odot}$. This result, discovered in 1970 ([6, 7]), is not only evidence of a discontinuity (named vDVZ for its original discoverers) between the the linearised massless graviton theory and the zero mass limit of the Fierz-Pauli theory but also that the latter cannot correspond to reality.

The reason for this discontinuity can be easily understood by studying the several modes of the massive graviton and their respective couplings to the stress-energy tensor, as we did in section 2.1 for the Proca field. For a free graviton the field and gauge equations reduce to

$$(\Box - m_g^2)h_{\mu\nu} = 0 \quad , \quad \partial^\mu h_{\mu\nu} \quad , \quad h = 0\,. \tag{3.22}$$

The general solution of the field equations is, then, $h_{\mu\nu} = \epsilon_{\mu\nu}(k)e^{ik\cdot x}$, where, again, $\mathbf{k} = (\omega, 0, 0, k_3)$ $\epsilon_{\mu\nu}(k)$ are symmetric 2-tensors, restricted by the gauge equations in the following manner:

$$k^\mu \epsilon_{\mu\nu} = 0 \quad , \quad \epsilon = \eta^{\mu\nu}\epsilon_{\mu\nu} = 0\,. \tag{3.23}$$

Two of the solutions we can build are the same as for the massless graviton,

$$\epsilon^{(a)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \epsilon^{(b)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,, \tag{3.24}$$

while the other three, orthonormal modes are

$$\epsilon^{(c)} = -\frac{1}{\sqrt{2}m_g}\begin{pmatrix} 0 & k_3 & 0 & 0 \\ k_3 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \end{pmatrix} \quad , \quad \epsilon^{(d)} = -\frac{1}{\sqrt{2}m_g}\begin{pmatrix} 0 & 0 & k_3 & 0 \\ 0 & 0 & 0 & 0 \\ k_3 & 0 & 0 & -\omega \\ 0 & 0 & -\omega & 0 \end{pmatrix}\,, \tag{3.25}$$

$$\epsilon^{(e)} = \frac{1}{\sqrt{2}m_g^2}\begin{pmatrix} k_3^2 & 0 & 0 & -\omega k_3 \\ 0 & -\frac{m_g^2}{2} & 0 & 0 \\ 0 & 0 & -\frac{m_g^2}{2} & 0 \\ -\omega k_3 & 0 & 0 & \omega^2 \end{pmatrix}\,, \tag{3.26}$$

where we identify $\epsilon^{(c)}$ and $\epsilon^{(d)}$ with the vector and $\epsilon^{(e)}$ with the scalar modes. We now calculate, as before, the coupling with the stress-energy of each mode:

$$\epsilon^{(a)}_{\mu\nu}T^{\mu\nu} = \frac{1}{\sqrt{2}}(T_{11} - T_{22})\,, \tag{3.27a}$$

$$\epsilon^{(b)}_{\mu\nu}T^{\mu\nu} = \sqrt{2}T_{12}\,, \tag{3.27b}$$

$$\begin{aligned} \epsilon^{(c)}_{\mu\nu}T^{\mu\nu} &= -\frac{1}{\sqrt{2}m_g}(-2k_3T_{01} - 2\omega T_{13}) = \frac{\sqrt{2}}{m_g\omega}(\omega k_3 T_{01} + k_3^2 T_{13} + m_g^2 T_{13}) \\ &= \frac{\sqrt{2}}{\omega m_g}k_3(\omega T_{01} + k_3 T_{13}) + \frac{\sqrt{2}m_g}{\omega}T_{13} = \frac{\sqrt{2}m_g}{\omega}T_{13}\,, \end{aligned} \tag{3.28a}$$

$$\epsilon^{(d)}_{\mu\nu}T^{\mu\nu} = \frac{\sqrt{2}m_g}{\omega}T_{23}\,, \tag{3.28b}$$

$$\epsilon^{(e)}_{\mu\nu}T^{\mu\nu} = \frac{1}{\sqrt{2}m_g^2}\left(k_3^2 T_{00} + \omega^2 T_{33} + 2\omega k_3 T_{03} - \frac{m_g^2}{2}(T_{11}+T_{22})\right) = \\ = \frac{1}{\sqrt{2}m_g^2}\left(\frac{1}{k_3^2}(k_3^2-\omega^2)^2 T_{00} - \frac{m_g^2}{2}(T_{11}+T_{22})\right) = \frac{m_g^2}{\sqrt{2}k_3^2}T_{00} - \frac{1}{2\sqrt{2}}(T_{11}+T_{22})\,. \tag{3.29}$$

As it had happened for the Proca case, the coupling of the modes we identified with the massless ones remains when taking the zero mass limit. Also as before, two of the new modes, the ones corresponding to $s = \pm 1$, go to zero with the mass. However, the coupling of the fifth, scalar mode does not fully disappear with the mass. This means that the zero mass limit of a massive graviton is not, as expected, a massless graviton but is, in fact, a massless graviton plus a scalar field (usually denominated galileon). It is this remaining field that causes the previously mentioned discontinuity.

The reason why this field does not disappear in said limit was discovered in 1972 [10] by Vainshtein. In his work he again studied the deflection of light by a massive body and discovered that inside a given region, defined by what is called the Vainshtein radius, the linear theory was not valid, as the nonlinear terms on the fields have values large enough to affect the final results. Moreover, these extra terms compensate the factors that caused the discontinuity (a process which is usually called the ”Vainshtein screening” of the galileon), giving, in the end, the correct value for the deflection.

Assuming the well functioning of this Vainshtein screening, we can, based on what we have already seen and on experimental observations, impose bounds on the mass of the graviton, which we present in table 3.1. One such bound comes from the dispersion relation for gravitational waves, calculated through the analysis of data of the recent events GW150914, GW151226 and GW170104 [25]. Other bounds come from precise measurements of the advance of the perihelion not only of Mercury (as in the classical test of GR) but also of Saturn and Mars [18], which provides the strongest bound on the mass of the graviton. For historical reasons, we also present the bound obtained by Finn and Sutton through the analysis of the orbital decay of binary pulsars [20]. This last one was obtained through a theory with the choice $\kappa = \frac{1}{2}$ in the action (3.1), instead of the FP value of $\kappa = 1$. This was shown to lead exactly to the classical linearised Einstein equations, giving, therefore, the correct result for the light deviation. However, it also causes the appearance of the previously avoided ghost, losing its viability.

| Phenomenon | Upper bound on $m_g$ ($10^{-23}eV/c^2$) |
|---|---|
| Binary pulsars [20] | $7600$ |
| GWs [25] | $7.7$ |
| Perihelion of Mercury [18] | $5.6$ |
| Perihelion of Mars [18] | $1.0$ |
| Perihelion of Saturn [18] | $46.0$ |
| Superradiant instabilities [26] | $5.0$ |

Table 3.1: Bounds on $m_g$; Some of the presented values will be explained later on.

But this is not the end of the road. To really discover the effects, unrealistic or according to our observations, of a massive graviton we still need to build a working nonlinear theory. We do this by combining General Relativity and the idea behind the Fierz-Pauli theory: the new lagrangian should have a not necessarily linear term dependent on the perturbation metric $h_{\mu\nu}$ and its first and second derivatives, which we assume to be the Ricci scalar $R$, and a potential dependent only in combinations of $h_{\mu\nu}$ times a constant, the graviton mass:

$$\mathcal{L} = R - \frac{m_g^2}{4} U(\eta_{\mu\nu}, h_{\mu\nu}, h_{\mu\nu}^2, ...) \,. \tag{3.30}$$

However, still in the same year this idea arose, it got an apparently fatal blow. When building this theory we must, as in the Fierz-Pauli theory, be aware of how we pick the now nonlinear mass term, so as to avoid the $6^{\text{th}}$ degree of freedom. In 1972, Boulware and Deser showed in [11] that, for a general potential $U$, not only the $6^{\text{th}}$ degree of freedom existed but it also allowed the existence of modes with negative kinetic energy, this mode becoming known as the Boulware-Deser ghost. Faced with this unphysical feature, the idea of a massive graviton was mostly abandoned, and work on this subject halted until quite recently.

## 3.2 dRGT and the disappearance of the ghost

In 2010, de Rham, Gabadadze and Tolley [12, 13] first proposed the dRGT massive gravity theory. In its original formulation, the theory was built over a Minkowski background and already in the decoupling (or zero mass) limit, meaning that we are considering *a priori* separate fields for the massless graviton ($h_{\mu\nu}$) and the galileon ($\pi$). The lagrangian of the theory is the same as in (3.30) but now with a potential given by $U = U(\eta_{\mu\nu}, H_{\mu\nu})$, $H_{\mu\nu}$ being a combination of the massless graviton and derivatives of the galileon. The idea is then to, for each order in the considered fields, pick the coefficient of each different combination of fields in such a way that the Boulware-Deser ghost does not arise. In other words, the nonlinear mass term is built by taking the process through which the Fierz-Pauli theory itself avoided the $6^{\text{th}}$ mode and generalizing it to all orders.

While they proved that this theory was ghost-free, it was not yet a full generalization of general relativity, as the theory assumes 1) to be already in the decoupling limit and 2) a flat, Minkowskian background. These two issues were soon solved by connecting massive gravity with another, apparently unrelated theory, that of bimetric gravity. In bimetric gravity one considers two (possibly) dynamical metrics, both with the usual general relativistic lagrangian, the Ricci scalar:

$$S = \int d^4x \sqrt{-g} \left[ R_g + \frac{\sqrt{-f}}{\sqrt{-g}} R_f \right] . \tag{3.31}$$

where $R_f$ refers to the $\mathbf{f}$ metric and likewise for $R_g$.

However, dRGT massive gravity assigns different roles to each metric. In our case of interest, $f_{\mu\nu}$ plays the role of the background metric (so far fixed to Minkowski, now completely generic), which, in this work, is assumed to be non-dynamical for simplicity. As for $g_{\mu\nu}$, it refers to a perturbation over said background (including, therefore, what so far we have been calling $h_{\mu\nu}$), being fully dynamical. Besides this, there is an extra term, corresponding to the mass term, built from a specific combination of both metrics, $\sqrt{g^{-1}f}$, defined by the relation $\sqrt{g^{-1}f}\sqrt{g^{-1}f} = g^{\mu\lambda} f_{\lambda\nu}$. All this results in the action (3.32):

$$S = \int d^4x\sqrt{-g}\Big[M_g^2 R_g - 2M_v^4 \sum_{n=0}^{4} \beta_n V_n(\sqrt{g^{-1}f})\Big] + S_m \,, \tag{3.32}$$

where $S_m$ is the matter action, $M_g^2$ and $M_v^2$ are coupling constants and:

$$V_0(\Pi) = 1 \,, \tag{3.33}$$

$$V_1(\Pi) = [\Pi] \,, \tag{3.34}$$

$$V_2(\Pi) = \frac{1}{2}\big([\Pi]^2 - [\Pi^2]\big) \,, \tag{3.35}$$

$$V_3(\Pi) = \frac{1}{6}\big([\Pi]^3 - 3[\Pi][\Pi^2] + 2[\Pi^3]\big) \,, \tag{3.36}$$

$$V_4(\Pi) = \text{det}(\Pi) \,, \tag{3.37}$$

$[\Pi]$ being the trace of the tensor $\Pi$.

