Gravitational waves and massive gravitons

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Abstract

Massive graviton theories are of interest as an alternative to General Relativity (GR) for their self-accelerating cosmological solutions. They are also conceptually interesting for several other aspects and phenomena they provide. In this thesis we study one such aspect of massive gravity, that of the emission of gravitational waves. We analyse this phenomenon for the case of a Schwarzschild black hole perturbed by a point particle in one of two geodesic trajectories: highly relativistic radial infall and circular orbit. Simplifying the field equations of this system through the decomposition in tensor spherical harmonics, we show that the monopolar and dipolar perturbations both lead to the emission of gravitational waves, unlike in GR. This emission corresponds to the new excitation modes introduced by the mass of the graviton. Finally, we explicitly compute the solutions for the monopolar mode, finding that, although it causes oscillations of a measurable amplitude, their frequency precludes detection by current and near future experiments. This and results for the dipolar modes are further discussed in reference [1].
Keywords: Proca theory, dRGT massive gravity, black holes, perturbation theory, gravitational waves

1. Introduction

The idea of a non-zero mass of the graviton first arose in 1939, through the Fierz-Pauli (FP) theory[2]. Knowing that the equations of motion coming from the linear theory of a spin-2 particle, tentatively called a graviton, were exactly the linearized Einstein equations, Markus Fierz and Wolfgang Pauli attempted to extend this theory by adding a mass to the field it was describing. An immediate result of this addition is that the low-energy potential associated with such field, Newtonian for zero mass, is now of the Yukawa type. For low enough graviton masses, the Yukawa factor modifies the potential solely at large distances. These modifications of the Newtonian potential have been shown to be a possibly viable alternative to dark energy as a solution to the current accelerated expansion of the universe, demonstrating the importance of studying more carefully massive gravity theories.

However, such theories can be quite problematic. In 1970, it was found that the zero mass limit of the FP theory did not, as expected, correspond to linearized GR. In fact, when considered in such limit, certain predictions of the theory concerning classical tests of GR turned out to be incorrect when compared with experimental observation. A major example of this is the value for the deflection of light by the Sun, first measured by Eddington in 1919. FP theory predicts this value to be $\frac{3}{4}$ of its real, measured value, this issue being known as the vDVZ discontinuity[3, 4]. An explanation for such difference was soon found: the FP theory, corresponding to a linear massive graviton, propagates five degrees of freedom, out of which, in the zero mass limit, two, $s = \pm 2$, were the ones of the massless graviton, two other, $s = \pm 1$, decoupled off matter, and the scalar mode (also called galileon in the literature), of $s = 0$, remained. It is this extra scalar mode that provokes the discontinuity existent in the limit.

Soon after, Vainshtein[5] discovered a possible solution to this issue. In the specific case of the light deflection, and for a given region of space, defined by the so called Vainshtein radius, the linear theory is not valid, the higher order terms of the galileon being comparable to the 1$^{\text{st}}$ order ones. Therefore, one needs to consider a fully nonlinear theory (and its zero mass limit) to find the true implications of a massive graviton. This was found to raise yet another issue[6] by Boulware and Deser. A generalization of the mass term of the FP theory to higher order on the spin-2 field leads to the appearance of a 6$^{\text{th}}$ mode, called the Boulware-Deser ghost. Moreover, this mode allows states with negative kinetic energy, something that is physically forbidden. While all this is trivially avoided in the FP theory, is not as trivial to eliminate in the nonlinear theory.

The problem was finally settled in 2011, with the construction of the dRGT massive gravity theory[7, 8]. By carefully building the generalized mass term at each order in the field, the theory was proven to be ghost-free, while, at the same time, solving the dVDZ discontinuity. The success of the theory quickly led to its study in other topics besides cosmology, originally its main motivation. Among these, there were several studies of black hole solutions, of which a greater variety than in GR was found[9]. Furthermore, the classical BH solutions of GR, the Schwarzschild and Kerr metrics, were found to also be solutions of dRGT massive gravity and their vacuum perturbations were studied to some detail[10].

Our purpose with this thesis was to study yet another phenomenom of GR in dRGT massive gravity, that of the emission of gravitational waves. This topic has been previously broached in the decoupling limit of the theory in [11], that is, by studying a theory of a massless graviton plus a galileon. The setup considered in this work was that of a Minkowskian background for the massless graviton and a spherical symmetric background solution for the galileon field, performing perturbation theory over them. However, this perturbation method was found to lead to negligible corrections to GR in extreme mass-ratio systems, while being non-valid in other situations. Instead, our approach was to perturb a Schwarzschild black hole directly in dRGT massive gravity theory, the perturbation being that of a point particle following a background geodesic.