This formulation of massive gravity was found to be equivalent to the original one and similarly ghost-free (we recommend [27, 28, 29] for more details). It is now the most commonly used for practical calculations, being the one we adopted for this work.

## 3.3 Features of dRGT

With the existence of a working theory of massive gravity confirmed, the next logical step would be to study its phenomenology, both in terms of the evolution of the universe, and of new features (when compared with GR) there might appear and with which we can test it. The former were the initial motivation of massive gravity, work on it producing promising results [3, 4]. Here in this work we will focus on the latter.

A major point of interest in dRGT massive gravity has been the study of black hole solutions. This is a much vaster topic than in GR, there being, for instance, non-Schwarzschild spherically symmetric metrics [30], among other variations. For an extensive review on this matter see [31]. Of greater interest to us, some of the solutions the dRGT theory has in common with GR, such as the Schwarzschild and the Kerr metrics, were found to be unstable [26], possibly compromising studies of phenomena related with such objects, like our own. In the Schwarzschild case, it was also found that these instabilities occur over very large timescales, larger than what would be worthy of concern. As for the Kerr case, superradiance was found to also lead to instabilities of the black hole. The experimental observation of such black holes, however, suggests that these instabilities must be somewhat controlled. This was found to impose a bound on the mass of the graviton, shown in the previous table 3.1.

Another point of interest to us is perturbation theory applied to dRGT massive gravity. Assuming a generic background, we can obtain the linearised "Einstein" equations, as done in [32]. We start by writing the full equations of motion, given by (dropping the $_g$ index from the curvature tensors and scalars):

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \frac{M_v^4}{M_g^2}\sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\rho}Y^{\rho}{}_{(n)\nu}(\sqrt{g^{-1}f}) = \frac{1}{2M_g^2}T_{\mu\nu}\,, \tag{3.38}$$

where $Y^{\rho}_{(n)\nu}(\Pi) = \sum_{r=0}^{n}(-1)^r(\Pi^{n-r})^{\rho}{}_{\nu}V_r(\Pi)$. This comes directly from varying the action (3.32) in order to the metric $g_{\mu\nu}$ (and remembering that we assume the second metric, $f_{\mu\nu}$ to be non-dynamical).

To linearise, we pose, as said before, $g_{\mu\nu} = f_{\mu\nu} + h_{\mu\nu}$ and, therefore, $(\sqrt{g^{-1}f})^{\rho}{}_{\nu} \simeq \delta^{\rho}_{\nu} - \frac{1}{2}f^{\rho\mu}h_{\mu\nu}$. From this, simple (although lengthy) substitution gives us:

$$\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma} + \frac{\mu^2}{2}(h_{\mu\nu} - f_{\mu\nu}h) = \frac{1}{2M_g^2}T^{(1)}_{\mu\nu} \quad ,\mu^2 = \frac{M_v^4}{M_g^2}(\beta_1 + 2\beta_2 + \beta_3)\,, \tag{3.39}$$

where $\mu$ is what we call the mass of the graviton, $T^{(1)}_{\mu\nu}$ is the first order part of the stress energy tensor and $\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}$ is the linearised Einstein equation differential operator for a generic background:

$$\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma} = -\frac{1}{2}\left[\overline{\Box}h_{\mu\nu} + \overline{\nabla}_{\mu}\overline{\nabla}_{\nu}h - \overline{\nabla}^{\sigma}\overline{\nabla}_{\nu}h_{\mu\sigma} - \overline{\nabla}^{\sigma}\overline{\nabla}_{\mu}h_{\nu\sigma} - \overline{g}_{\mu\nu}\overline{\Box}h + \overline{g}_{\mu\nu}\overline{\nabla}^{\rho}\overline{\nabla}^{\sigma}h_{\rho\sigma}\right]. \tag{3.40}$$

This corresponds to the equations of motion for a linearised massive graviton theory with $\kappa = \frac{1}{2}$ instead of the $\kappa = 1$ of the FP theory, as in the situation analyzed in [20], leading, as mentioned previously, to the clasical linearized Einstein equations. As this theory is now assured to be free of ghosts at all orders, this proves that the vDVZ discontinuity does not occur for dRGT massive gravity, meaning that the light deflection case will give the correct value.

Taking the covariant derivative $\overline{\nabla}^{\mu}$ and the trace of equation (3.39) we obtain, respectively:

$$\overline{\nabla}^{\mu}h_{\mu\nu} = \alpha\overline{\nabla}_{\nu}T\,, \tag{3.41a}$$

$$h = \alpha T\,, \tag{3.41b}$$

with $\alpha = -\frac{2M_g^2}{3\mu^2}$ which reduces the initial 10 degrees of freedom of the metric perturbation $h_{\mu\nu}$ to the 5 expected degrees of freedom of the massive graviton. It is this system of equations, (3.39) and (3.41), which will be our focus for the remainder of this work.

# Chapter 4

# Perturbation of Schwarzschild black hole in dRGT massive gravity

This chapter represents the main part of this work, which is the study of general perturbations of a Schwarzschild black hole in dRGT massive gravity. While the mass of the graviton imposes at least a slight correction in all multipole orders of the perturbation, here we focus chiefly on the most drastic alterations to the system in study. These are the excitation of modes that previously did not exist, namely in the polar monopole and dipole modes. We also present the perturbation equations of the axial sector for $l \geq 2$. In the end we specify the equations for two specific physical cases: those of a point-particle falling radially into the black hole and orbitting circularly around it.

## 4.1 Setup

Having proved that dRGT massive gravity is free of ghosts, we can now proceed to study the theory itself and, in particular, possible deviations from the GR theory. With this in mind, the system we wish to study is that of a body in free fall in the vicinity of a black hole. We will consider this body to be a point particle in geodesic motion (which will be specified later) and the black hole to correspond to the Schwarzschild solution. This system is described by equations (3.39) and (3.41) with the non-dynamical, background metric $f_{\mu\nu}$ coresponding to the line element:

$$ds^2 = -F(r)dt^2 + F(r)^{-1}dr^2 + r^2 d\Omega^2 \quad , \begin{cases} d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \\ F(r) = \left(1 - \frac{r_s}{r}\right) \end{cases} \quad , \tag{4.1}$$

where $r_s = 2M$ is the Schwarzschild radius and $M$ is the mass of the black hole.

Solving this system is not, however, a trivial matter. To do it, we will follow the formalism used originally by Regge and Wheeler [16] and, later, by Zerilli [15], and decompose the perturbation metric $h_{\mu\nu}$ in spherical harmonics. This is also similar to what we did in section 2.2, except now the spherical harmonic functions are of a tensorial nature instead of vectorial, and there being 10 orthonormal spherical harmonics[1] instead of 4. The perturbation metric is then given by:

$$
\begin{aligned}
\mathbf{h} = &= \sum_{lm} \Bigg[ F(r) H_{0lm}(t,r) \mathbf{a}^{(0)}_{lm} - i\sqrt{2} H_{1lm}(t,r) \mathbf{a}^{(1)}_{lm} + \frac{1}{F(r)} H_{2lm}(t,r) \mathbf{a}_{lm} \\
&- \frac{i}{r}\sqrt{2l(l+1)}\eta_{0lm}(t,r)\mathbf{b}^{(0)}_{lm} + \frac{1}{r}\sqrt{2l(l+1)}\eta_{1lm}(t,r)\mathbf{b}_{lm} \\
&+ \sqrt{\frac{1}{2} l(l+1)(l(l+1)-2)} G_{lm}(t,r)\mathbf{f}_{lm} + \left(\sqrt{2}K_{lm}(t,r) - \frac{l(l+1)}{\sqrt{2}} G_{lm}(t,r)\right)\mathbf{g}_{lm} \\
&- \frac{\sqrt{2l(l+1)}}{r} h_{0lm}(t,r)\mathbf{c}^{(0)}_{lm} + \frac{i\sqrt{2l(l+1)}}{r} h_{1lm}(t,r)\mathbf{c}_{lm} + \frac{\sqrt{2l(l+1)(l(l+1)-2)}}{2r} h_{2lm}(t,r)\mathbf{d}_{lm} \Bigg],
\end{aligned}
\tag{4.2}
$$

where the tensor spherical harmonics form an orthonormal basis with respect to te Minkowski metric. Their expressions, taken from [33], are:

$$
\mathbf{a}^{(0)}_{lm} = \begin{pmatrix} Y_{lm} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3a}
$$

$$
\mathbf{a}_{lm} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Y_{lm} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3b}
$$

$$
\mathbf{a}^{(1)}_{lm} = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & Y_{lm} & 0 & 0 \\ Y_{lm} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.3c}
$$

$$
\mathbf{b}^{(0)}_{lm} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & \partial_\theta Y_{lm} & \partial_\phi Y_{lm} \\ 0 & 0 & 0 & 0 \\ \partial_\theta Y_{lm} & 0 & 0 & 0 \\ \partial_\phi Y_{lm} & 0 & 0 & 0 \end{pmatrix}, \tag{4.3d}
$$

$$
\mathbf{b}_{lm} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta Y_{lm} & \partial_\phi Y_{lm} \\ 0 & \partial_\theta Y_{lm} & 0 & 0 \\ 0 & \partial_\phi Y_{lm} & 0 & 0 \end{pmatrix}, \tag{4.3e}
$$

[1]instead of 16, as the perturbation metric is symmetric

$$\mathbf{g}_{lm} = \frac{r^2}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & Y_{lm} & 0 \\ 0 & 0 & 0 & \sin^2\theta Y_{lm} \end{pmatrix}, \tag{4.3f}$$

$$\mathbf{f}_{lm} = \frac{r^2}{\sqrt{2l(l+1)(l(l+1)-2)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_{lm} & X_{lm} \\ 0 & 0 & X_{lm} & -\sin^2\theta W_{lm} \end{pmatrix}, \tag{4.3g}$$

$$\mathbf{c}^{(0)}_{lm} = \frac{r}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta}\partial_\phi Y_{lm} & -\sin\theta\partial_\theta Y_{lm} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sin\theta}\partial_\phi Y_{lm} & 0 & 0 & 0 \\ -\sin\theta\partial_\theta Y_{lm} & 0 & 0 & 0 \end{pmatrix}, \tag{4.3h}$$

$$\mathbf{c}_{lm} = \frac{ir}{\sqrt{2l(l+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sin\theta}\partial_\phi Y_{lm} & -\sin\theta\partial_\theta Y_{lm} \\ 0 & \frac{1}{\sin\theta}\partial_\phi Y_{lm} & 0 & 0 \\ 0 & -\sin\theta\partial_\theta Y_{lm} & 0 & 0 \end{pmatrix}, \tag{4.3i}$$

$$\mathbf{d}_{lm} = \frac{ir^2}{\sqrt{2l(l+1)(l(l+1)-2)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sin\theta}X_{lm} & \sin\theta W_{lm} \\ 0 & 0 & \sin\theta W_{lm} & \sin\theta X_{lm} \end{pmatrix}, \tag{4.3j}$$

with

$$X_{lm} = 2\left(\frac{\partial^2 Y_{lm}}{\partial\phi\partial\theta} - \cot\theta\frac{\partial Y_{lm}}{\partial\theta}\right), \tag{4.4}$$

$$W_{lm} = \frac{\partial^2 Y_{lm}}{\partial\theta^2} - \cot\theta\frac{\partial Y_{lm}}{\partial\theta} - \frac{1}{\sin^2\theta}\frac{\partial^2 Y_{lm}}{\partial\phi^2} \tag{4.5}$$

and the orthonormalisation being with respect to the inner product

$$(\mathbf{A}, \mathbf{B}) = \int d\Omega\, A^*_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta}B_{\alpha\beta}\,, \tag{4.6}$$

$\eta$ being the Minkowski metric and $\mathbf{A}$ and $\mathbf{B}$ being any two 2nd order tensors. The orthonormalisation is such that

$$\left(\mathbf{a}^{(\mathrm{sph})}_{lm}, \mathbf{a}^{(\mathrm{sph})}_{l'm'}\right) = \delta_{ll'}\delta_{mm'}\,. \tag{4.7}$$

With this decomposition the angular part of the equations is automatically solved and we need only solve our original equations for the 10 functions $H_{0lm}$, $H_{1lm}$, $H_{2lm}$, $\eta_{0lm}$, $\eta_{1lm}$, $K_{lm}$, $G_{lm}$, $h_{0lm}$, $h_{1lm}$, $h_{2lm}$, dependent on the time and radial coordinates only. When doing it, we can also take into account the parity of the spherical harmonics. The first seven harmonics that were presented form what is called the polar sector, having parity $(-1)^{l+1}$ under the the transformation $(\theta,\phi)\rightarrow(\pi-\theta,\phi+\pi)$, while the other three form the axial sector, with parity $(-1)^l$.These two sectors of harmonics (and corresponding perturbation functions) can be proved to be independent from each other, reason for which we will treat them separately *a priori*.

We also decompose the perturbation stress-energy tensor $T^{(1)}_{\mu\nu}$ into spherical harmonics:

$$\mathbf{T}=\sum_{lm}\big[A^{(0)}_{lm}(t,r)\mathbf{a}^{(0)}_{lm}+A^{(1)}_{lm}(t,r)\mathbf{a}^{(1)}_{lm}+A_{lm}(t,r)\mathbf{a}_{lm}+B^{(0)}_{lm}(t,r)\mathbf{b}^{(0)}_{lm}+B_{lm}(t,r)\mathbf{b}_{lm}+G^{(s)}_{lm}(t,r)\mathbf{g}_{lm}+F_{lm}(t,r)\mathbf{f}_{lm}$$
$$+Q^{(0)}_{lm}(t,r)\mathbf{c}^{(0)}_{lm}+Q_{lm}(t,r)\mathbf{c}_{lm}+D_{lm}(t,r)\mathbf{d}_{lm}\big] , \tag{4.8}$$

where the functions of $t$ and $r$, obtained from $T^{(1)}_{\mu\nu}$, will be specified later, in section 4.6.

We can compare all this with the GR case, where the equations under study are merely the Einstein equations:

$$\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma}=8\pi T^{(1)}_{\mu\nu}\,. \tag{4.9}$$

In this case there is a further simplification we can make: we have gauge freedom, as equations (4.9) are invariant under the transformation $h_{\mu\nu}\rightarrow h_{\mu\nu}-(\nabla_\mu\xi_\nu+\nabla_\nu\xi_\mu)$, which we can use to reduce the number of unknown functions to solve for. For $l\geq 2$ the usual gauge to pick is the Regge-Wheeler gauge, introduced in [16], which eliminates the functions $\eta_{0lm}$, $\eta_{1lm}$, $G_{lm}$ and $h_{2lm}$. However, in dRGT massive gravity there is no such gauge freedom, reason for which we will have to work, at first, with all unknown functions.

## 4.2 Polar perturbations, $l=0$

The simplest case for which we studied equations (3.39) and (3.41) was that of $l=0$ (and, consequently, $m=0$). In this case, we have that $W_{00}=X_{00}=0$ and $Y_{00}=\frac{1}{\sqrt{4\pi}}$ (hence, its derivatives are also zero). For these reasons the axial sector does not give any contribution (as $\mathbf{c}_{lm}$, $\mathbf{c}^{(0)}_{lm}$ and $\mathbf{d}_{lm}$ are all undefined) and from the polar sector we only have four defined spherical harmonics: $\mathbf{a}^{(0)}_{lm}$, $\mathbf{a}^{(1)}_{lm}$, $\mathbf{a}_{lm}$ and $\mathbf{g}_{lm}$. This also means that there are four unknown functions[2] we wish to calculate, $H_0$, $H_1$, $H_2$ and $K$. The full $l=0$ perturbation tensor is shown in equation (4.10), where we can see that the resulting metric is already spherically symmetric, as expected from the monopolar perturbation.

The calculations that ensue (as all further ones in this section) have been made through the software *Mathematica* and the package *xAct* [34]. The corresponding notebooks are available in [35]. Nonetheless, a sketch of such calculations is presented.

[2]from hereon, we only use the $lm$ indices in the perturbation functions when not doing so might raise some confusion

$$
\mathbf{h}_{l=0,m=0} = \begin{pmatrix} \frac{1}{\sqrt{4\pi F(r)}} H_0(t,r) & \frac{1}{\sqrt{4\pi}} H_1(t,r) & 0 & 0 \\ \frac{1}{\sqrt{4\pi}} H_1(t,r) & \frac{F(r)}{\sqrt{4\pi}} H_2(t,r) & 0 & 0 \\ 0 & 0 & \frac{1}{4\pi} r^2 K(t,r) & 0 \\ 0 & 0 & 0 & \frac{1}{4\pi} r^2 \sin^2\theta K(t,r) \end{pmatrix} . \tag{4.10}
$$

For the 4 unknown functions there are 6 independent equations: the trace, the $t$ and $r$ components of the Lorentz gauge equations and the $tt$, $tr$ and $rr$ components of the field equations, of which the $\theta\theta$ and $\phi\phi$ components, while non-zero, can be obtained through combinations of some of the others. We then took their Fourier transforms, obtaining, like in the Proca case, coupled ODEs on the $r$ coordinates. This set of equations is still not trivial to solve numerically. Therefore, it would be easier to manipulate these ODEs in order to obtain a single equation, for a single unknown function, from which all others would be defined[3].

It is, in fact, possible to do so for the $l = 0$ case. We can write, through the $t$ gauge equation, $H_0$ in terms of $H_1$ and $H_1'$ (and source functions, whose presence, being irrelevant to this current discussion, we will leave as implicit) and, with this and the trace equation, $H_2$ in terms of $H_1$, $H_1'$ and $K$. This gives us now 4 coupled ODEs for $H_1$ and $K$, of which the $r$ gauge equation and the $tt$ and $tr$ field equations can be solved for $H_1$ and its derivatives in terms of $K$. Substituting all this in our last available equation, the $rr$ component, we finally obtain a 2[nd] order ODE for $K$, with the three other functions written in terms of it and its derivatives. Both for a more compact presentation and to aid its eventual solving method, we transform $K$ into another function $\varphi_0$ through

$$
K(r) = \frac{\sqrt{-4\mu^2 M + \mu^4 r^3 + 2\mu^2 r + 4r\omega^2}}{r^{5/2}} \varphi_0(r) \,, \tag{4.11}
$$

in such a way that we end up with a wavelike equation

$$
\frac{d^2\varphi_0}{dr_*^2} + (\omega^2 - V_{\text{pol}}^{l=0}(\omega,r))\varphi_0(r) = S_{\text{pol}}^{l=0}(r) \,, \tag{4.12}
$$

where $r_*$ is the tortoise coordinate, defined by

$$
\frac{dr_*}{dr} = \frac{1}{F(r)} \,. \tag{4.13}
$$

The potential of equation (4.12) is of the form

$$
\begin{aligned}
V_{\text{pol}}^{l=0}(\omega,r)) = & \frac{(2M-r)}{r^4\left(-4\mu^2 M + \mu^4 r^3 + 2\mu^2 r + 4r\omega^2\right)^2} (4\mu^4 (2M-r)\left(5M^2 - 6Mr + 2r^2\right) - 2\mu^6 r^3 \left(6M^2 - 10Mr + 3r^2\right) \\
& - 4\mu^2 r\omega^2 \left(28M^2 - 22\mu^2 M r^3 - 32Mr + 3\mu^4 r^6 + 7\mu^2 r^4 + 8r^2\right) + 6\mu^8 r^6 (3M - r) \\
& - 32r^2\omega^4 \left(-3M + \mu^2 r^3 + r\right) - \mu^{10} r^9) \,,
\end{aligned} \tag{4.14}
$$

which behaves like $V_{\text{pol}}^{l=0} \to \mu^2$ at very large distances and vanishes at the horizon $r = 2M$. As for

[3]if possible, that is

$S^{l=0}_{\text{pol}}(r)$, it consists in a combination of source functions (the ones coming from the decomposition of the stress-energy tensor). The specific combination comes from the previous manipulations and is presented in the already mentioned notebook, along with the respective equation. In subsection 4.6 we write its expression (as the ones for the other source terms in this section) after substituting the explicit formulae of the source functions.

The fact that this equation even exists is already noteworthy, being in stark contrast with the same case in GR, which we will present here as well. Starting from the perturbation (4.10), we can define $\xi = \begin{pmatrix} \frac{M_0(t,r)}{\sqrt{4\pi}} & \frac{M_1(t,r)}{\sqrt{4\pi}} \end{pmatrix}$, which leads to the redefinition of the perturbation functions under the previously mentioned gauge transformation:

$$H_0^{(g)}(t,r) = H_0(t,r) + 2F(r)\frac{dM_0}{dt} + F'(r)M_1(t,r)\,, \tag{4.15a}$$

$$H_1^{(g)}(t,r) = H_1(t,r) - \frac{1}{F(r)}\frac{dM_1}{dt} + F(r)\frac{dM_0}{dr}\,, \tag{4.15b}$$

$$H_2^{(g)}(t,r) = H_2(t,r) + \frac{F'(r)}{F(r)^2}M_1 + 2\frac{1}{F(r)}\frac{dM_1}{dr}\,, \tag{4.15c}$$

$$K^{(g)}(t,r) = K(t,r) - 2M_1(t,r)\,. \tag{4.15d}$$

These expressions allow us to impose $K^{(g)} = H_1^{(g)} = 0$, as long as we choose $M_1$ and $M_0$ (up to a function of time) appropriately. The other two functions are defined by the components $tt$ and $rr$ of the Einstein equations:

$$\frac{d}{dr}\left((r-2M)H_2\right) = \frac{r^2}{\left(1-\frac{2M}{r}\right)}8\pi A^{(0)}_{lm}(t,r)\,, \tag{4.16a}$$

$$-\frac{1}{r}\frac{dH_0}{dr} - \frac{H_2(r)}{r(r-2M)} = 8\pi A_{lm}(t,r)\,, \tag{4.16b}$$

These equations lead, as we confirm in section 5.1, to a non-oscillatory and still spherically symmetric perturbation metric, very much unlike what is obtained in the dRGT theory. The reason for this can be easily seen from a particle point of view. For the massless graviton, the helicity states cannot be excited by this spherically symmetric perturbation. Contrarily, it was to be expected that the scalar mode of the massive graviton would be excited by such a perturbation, this being the reason for such a difference between theories for $l = 0$.

## 4.3 Polar perturbations, $l = 1$

Keeping ourselves to the polar sector and moving on to $l = 1$, there are now two more defined harmonics, $\mathbf{b}^{(0)}_{lm}$ and $\mathbf{b}_{lm}$, corresponding to two more functions $\eta_0$ and $\eta_1$ for which we need to solve our system.

For this case the maximum simplification possible is to define all functions in terms not of just one but of two unknown functions, defined by two coupled 2[nd] order ODEs. The process is similar to what we did before: from the trace equation and $r$ and $\theta$ components of the gauge equation we can define $H_0$,

$H_1$ and $\eta_0$ in terms of the other three functions. Afterwards, the $tr$ and $r\theta$ equations give a definition of $H_2$ and $H_2'$ in terms of $\eta_1$ and $K$. Substituting all this into the $tt$ and $rr$ equations we obtain two coupled ODEs for these functions,

$$\frac{d^2\eta_1}{dr_*^2} + p_{\eta\eta}\frac{d\eta_1}{dr_*} + p_{K\eta}\frac{dK}{dr_*} + q_{\eta\eta}\eta_1 + q_{K\eta}K = S^{l=1}_{pol,\eta} \tag{4.17}$$

and

$$\frac{d^2K}{dr_*^2} + p_{\eta K}\frac{d\eta_1}{dr_*} + p_{KK}\frac{dK}{dr_*} + q_{\eta K}\eta_1 + q_{KK}K = S^{l=1}_{pol,K}\,, \tag{4.18}$$

where we transformed the radial derivatives into tortoise derivatives and the $p_{ij}$ and $q_{ij}$ are functions of $\omega$ and $r$, which can be found in the above mentioned *Mathematica* notebook. Note that the source functions now can, for the same value of $l = 1$, be different according to $m \in \{-1, 0, 1\}$, as we will see, again, in subsection 4.6.

Once again, the mere fact that the polar $l = 1$ perturbation equations are such as these is a variation with respect to GR, where this perturbation is completely removed by the existing gauge freedom. As before, the explanation for this fact comes from the existence of extra modes in the massive graviton that can be excited.

## 4.4 Axial perturbations, $l = 1$

The first multipolar order at which the axial sector is defined is $l = 1$, for which we have two unknown functions, $h_0$ and $h_1$, the harmonic $\mathbf{d}_{lm}$ not being defined unless for $l \geq 2$. To define these functions we use two equations, the $\theta$ component of the gauge equations and the $(r\theta)$ component of the field equations.

To simplify them, we define $h_0$ through the $\theta$ gauge gauge equation in terms of $h_1$ and its first derivative. Plugging this new expression into the $(r\theta)$ equation we obtain a single 2nd order ODE for $h_1$. To obtain a wavelike equation we perform a transformation over the function $h_1$,

$$Q(r) = \frac{h_1(r)}{F(r)}\,, \tag{4.19}$$

and change the equation variable to the tortoise coordinate, obtaining:

$$\frac{d^2Q}{dr_*^2} + \left(\omega^2 - V^{l=1}_{\text{ax}}(r)\right)Q(r) = S^{l=1}_{\text{ax}}(\omega, r)\,, \tag{4.20}$$

where the potential is:

$$V^{l=1}_{\text{ax}}(r) = F(r)\left(\mu^2 + \frac{6}{r^2} - \frac{16M}{r^3}\right)\,. \tag{4.21}$$

Solving this equation also gives us the solution to $h_0$, given by:

$$h_0(r) = i\frac{\left(1 - \frac{2M}{r}\right)}{\omega r^2}\frac{d}{dr}\left(r^2 Q(r)\right)\,. \tag{4.22}$$

## 4.5 Axial perturbations, $l \geq 2$

For $l \geq 2$ we have all three axial tensor spherical harmonics defined, with the three corresponding unknown functions to determine, $h_0$, $h_1$ and $h_2$. The independent equations defining these functions are the $\theta$ component of the gauge equations and the $t\theta$, $r\theta$ and $\theta\theta$ field equations.

The simplification of this system of equations is still quite straightforward. We define $h_0$ from the $\theta$ gauge equation in terms of $h_1$, $h_1'$ and $h_2$ and, substituting this into the $r\theta$ and $\theta\theta$ equations obtain two coupled equations. Performing a transformation over the function $h_1$ (4.24), we can write them in the following simple notation:

$$\frac{d^2\mathbf{\Psi}}{dr_*^2} + [\omega^2 - \mathbf{V}_{ax}(r)]\mathbf{\Psi} = \mathbf{S}_{ax}\,, \tag{4.23}$$

where $\mathbf{\Psi} = \begin{pmatrix} Q & Z \end{pmatrix}^T$ and, bearing in mind that $\lambda = l(l+1)$:

$$Q(r) = \frac{h_1(r)}{F(r)} \quad , \quad Z(r) = h_2(r)\,, \tag{4.24}$$

$$\mathbf{V}_{ax} = F(r)\begin{pmatrix} \mu^2 + 2\frac{\lambda+3}{r^2} - \frac{16M}{r^3} & 2i\lambda\frac{3M-r}{r^3} \\ \frac{4i}{r^3} & \mu^2 + \frac{2\lambda}{r^2} + \frac{2M}{r^3}\,. \end{pmatrix} \tag{4.25}$$

Solving these ODEs is enough to find all three functions, considering the above mentioned definition of $h_0$:

$$h_0(r) = F(r)\left(\frac{2i}{\omega r}Q(r) + \frac{i}{\omega}Q'(r) + \frac{\lambda-2}{2\omega r}Z(r)\right). \tag{4.26}$$

## 4.6 Source terms

Having the equations necessary to define our system, the only things missing are their source terms. These, as said before, are combinations of the functions resulting of the decomposition of the stress-energy tensor, so we need first to specify what this tensor is.

The black hole perturbation we considered was that of a point-particle, that is, a body of some mass $m_0$ and no internal structure, free falling in the black hole metric, that is, following some geodesic motion. This leads to the expression for the stress-energy tensor:

$$T^{\mu\nu} = m_0\int_{-\infty}^{\infty} d\tau\delta^{(4)}(x - z(\tau))\frac{dz^\mu}{d\tau}\frac{dz^\nu}{d\tau} = m_0\frac{dT}{d\tau}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}\frac{\delta(r - R(t))}{r^2}\delta^{(2)}(\Omega - \Omega(t))\,, \tag{4.27}$$

where $\delta^{(2)}(\Omega-\Omega(t))$ is short for $\frac{\delta(\theta-\theta(t))}{\sin\theta}\delta(\phi-\phi(t))$, $\mathbf{z}(\tau) = (T(\tau), R(\tau), \Theta(\tau), \Phi(\tau))$ is the parametrization of the geodesic motion of the particle in terms of the affine parameter $\tau$ and $\frac{dT}{d\tau}$.

The next step is to decompose this tensor into spherical harmonics. The orthonormality of the basis (4.3a)-(4.3j) makes it so we can write each source function of (4.8) as the inner product between the stress-energy tensor and its respective spherical harmonic, for instance, $A^{(0)}_{lm}(t,r) = \left(\mathbf{a}^{(0)}_{lm}, \mathbf{T}\right)$, where the inner product between two tensors has been previously defined in (4.6)

With this method, it is straightforward to calculate the general expression of each source function. We refer to [33] for a complete (and correct) list of them.

For this work we considered two types of motion, radial infall and circular motion. These are defined by the quantities $\frac{dT}{d\tau}$ and $\frac{dz^i}{dt}$, which are, for the radial case:

$$\frac{dT}{d\tau} = \frac{\gamma}{\left(1-\frac{2M}{r}\right)} \quad , \quad \frac{dR}{dt} = -\left(1-\frac{2M}{r}\right) \quad , \quad \frac{d\Theta}{dt} = 0 \quad , \quad \frac{d\Phi}{dt} = 0\,. \tag{4.28}$$

Furthermore, besides being constant, the angles at which the particle is infalling are $\Theta = \Phi = 0$. We also assume that the motion is highly relativistic, as it has been seen that this still resembles closely the non-relativistic cases. With these definitions, we computed the source functions through a *Mathematica* notebook we developed [35], obtaining:

$$A^{(0)}_{lm}(t,r) = \frac{m_0\gamma}{r^2}F(r)\delta(r-R(t))Y^*_{lm}(0,0)\,, \tag{4.29}$$

$$A^{(1)}_{lm}(t,r) = -i\sqrt{2}\frac{m_0\gamma}{r^2}\delta(r-R(t))Y^*_{lm}(0,0)\,, \tag{4.30}$$

$$A_{lm}(t,r) = \frac{m_0\gamma}{r^2}\frac{1}{F(r)}\delta(r-R(t))Y^*_{lm}(0,0)\,, \tag{4.31}$$

where $T(r)$ is defined from $\frac{dT}{dr} = -\frac{1}{1-\frac{2M}{r}}$. The corresponding Fourier transforms are:

$$A^{(0)}_{lm}(\omega,r) = -\frac{m_0\gamma}{\sqrt{2\pi}r^2}e^{i\omega T(r)}Y^*_{lm}(0,0)\,, \tag{4.32}$$

$$A^{(1)}_{lm}(\omega,r) = i\frac{m_0\gamma}{\sqrt{\pi}r^2}\frac{1}{\left(1-\frac{2M}{r}\right)}e^{i\omega T(r)}Y^*_{lm}(0,0)\,, \tag{4.33}$$

$$A_{lm}(\omega,r) = -\frac{m_0\gamma}{\sqrt{2\pi}r^2}\frac{1}{\left(1-\frac{2M}{r}\right)^2}e^{i\omega T(r)}Y^*_{lm}(0,0)\,, \tag{4.34}$$

where the integral over time was done in the following fashion (we explicitly write the $A^{(0)}_{lm}$ case as an example):

$$A^{(0)}_{lm}(\omega,r) = \int_{-\infty}^{\infty} dt e^{-i\omega t}A^{(0)}_{lm}(t,r) = \int_{-\infty}^{\infty} dt e^{-i\omega t}\,. \tag{4.35}$$

Using these functions, we first calculated the source term for the Zerilli equation not in dRGT massive gravity but in GR, comparing it with [36] for a verification of our methods. Having obtained it successfully, we calculated the source terms for each of the previously derived equations on the polar sector:

$$S^{l=0}_{\text{pol}}(\omega,r) = \frac{8\sqrt{2}\gamma m_0(r-2M)\left(\mu^2 r+2i\omega\right)e^{i\omega T(r)}}{\sqrt{r}\left(-4\mu^2 M+\mu^4 r^3+2\mu^2 r+4r\omega^2\right)^{3/2}}\,, \tag{4.36}$$

$$S^{l=1,m}_{pol,\eta}(\omega,r) = m\frac{8i\sqrt{6}\gamma m_0\left(1-\frac{2M}{r}\right)\left(-i\mu^2 r^2+2r\omega-2i\right)e^{i\omega T(r)}}{r\left(4\mu^2 M r^2-8M-\mu^4 r^5-6\mu^2 r^3-4r^3\omega^2\right)}\,, \tag{4.37a}$$

$$S^{l=1}_{pol,K}(\omega, r) = \frac{r}{\left(1 - \frac{2M}{r}\right)} S^{l=1}_{pol,\eta}\,, \tag{4.37b}$$

while the source terms in the axial sector are zero, due to the angle we chose for the infalling particle.

As for the circular orbit geodesics, these are defined by:

$$\frac{dT}{d\tau} = \frac{1}{\sqrt{1 - \frac{3M}{r}}} \quad , \quad \frac{dR}{dt} = 0 \quad , \quad \frac{d\Theta}{dt} = 0 \quad , \quad \frac{d\Phi}{dt} = \sqrt{\frac{M}{r^3}} = \omega_c\,, \tag{4.38}$$

where we also assume that $\Theta = \frac{\pi}{2}$ and $R = R_c$. This leads to the following non-zero source functions:

$$A^{(0)}_{lm}(t, r) = \frac{m_0}{\sqrt{1 - \frac{3M}{r}} r^2} F(r)^2 \delta_r Y^*_{lm}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.39}$$

$$B^{(0)}_{lm}(t, r) = \frac{m_0}{\sqrt{1 - \frac{3M}{r}}} \sqrt{\frac{2M}{\lambda r^5}} F(r) \delta_r Y^*_{lm}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.40}$$

$$G_{lm}(t, r) = \frac{m_0 M}{\sqrt{2}\sqrt{1 - \frac{3M}{r}} r^3} \delta_r Y^*_{lm}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.41}$$

$$F_{lm}(t, r) = \frac{m_0 M}{\sqrt{2}\sqrt{1 - \frac{3M}{r}} r^3} \frac{\sqrt{\lambda(\lambda - 2)}}{\lambda} \delta_r Y^*_{lm}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.42}$$

$$D_{lm}(t, r) = \sqrt{2} \frac{m_0 M}{\sqrt{1 - \frac{3M}{r}} r^3} \frac{1}{\sqrt{\lambda(\lambda - 2)}} \delta_r \frac{dY^*_{lm}}{d\Theta}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.43}$$

$$Q^{(0)}_{lm}(t, r) = \frac{m_0}{\sqrt{1 - \frac{3M}{r}}} \frac{2M}{\lambda r^5} \delta_r \frac{dY^*_{lm}}{d\Theta}\left(\frac{\pi}{2}, \omega_c t\right)\,, \tag{4.44}$$

and subsequent Fourier transforms:

$$A^{(0)}_{lm}(\omega, r) = \frac{m_0 \left(1 - \frac{2M}{r}\right)^2}{\sqrt{2} r^2 \sqrt{1 - \frac{3M}{r}}} \delta_r \delta_\omega N_l P_l\left(\frac{\pi}{2}\right)\,, \tag{4.45}$$

$$B^{(0)}_{lm}(\omega, r) = \frac{m_0}{\sqrt{1 - \frac{3M}{r}}} \sqrt{\frac{M}{\lambda r^5}} \left(1 - \frac{2M}{r}\right) \delta_r \delta_\omega N_l P_l\left(\frac{\pi}{2}\right)\,, \tag{4.46}$$

$$G_{lm}(\omega, r) = \frac{m_0 M}{2\sqrt{1 - \frac{3M}{r}} r^3} \delta_r \delta_\omega N_l P_l\left(\frac{\pi}{2}\right)\,, \tag{4.47}$$

$$F_{lm}(\omega, r) = \frac{m_0 M}{\sqrt{1 - \frac{3M}{r}} r^3} \sqrt{\frac{(\lambda - 2)}{4\lambda}} \delta_r \delta_\omega N_l P_l\left(\frac{\pi}{2}\right)\,, \tag{4.48}$$

$$D_{lm}(\omega, r) = \frac{m_0 M}{\sqrt{1 - \frac{3M}{r}} r^3} \frac{1}{\sqrt{\lambda(\lambda - 2)}} \delta_r \delta_\omega N_l \frac{dP_l}{d\Theta}\left(\frac{\pi}{2}\right)\,, \tag{4.49}$$

$$Q^{(0)}_{lm}(\omega, r) = \frac{m_0}{\sqrt{1 - \frac{3M}{r}}} \sqrt{\frac{M}{\lambda r^5}} \delta_r \delta_\omega N_l \frac{dP_l}{d\Theta}\left(\frac{\pi}{2}\right)\,, \tag{4.50}$$

where $\delta_r$ and $\delta_\omega$ stand, respectively, for $\delta(r - R_c)$ and $\delta(\omega - m\omega_c)$, and $N_l = \sqrt{(2l+1)\frac{(l-m)!}{(l+m)!}}$

In the $l = 0$ case, we also have $m = 0$, so that $\delta_\omega$ becomes simply $\delta(\omega)$. This means that this mode is not excited by the perturbation that is a particle orbiting it, the only effect being a constant correction. Therefore, the polar sector for $l = 1$ is the only case for which we obtain a contribution that is not merely a small correction to GR. The corresponding source terms are:

$$S^{l=1}_{pol,\eta}(\omega,r) = A_\eta(\omega,r)\delta(r-R_c)+B_\eta(\omega,r)\delta'(r-R_c)\,, \qquad , S^{l=1}_{pol,K}(\omega,r) = A_K(\omega,r)\delta(r-R_c)+B_K(\omega,r)\delta'(r-R_c)\,, \tag{4.51}$$

with

$$A_\eta(\omega,r) = -\frac{4\sqrt{2}\pi m_0 P_l\left(\frac{\pi}{2}\right)\delta(\omega-m\omega_c)}{3\mu^2 r^3\sqrt{1-\frac{3M}{r}}\left(M\left(8-4\mu^2r^2\right)+r^3\left(6\mu^2+\mu^4r^2+4\omega^2\right)\right)}\Big(24M^3\left(6-7\mu^2r^2\right)+M^2\big(78\mu^4r^5 + 216r^3\left(2\mu^2+\omega^2\right)-56r\big)-Mr^4\left(238\mu^2+57\mu^4r^2+148\omega^2\right)+2r^5\left(18\mu^2+5\mu^4r^2+12\omega^2\right)\Big), \tag{4.52}$$

$$A_K(\omega,r) = -\frac{4\sqrt{2}\pi m_0 P_l\left(\frac{\pi}{2}\right)(1-\frac{2M}{r})\delta(\omega-m\omega_c)}{3\mu^2 r^6\sqrt{1-\frac{3M}{r}}\left(M\left(8-4\mu^2r^2\right)+r^3\left(6\mu^2+\mu^4r^2+4\omega^2\right)\right)}\Big(24M^3\left(2-5\mu^2r^2\right)+2M^2r\big(39\mu^4r^4 + 4r^2\left(40\mu^2+21\omega^2\right)-12\big)-Mr^4\left(3\mu^6r^4+79\mu^4r^2+6\mu^2\left(2r^2\omega^2+33\right)+132\omega^2\right) + 2r^5\left(4\omega^2\left(\mu^2r^2+3\right)+\mu^2\left(\mu^2r^2+2\right)\left(\mu^2r^2+9\right)\right)\Big), \tag{4.53}$$

$$B_\eta(\omega,r) = -\frac{8\pi m_0 P_l\left(\frac{\pi}{2}\right)\sqrt{2-\frac{6M}{r}}(r-2M)^2}{3\mu^2r^4}\delta(\omega-m\omega_c)\,, \tag{4.54}$$

$$B_K(\omega,r) = -B_\eta(\omega,r)\,. \tag{4.55}$$

These source terms cause excitations of some of the modes of the graviton only for $m = \pm 1$, the $m = 0$ being, again, leading to a constant correction.

# Chapter 5

# Monopolar gravitational radiation emission

With the equations fully specified, we can finally attempt to solve them. We will do so only for the polar $l = 0$ mode, solving first for the GR case and then for the dRGT theory, explaining the method employed, showing the resulting waveform and extracting what information we can from it.

## 5.1 Monopolar perturbation in GR

Having the source terms for the radial infall, we can now write explicitly equation (4.16a):

$$\frac{d}{dr}\left((r-2M)H_2\right) = 4\sqrt{\pi}m_0\gamma\delta(r-R(t))\,. \tag{5.1}$$

If we integrate this equation until some $r < R(t)$, that is, until some radius closer to the black hole than the infalling particle, we obtain no perturbation at all. Otherwise, we obtain that

$$H_2(t,r) = 2\gamma m_0 \frac{1}{r}\frac{1}{1-\frac{2M}{r}} \quad , r > R(t)\,. \tag{5.2}$$

Similarly, from (4.16b) we obtain

$$\frac{dH_0}{dr} = -\frac{H_2(t,r)}{r-2M} - 4\sqrt{\pi}\gamma m_0 \frac{1}{r-2M}\delta(r-R(t))\,. \tag{5.3}$$

Again, for $r < R(t)$ the function $H_0$ is null. For $r > R(t)$ the first term gives exactly $H_2(t,r)$, while the second gives us a function of the time coordinate only. Reabsorbing this function into the gauge function $M_0$, we finally end up with

$$H_0(t,r) = H_2(t,r) = \begin{cases} 0 & , r < R(t) \\ 2\gamma m_0 \frac{1}{r}\frac{1}{1-\frac{2M}{r}} & , r > R(t) \end{cases}\,. \tag{5.4}$$

Adding this to the background metric we obtain:

$$g_{00} = -\left(1 - \frac{2M}{r}\right)(1 + H_0(t,r)Y_{00}) = \begin{cases} -\left(1 - \frac{2M}{r}\right) & , r < R(t) \\ -\left(1 - \frac{2(M+\gamma m_0)}{r}\right) & , r > R(t) \end{cases} , \tag{5.5}$$

$$g_{rr} = \frac{1}{\left(1 - \frac{2M}{r}\right)}(1 + H_2(t,r)Y_{00}) = \begin{cases} \frac{1}{\left(1 - \frac{2M}{r}\right)} & , r < R(t) \\ \simeq \left(1 + \frac{2(M+\gamma m_0)}{r}\right) & , r > R(t) \end{cases} . \tag{5.6}$$

We see that, as expected, for radii closer to the black hole than the infalling particle the solution is still a Schwarzschild black hole, while for larger radii the black hole seems to have a mass of $M + \gamma m_0$, receiving a contribution from the perturbation. This result was inevitable in GR, as the spherical symmetry of both the background solutions and the monopolar perturbation implied that the perturbed metric would also be spherical symmetric, under which conditions, following Birkhoff's theorem, the only possible solution is Schwarzschild-like. This theorem does not apply in dRGT massive gravity, reason for which the solution to the monopolar perturbation is radically different.

## 5.2 Numerical integration

As derived in 4.2, the polar $l = 0$ master equation is an inhomogeneous wave equation, that is, of the form:

$$\frac{d^2\varphi_0}{dr_*^2} + \left(\omega^2 - V\right)\varphi_0 = S\,. \tag{5.7}$$

We solved this equation through variation of constants, whose setup is presented in [37]. This method consists on solving first the corresponding homogeneous equation (i.e. without source term) and then obtaining the general solution to the inhomogeneous one through them.

To obtain the two[1] homogeneous solutions we need first to understand how they behave at the boundaries $r_* \to -\infty$ and $r_* \to +\infty$. Knowing that $\lim_{r_* \to -\infty} V = \lim_{r \to r_s} V = 0$ and $\lim_{r_* \to \infty} V = \lim_{r \to \infty} V = \mu^2$, we see that equation (5.7) becomes

$$\frac{d^2\psi_h}{dr_*^2} + \omega^2\psi_h = 0\,, \tag{5.8}$$

$$\frac{d^2\psi_h}{dr_*^2} + (\omega^2 - \mu^2)\psi_h = 0\,, \tag{5.9}$$

which shows us that in these asymptotic limits the homogeneous solutions must, in general, behave like $Ae^{i\omega r_*} + Be^{-i\omega r_*}$ and $Ae^{k_\omega r_*} + Be^{-k_\omega r_*}$, $k_\omega = \sqrt{\mu^2 - \omega^2}$. This is the same as saying that they are a sum of ingoing and outgoing waves. It is here that we impose our boundary conditions. We say that one of the solutions, $\psi_h$ must be composed only by ingoing waves at the horizon radius $r_s$ and the other, $\psi_\infty$, must be composed only by outgoing waves at infinity. Then, to sum up, we have that:

[1] due to the equation being a 2$^{\text{nd}}$ order ODE

$$\lim_{r_* \to -\infty} \psi_h = e^{-i\omega r_*} \quad , \quad \lim_{r_* \to \infty} \psi_h = A_{\text{in}} e^{-k_\omega r_*} + A_{\text{out}} e^{k_\omega r_*} \,, \tag{5.10}$$

$$\lim_{r_* \to -\infty} \psi_\infty = B_{\text{out}} e^{i\omega r_*} + B_{\text{in}} e^{-i\omega r_*} \quad , \quad \lim_{r_* \to \infty} \psi_\infty = e^{k_\omega r_*} \,. \tag{5.11}$$

Imposing the previous boundary conditions, we solved numerically the homogeneous solutions of the equation under study through the default methods of the *NDSolve* function of the software *Mathematica*.

Knowing the homogeneous solutions, we can now pose the following ansatz to the general solution:

$$\varphi_0(r) = C_h(r)\psi_h(r) + C_\infty(r)\psi_\infty(r) \,, \tag{5.12}$$

where $C_h$ and $C_\infty$ must be such that

$$C_h'(r)\psi_h(r) + C_\infty'(r)\psi_\infty(r) = 0 \,. \tag{5.13}$$

From plugging (5.12) into equation (5.7) and (5.13) we obtain a system for $C_\infty'(r)$ and $C_h'(r)$:

$$\begin{pmatrix} \psi_h & \psi_\infty \\ \psi_h' & \psi_\infty' \end{pmatrix} \begin{pmatrix} C_h' \\ C_\infty' \end{pmatrix} = \begin{pmatrix} 0 \\ S \end{pmatrix} \,, \tag{5.14}$$

which we can solve, giving us expressions for the derivatives of the coefficients $C$:

$$C_\infty'(r) = \frac{\psi_h S}{W} \quad , \quad C_h'(r) = -\frac{\psi_\infty S}{W} \,, \tag{5.15}$$

where $W = \psi_h \psi_\infty' - \psi_\infty \psi_h'$ is the Wronskian of the two homogeneous solutions. With this, we can finally write the expression for the Fourier transform of the gravitational waves emitted by our system (now writing explicitly the radius and frequency dependencies):

$$\begin{aligned} \Psi_{\text{out}}(\omega, r_*) &= \psi_\infty(\omega, r_*) \int_{-\infty}^{r_*} dr_*' \, \frac{\psi_h(\omega, r_*')S(\omega, r_*')}{W(\omega, r_*')} \\ &= e^{k_\omega r_*} \int_{-\infty}^{\infty} dr_*' \, \frac{\psi_h(\omega, r_*')S(\omega, r_*')}{W(\omega, r_*')} \end{aligned} \,, \tag{5.16}$$

where in the last equality we have assumed to be in the limit $r_* \to +\infty$, and that of the gravitational waveform, in the time domain:

$$\Psi_{\text{out}}(t, r_*) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \Psi_{\text{out}}(\omega, r_*) e^{-i\omega t} \,. \tag{5.17}$$

To numerically evaluate this quantity we must, however, tread carefully. First of all, unlike GR, the outgoing frequencies are not merely $\omega$ but $\sqrt{\omega^2 - \mu^2}$. This means that for frequencies lower than $\mu^2$ the metric will not oscillate but have an exponential decay. For this reason, when perfoming the integral in (5.17) we make an approximation and assume the integrand to be zero in the interval $[-\mu, \mu]$. Also, we want to evaluate $\Psi_{\text{out}}$ far from the source, at a radius $r_u$ defined through the so called extraction radius $R = \mu r_u$. This radius is picked so that it is big enough that we reach a region where the coefficient $C_\infty$

becomes constant with respect to $r$. In practical terms, this means that the upper limit of (5.16) $r_u$ is irrelevant as long as the integral becomes constant before it.

Another feature important to have under consideration is that (5.16) is even with respect to $\omega$, as in GR. Therefore, we need only to compute the integral (5.17) on the interval $(\mu, +\infty)$ and double this quantity to obtain the waveform in the time domain.

This was the method we used to compute the spectra and waveforms presented in the next subsections. A simple yet not straightforward generalisation of these methods could also be used to solve systems of coupled equations such as the one for the $l = 1$ polar mode. For this result we point the reader to [1]. We note as well that this same method can also be applied to GR, with the sole modification of setting $\mu$ to zero.

## 5.3 Energy spectrum

After calculating the metric components in the frequency domain (defined the master function $\varphi_0$) using the above method we can extract some information about the gravitational waves that are being emitted, namely the energy content of the gravitational waves emitted. To do so, we need to reverse to the lagrangian description of the dRGT theory.

It is easy to see that the linearized lagrangian of dRGT massive gravity present in (3.32) reduces to (3.30) with $\kappa = \frac{1}{2}$. Noether's theorem states that the stress-energy tensor of the field[2] is given by:

$$T^{GW}_{\mu\nu} = \left\langle \frac{\delta \mathcal{L}}{\delta (h^{\alpha\beta},_\mu)} h^{\alpha\beta},_\nu - \eta_{\mu\nu} \mathcal{L} \right\rangle , \tag{5.18}$$

which, for the mentioned lagrangian, is equal to:

$$T^{GW}_{\mu\nu} = \frac{1}{32\pi} \left\langle h_{\alpha\beta,\mu} h^{\alpha\beta}{}_{,\nu} - h_{,\mu} h_{,\nu} \right\rangle = \left\langle h_{\alpha\beta,\mu} h^{\alpha\beta}{}_{,\nu} \right\rangle , \tag{5.19}$$

where the last equality comes from the fact that, at large distances, the trace of the metric goes to zero (along with its derivative) due to the previous consideration we made about the stress-energy tensor. From the stress-energy tensor we can write the energy loss $\frac{dE}{dt}$ of the radiating source, given by:

$$\frac{dE}{dt} = \int dS^i \, T^{\mathrm{GW}}_{0i} = \int d\Omega \, T^{\mathrm{GW}}_{0i} n^i r^2 = \int d\Omega \, T^{\mathrm{GW}}_{0r} r^2 , \tag{5.20}$$

where $n^i$ is the normal of the surface element $dS^i$. Combining this with the expression for the gravitational stress-energy tensor, we get:

$$\frac{dE}{dt} = \frac{r^2}{32\pi} \sum_{l,m} \sum_{l',m'} \int d\Omega \left\langle h_{\alpha\beta,0} h^{\alpha\beta}{}_{,r} \right\rangle . \tag{5.21}$$

To evaluate this, we recall the expression for the perturbation metric (4.2) and that the tensorial spherical harmonics with which we define it are orthonormalized according to (4.7). All this implies the following general formula for the lost energy (that is, after integrating over all time):

[2] ignoring the matter term, as we are analysing the field far enough that the matter distribution has direct influence

$$E = \frac{r^2}{32\pi}\int_{-\infty}^{\infty} dt \Bigg( H_{0,t}^* H_{0,r} - 2H_{1,t}^* H_{1,r} + H_{2,t}^* H_{2,r} - \frac{2l(l+1)}{r^2}\eta_{0,t}^*\eta_{0,r} + \frac{2l(l+1)}{r^2}\eta_{1,t}^*\eta_{1,r}$$
$$+ \frac{1}{2}l(l+1)(l(l+1)-2)G_{,t}^* G_{,r} + 2K_{,t}^* K_{,r} + \frac{l(l+1)}{2}G_{,t}^* G_{,r} - l(l+1)(K_{,t}^* G_{,r} + G_{,t}^* K_{,r})$$
$$- \frac{2l(l+1)}{r^2}h_{0,t}^* h_{0,r} + \frac{2l(l+1)}{r^2}h_{1,t}^* h_{1,r} + \frac{l(l+1)(l(l+1)-2)}{2r^2}h_{2,t}^* h_{2,r}\Bigg) . \tag{5.22}$$

We can simplify this further by analysing the behaviour of the perturbation functions. We can write any of them (we will represent them by a generic function $A_{lm}$) in terms of their Fourier transforms:

$$A_{lm}(t,r) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} A_{lm}(\omega, r) = \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} e^{i\sqrt{\omega^2-\mu^2}r} A_{lm}(\omega) , \tag{5.23}$$

where the last equality is an approximation valid at long distances from the source, where the Fourier transform of each perturbation function both decays with some power of the radius and is oscillatory, with frequency $\sqrt{\omega^2-\mu^2}$. Armed with this expression, we take the above derivatives of such functions:

$$\frac{dA_{lm}}{dt} = \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} e^{i\sqrt{\omega^2-\mu^2}r} A_{lm}(\omega)(-i\omega) , \tag{5.24}$$

$$\frac{dA_{lm}}{dr} \simeq \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} e^{i\sqrt{\omega^2-\mu^2}r} A_{lm}(\omega)\left(i\sqrt{\omega^2-\mu^2}\right) . \tag{5.25}$$

Using these results in equation (5.22) we obtain

$$\int_{-\infty}^{\infty} dt \left\langle \frac{dA_{lm}^*}{dt}\frac{dA_{lm}}{dr}\right\rangle = -\frac{1}{2\pi}\frac{1}{r^{2\alpha}}\int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega'\, e^{i(\omega'-\omega)t} e^{i\left(\sqrt{\omega^2-\mu^2}-\sqrt{\omega'^2-\mu^2}\right)} A_{lm}^*(\omega')A_{lm}(\omega)\omega'\sqrt{\omega^2-\mu^2}$$
$$= -\frac{1}{r^{2\alpha}}\int_{-\infty}^{\infty} d\omega\, |A_{lm}(\omega)|^2 \omega\sqrt{\omega^2-\mu^2} , \tag{5.26}$$

where we used the definition of the Dirac delta:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega'-\omega)t} = \delta(\omega'-\omega) . \tag{5.27}$$

The decay with the radius can be shown to correspond, at the lowest order, to $\alpha = 0$ for $\eta_0$, $\eta_1$, $h_0$, $h_1$ and $h_2$ and to $\alpha = 1$ for $H_0$, $H_1$, $H_2$, $K$ and $G$. Therefore, defining the energy spectrum $\frac{dE}{d\omega}$ through $E = \int_{-\infty}^{\infty} d\omega\, \frac{dE}{d\omega}$, we obtain its general expression:

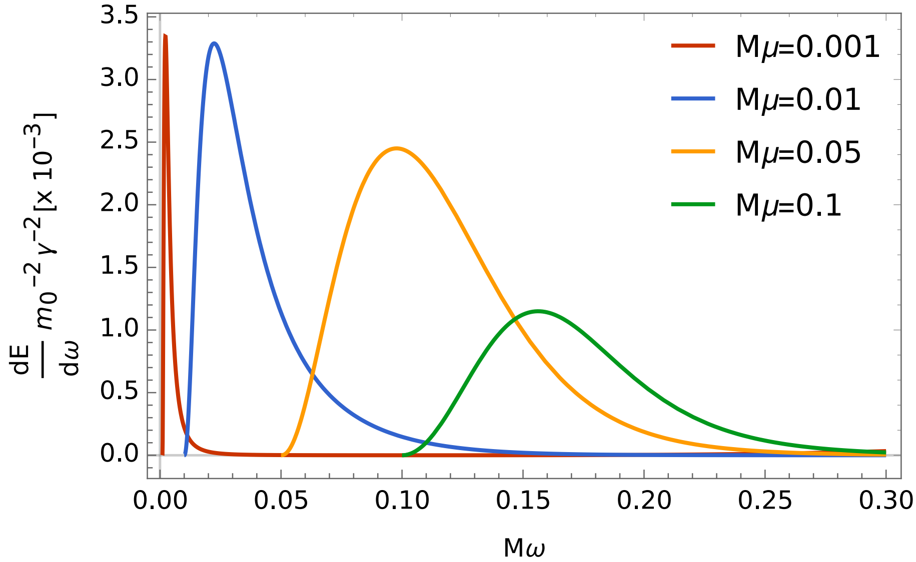


Figure 5.1: GW energy spectra for the $l = 0$ polar mode for a radially infalling particle and $M\mu = \{0.01, 0.05, 0.1\}$

$$\frac{dE}{d\omega} = -\frac{\omega\sqrt{\omega^2-\mu^2}}{32\pi}\Big(|H_0(\omega)|^2 - 2|H_1(\omega)|^2 + |H_2(\omega)|^2 - 2\lambda|\eta_0(\omega)|^2 + 2\lambda|\eta_1(\omega)|^2 + \frac{1}{2}\lambda(\lambda-1)|G(\omega)|^2 + 2|K(\omega)|^2 - \lambda(K^*(\omega)G(\omega) + G^*(\omega)K(\omega)) \cdot - 2\lambda|h_0(\omega)|^2 + 2\lambda|h_1(\omega)|^2 + \frac{1}{2}\lambda(\lambda-2)|h_2(\omega)|^2\Big) \tag{5.28}$$

We specified this quantity for the two cases of most interest, those of $l = 0$ and $l = 1$ in the polar sector. In both cases we can write all the perturbation functions in terms of some key function(s) through the relations derived in sections 4.2 and 4.3. Keeping, as done throughout this calculation, only the terms of lowest order in the radius in each relation, we obtain the energy spectrum for both polar $l = 0$ and $l = 1$ modes:

$$\left.\frac{dE}{d\omega}\right|_{\text{polar},l=0} = -3\frac{|\varphi_0(\omega)|^2}{8\pi}\omega\sqrt{\omega^2-\mu^2}\mu^4\,, \tag{5.29}$$

$$\left.\frac{dE}{d\omega}\right|_{\text{polar},l=1} = -\frac{\omega\sqrt{\omega^2-\mu^2}}{16\pi}\Big(2|K(\omega)|^2 + 4\mu^2/\omega^2|\eta_1(\omega)|^2 + 9|K(\omega)+\eta_1(\omega)|^2 + (2\mu^2/\omega^2-1)|K(\omega)+3\eta_1(\omega)|^2\Big)\,. \tag{5.30}$$

In the $l = 0$ mode, for which we solved the equations numerically, we obtained the corresponding energy spectra, which we show in figure 5.1.

The peak of these distributions is found close to the point where $\omega = \mu$ and after it the spectra rapidly fall off, indicating that the frequencies that contribute the most to the dissipation of energy by the perturbed black hole are the ones slightly above $\mu$. On the other hand, the distribution is null for $\omega < \mu$. As explained before, this is due to the fact that such modes correspond to exponentially decreasing solutions, not contributing to what can be observed at infinity coming from the system in study. However,

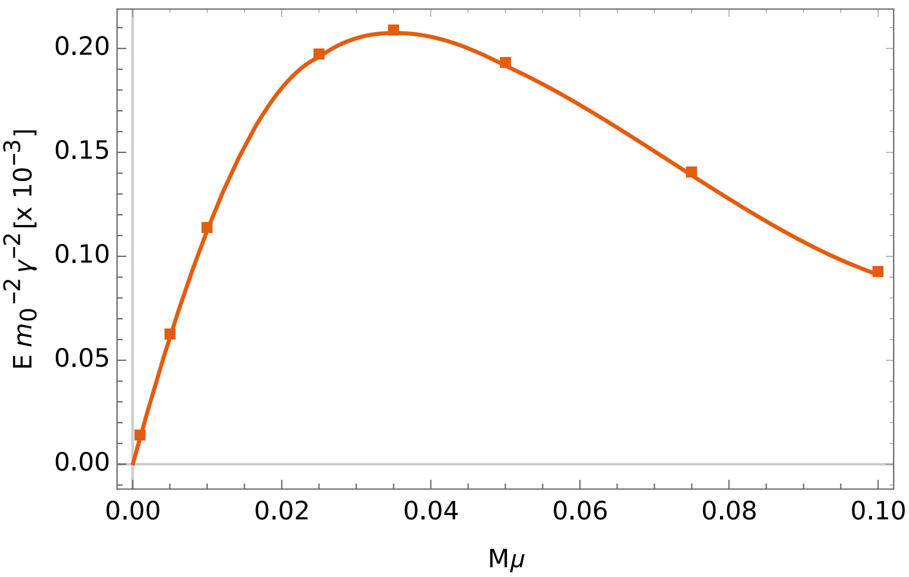


Figure 5.2: Total energy of the polar $l = 0$ mode. The presented curve is an interpolation of the calculated points, represented by the squares.

these modes may affect the black hole itself and change its evolution in time, meritting some further study.

What is also noteworthy is that the value of the peak of the spectrum for each value of $\mu$ seems to increase towards a constant value as the graviton mass decreases. As the area of the spectra decreases with $\mu$, this seems to indicate that the total energy of the polar $l = 0$ gravitational waves decreases as well, something that is confirmed by figure 5.2. This means that the emission of energy through the excitation of this mode, created by the addition of mass to the graviton, goes to zero with this same mass, which is exactly what one would expect from a working theory of a massive graviton.

## 5.4 Waveforms

Going now to the solutions in the time domain, we can study them by varying two different parameters: the quantity $M\mu$ and the extraction radius $R = r_u\mu$. All these solutions were calculated for the same boost factor $\gamma = 1$, perturbation mass $m_0 = 1$ and BH mass $M = 1$ and were plotted as function of the retarded time $t - r$. We chose to present here (5.3-5.6) the waveforms corresponding to $\mu = \{0.1, 0.01\}$ and $R = \{10, 100\}$.

Through these waveforms we analysed the variation of their maximum amplitude $\varphi_0^{\text{peak}}$ and the retarded time $(t - r)^{\text{peak}}$ at which such peak occurs. We did this with respect to both of the dimensionless quantities, $M\mu$ and $R$, which we varied, respectively, in the ranges $\{0.01, 0.025, 0.05, 0.075, 0.1\}$ and $\{10, 25, 50, 75, 100, 150, 200, 250\}$. Fitting this data to powers of $M\mu$ and $R$ through the software *gnuplot*,

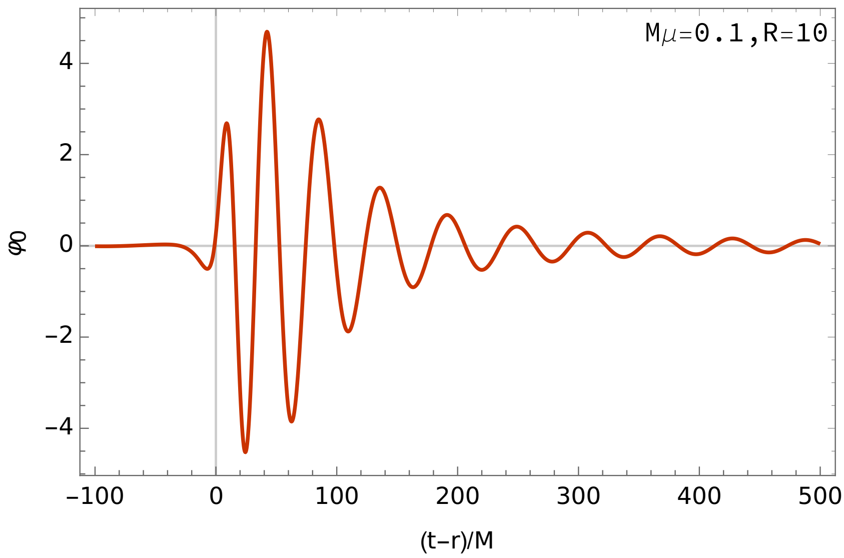


Figure 5.3: Waveform of $\varphi_0$ for $M\mu = 0.1$, $R = 10$

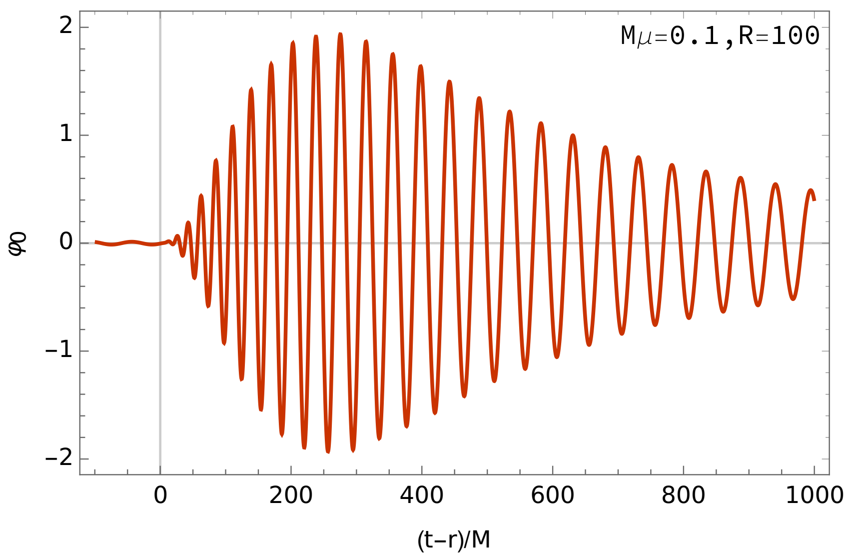


Figure 5.4: Waveform of $\varphi_0$ for $M\mu = 0.1$, $R = 100$

we obtained:

$$\varphi_0^{\text{peak}} \sim 0.068 m_0\gamma \frac{M^2}{(M\mu)^{2.40}R^{0.44}} \sim 0.068 \frac{0.01 m_0\gamma}{0.01M} \frac{M^{0.60}}{\mu^{2.88}{r_u}^{0.44}}\,, \tag{5.31a}$$

$$\begin{aligned}\Rightarrow K^{\text{peak}} &\sim 0.068 \frac{0.01 m_0\gamma}{0.01M} \frac{M^{0.60}}{\mu^{0.88}{r_u}^{1.44}} \\ &\sim 8.1\times 10^{-17} \frac{m_0\gamma}{0.01M}\left(\frac{10^{-23}\text{eV}}{\mu}\right)^{0.84}\left(\frac{M}{M_\odot}\right)^{0.60}\left(\frac{8\text{kpc}}{r}\right)^{1.44} \\ &\sim 3.7\times 10^{-23} \frac{m_0\gamma}{0.01M}\left(\frac{10^{-23}\text{eV}}{\mu}\right)^{0.84}\left(\frac{M}{M_\odot}\right)^{0.60}\left(\frac{1\text{Gpc}}{r}\right)^{1.44}\,,\end{aligned} \tag{5.31b}$$

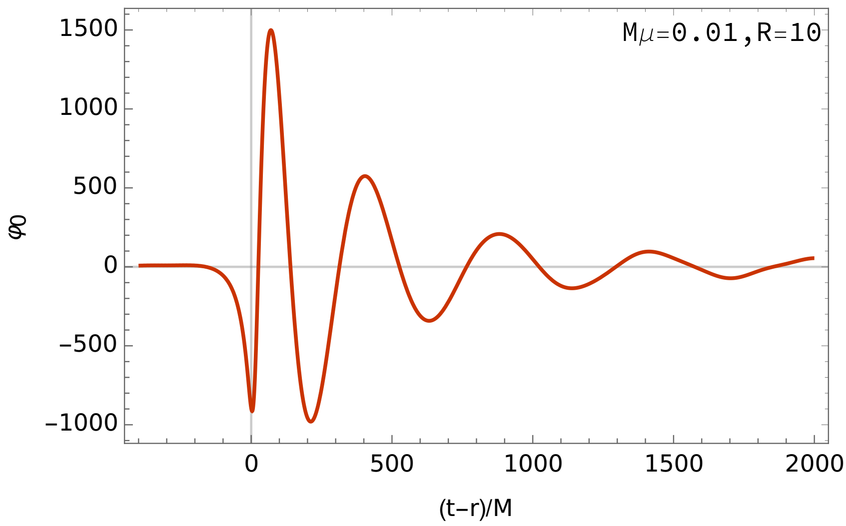


Figure 5.5: Waveform of $\varphi_0$ for $M\mu = 0.01$, $R = 10$

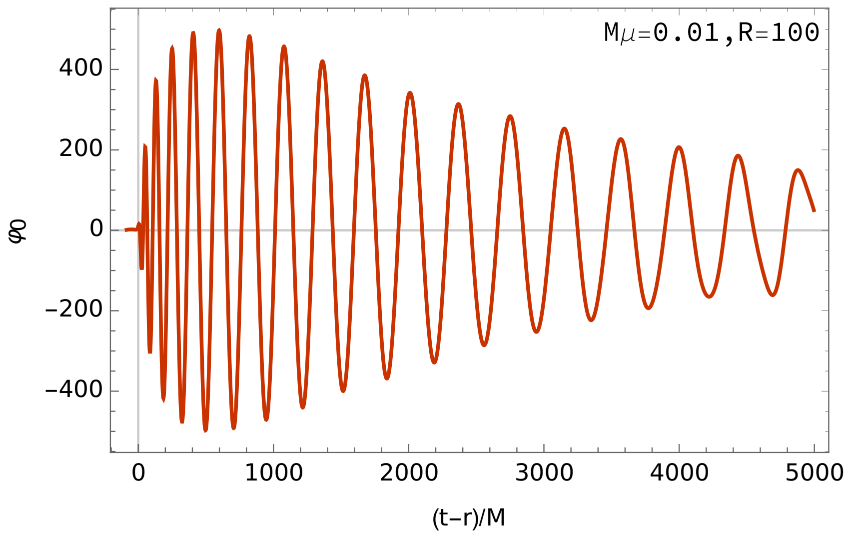


Figure 5.6: Waveform of $\varphi_0$ for $M\mu = 0.01$, $R = 100$

$$\begin{aligned}(t-r)^{\text{peak}} &\sim 1.976(M\mu)^{-0.285}R^{0.895}M \sim 1.976M^{0.715}\mu^{0.610}{r_u}^{0.895}\\ &\sim 133\left(\frac{\mu}{10^{-23}\text{eV}}\right)^{0.610}\left(\frac{M}{M_\odot}\right)^{0.715}\left(\frac{r}{8\text{kpc}}\right)^{0.895} s\\ &\sim 4.85\times 10^{6}\left(\frac{\mu}{10^{-23}\text{eV}}\right)^{0.610}\left(\frac{M}{M_\odot}\right)^{0.715}\left(\frac{r}{1\text{Gpc}}\right)^{0.895} s\,,\end{aligned} \tag{5.32}$$

where the $M$ dependency was adjusted with dimensional analysis considerations and the distance of $8$kpc was picked to correspond to Saggitarius A*, thought to be a supermassive black hole at the centre of our galaxy.

The first expression indicates that the amplitude of the perturbation to be detected on Earth is on a range that is feasible to measure, be it by LIGO or by the future detector LISA. As for the second expression, it tells us that the delay between the beginning of the signal and its peak is not too long, even decreasing with the mass of the graviton. Both these features give us hope that such signals (or similar ones, corresponding to the same mode excitation but in different systems) might be detected in association with the classical gravitational wave polarizations. Unlike these, the $l = 0$ mode would correspond to identical oscillations of spacetime at each radius, the signal between the two LIGO detectors, for instance, differing only in its amplitude due to the possibly different distances to the source.

However, another characteristic of the signal extinguishes our hope of detection. As we have seen, the frequencies $\omega$ for which the polar $l = 0$ gravitational wave carries most energy are the ones such that

$$\omega \sim \mu < 1.0 \times 10^{-23} \text{eV}/c^2 = 2.42 \times 10^{-9} \text{Hz} \,. \tag{5.33}$$

This frequency lies well below the ranges of both the LIGO and LISA detectors[25, 38], meaning that this signal, if it exists, cannot be detected by us with the experiments currently (or even in the near future).

# Chapter 6

# Conclusions

In this thesis, we studied a phenomenon of emission of gravitational waves in a theory of a nonlinear massive graviton, namely the dRGT massive gravity theory. We did so through the perturbation of a Schwarzschild black hole by a point particle following its geodesics. Although extensively analysed in GR[16, 15], this topic had, to our knowledge, never before been broached in this particular theory.

The interest in dRGT massive gravity lies in its potential as an extension of GR. Theories of a massive graviton with a small mass lead, in general, to a long-range behaviour of the gravitational potential in the classical limit, which is desirable to explain the accelerated expansion of the universe. In these theories, the fact that the field is propagated by a massive particle leads to a Yukawa-like potential instead of a Newtonian one. This dramatically decreases the value of the potential at far enough regions, how far depending on how small the mass of the graviton is. This would reduce the total gravitational attraction, possibly justifying the expansion as we see it today and providing an alternative to the current explanation, that of the cosmological constant.

However, the theory has to be valid not only cosmologically but also in its other predictions. For this reason, the study of black holes and associated phenomena, of which there has been plenty in GR, is essential, for comparison with the theory we know to be correct to a great extent. To add to this, the fact that gravitational wave experiments are now, for the first time, producing results is a motivation to study such phenomena in these alternative theories, as to ascertain their validity in what concerns them, fully motivating our work.

We started by analysing the formalism to be used in the main problem in a simpler one, that of a Proca field, i.e. a massive spin-1 particle, on a Schwarzschild background. We obtained the equations ruling this system by decomposing the solution in vector spherical harmonics, making use of the existent spherical symmetry. We also showed what are the source terms of such equations, specifying them to a particle following a radial infall geodesic, the case we most studied in the massive spin-2 case. In the end, we obtained a single inhomogeneous wavelike equation, called a master equation, from which all perturbation functions were defined, resembling, minus some technical complexity, the process used later on.

We then proceeded to presenting the linear theory of a massive graviton that was proposed by

Fierz and Pauli[5] and the path until the nonlinear theory of dRGT massive gravity. In particular, we explicitly showed how the FP theory keeps its scalar mode in the decoupling limit, leading to the vDVZ discontinuity, whose wrong result for the light deflection by the Sun we also calculated. We eventually showed, as in [32], the linearised equations of motion for dRGT massive gravity, which lead to the correct value of the deflection of light and were the central equations of all this work.

In section 4 we finally studied the perturbation of a static black hole by a point particle. To do so we decomposed both the perturbation and the stress-energy tensors in tensor spherical harmonics, substantially simplifying the equations to solve. With this decomposition we managed to not only separate the equations for each multipolar order but also in two sectors dependent on the parity of the spherical harmonics, the axial and polar. From this setup we obtained master equations for several different cases. While the axial $l \geq 2$ (and, although not presented here, the polar as well) correspond to mere corrections to the GR analogue and the axial $l = 0$ was not excited by any studied perturbations, the other lower multipoles proved to be much more interesting.

In GR, the polar $l = 0$ and $l = 1$ modes correspond, respectively, to an increase on the mass of the black hole and a Lorentz transformation in the center of mass of the whole system, while the axial $l = 1$ is an increase of angular momentum. In dRGT massive gravity, on the other hand, the master equations ruling all these modes are wavelike, meaning that they all correspond to excitations of some mode of the massive graviton. These correspond exactly to the vector and scalar modes absent in the massless graviton, which possesses only its helicity states, of a 2-rank tensorial nature. This is a main result of our work, showing that a perturbed black hole in this nonlinear massive graviton theory can emit gravitational radiation in more varied ways than its GR counterpart.

Having the equations, we showed what their corresponding source terms were for two types of geodesic motion: highly relativistic radial infall and circular orbit. From this point on, we would need only to solve the fully specified master equations. We did so for the polar $l = 0$ equations in the radial infall case, obtaining, for a range of values of graviton mass $\mu$, the energy spectrum, the total emitted energy and the waveforms of the master function at different radii of distance from the black hole itself. These fully characterised this mode. Starting by the energy, the values we obtained denote that less of it is emitted in this mode as the graviton mass decreases, which would make sense, as in such limit we would gradually recover the GR results, in which theory this mode does not exist. As for the rest, we found the peak amplitude of the signal and the delay between it and the beginning of the signal to be such that, based solely on this, one would expect our detectors (or at least future ones) to be able to measure these signals. However, according to the obtained spectra, the frequencies of oscillation that carry the most energy are the ones close to $\mu \sim 10^{-9} Hz$. Neither LIGO nor LISA are or will be able to detect signals of such frequency, meaning that the direct measurement of the polar $l = 0$ mode oscillations is, for now and the foreseeable future, unrealizable.

Future work in this topic would, of course, pass by solving the dipolar perturbations both in the axial and polar sectors, which we did not treat numerically. These still correspond to previously unexistent excitation modes and could still hold precious information about the emission of gravitational waves in dRGT massive gravity. Further analysis of this topic could still lead either to the discovery of gravitational waves liable of being detected by our current experiments or to phenomena that could rule out the theory entirely. A full treatment of the polar $l = 1$ mode for the two types of motion presented here can be found in [1].

Another related topic that might be of interest would be the fate of the black hole after the perturbation. As was seen in this work, gravitational waves of frequency lower than the mass $\mu$ are not emitted, decaying exponentially near to the horizon. While these can be ignored for our purposes of finding the radiation that arrives far away from the black hole, these very low-frequency modes will undoubtedly affect in some way the black hole probably causing its evolution into a solution different than the Schwarzschild one.

Finally, one can also consider our work as a launching pad for the more complex case of the perturbation of a Kerr black hole. The study of this system would certainly lead to new phenomena when compared to the GR case, as happened when considering a Schwarzschild background. Taking rotation into consideration, it would also correspond more closely to realistic black holes, leading to a better hope of a comparison with experimental data from gravitational wave detectors. This would be a major step in the process of settling whether dRGT massive gravity could correspond to reality and, if not, where does it fail.

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