2. BH perturbation and master equations

2.1. BH perturbation in dRGT massive gravity

To study the 1^{st} order perturbation of a Schwarzschild black hole in dRGT massive gravity is, as shown in Appendix A, to study the dRGT field equations (A.9) and the gauge and trace equations (A.10). The stress-energy tensor, being given by a point particle in geodesic motion, is

$$T^{(1)\mu\nu} = m_0 \int_{-\infty}^{\infty} d\tau \delta^{(4)}(x - z(\tau)) \frac{dz^\mu}{d\tau} \frac{dz^\nu}{d\tau} , \quad (1)$$

where m_0 is the mass of the point particle. We studied the two types of motion, the radial infall of a particle and its circular orbit around the black hole. The quantities that define them are also presented in Appendix C

To solve these equations we started by decomposing both the perturbation metric and the stress-energy tensor in tensor spherical harmonics, as demonstrated in Appendix B. This results in systems of coupled PDEs on the radius and time coordinates. We also separated the perturbation functions according to the parity of the corresponding harmonic metric. This originates two independent sectors of perturbations, which can be solved independently *a priori*. However, unlike the GR case treated originally by Regge and Wheeler[12] and afterwards by Zerilli[13], these are the sole simplifications we can make, there being no gauge freedom left to further reduce the number of unknown functions.

Our approach was then to Fourier transform the system of equations on the time coordinate, obtaining coupled ODEs on the radius. Then, for each multipolar order and parity sector we rearranged the equations in order to describe all functions in terms of the minimum possible number of 2^{nd} order ODEs, called the master equations. These are usually made to be inhomogeneous wavelike equations in the tortoise coordinate r_*, defined by

$$\frac{dr_*}{dr} = \left(1 - \frac{2M}{r}\right)^{-1} , \quad (2)$$

where M is the mass of the black hole. In the next sections we present a brief overview of the equations for the more relevant cases we studied, as well as their source functions. The details concerning the explicit rearrangements of the field and gauge equations into the master equations are available in a *Mathematica* notebook in [14].

2.2. Polar sector

In the GR case, the monopolar and dipolar perturbations are nonradiative, contributing, respectively, to an increase on the mass of the black hole and a Lorentz transformation of its center of mass. Therefore, the only interesting solutions in terms of gravitational waves are those for $l \geq 2$. The introduction of a massive graviton leads to small, μ-dependent corrections on the classical gravitational radiation.

However, in dRGT massive gravity these smaller l perturbations excite the system, leading to the emission of gravitational radiation. From a particle point of view, this makes sense: we now have three extra degrees of freedom that can be excited, when comparing with the massless graviton. Due to its classical nonexistence, it was on these two types of perturbations that we focused in the polar sector.

2.2.1 $l = 0$

For $l = 0$ the polar sector has four unknown perturbation functions, H_0, H_1, H_2 and K, the spherical harmonics of the other three not being defined. Through the field, gauge and trace equations we can write all these in terms of the function

K. Transfoming K through

$$K(r) = \frac{\sqrt{-4\mu^2 M + \mu^4 r^3 + 2\mu^2 r + 4r\omega^2}}{r^{5/2}}\varphi_0(r) \tag{3}$$

we obtain a wavelike equation,

$$\frac{d^2\varphi_0}{dr_*^2} + (\omega^2 - V_{\rm pol}^{l=0}(\omega, r))\varphi_0(r) = S_{\rm pol}^{l=0}(\omega, r)\,, \tag{4}$$

with some potential $V_{\rm pol}^{l=0}(\omega, r)$ (available on the *Mathematica* notebook). We note that the potential goes to zero at the horizon, as for the potentials of the Zerilli and Regge-Wheeler equations, but goes to μ^2 at infinity. This means that the frequency of the outgoing waves will be changed by this same factor with regard to the waves obtained in GR.

As for the source term, it is dependent on the type of motion under consideration. While a point particle in circular orbit does not give any contribution to the monopolar perturbation, the radial infall does, leading to the source term

$$\begin{aligned} &S_{\rm pol}^{l=0}(\omega, r) = \\ &= \frac{8\sqrt{2}\gamma m_0(r-2M)\left(\mu^2 r + 2i\omega\right)e^{i\omega T(r)}}{\sqrt{r}\left(-4\mu^2 M + \mu^4 r^3 + 2\mu^2 r + 4r\omega^2\right)^{3/2}}\,, \end{aligned} \tag{5}$$

where γ is the boost factor which goes to zero both at the horizon and at infinity, in what concerns the radius, as was expected. In our work we computed the numerical solution of this equation and we present the method and the results obtained in section 3.

2.2.2 $l = 1$

For $l = 1$ we have two more defined spherical harmonics, meaning two more unknown functions. We can write all six through two perturbation functions, K and η_1, obtaining the master equations

$$\frac{d^2\eta_1}{dr_*^2} + p_{\eta\eta}\frac{d\eta_1}{dr_*} + p_{K\eta}\frac{dK}{dr_*} + q_{\eta\eta}\eta_1 + q_{K\eta}K = S_{pol,\eta}^{l=1}\,, \tag{6}$$

$$\frac{d^2K}{dr_*^2} + p_{\eta K}\frac{d\eta_1}{dr_*} + p_{KK}\frac{dK}{dr_*} + q_{\eta K}\eta_1 + q_{KK}K = S_{pol,K}^{l=1}\,, \tag{7}$$

where the p_{ij} and q_{ij} are r and ω coefficientes defined in the aforementioned *Mathematica* notebook. This mode is excited both by the radial infall of a point particle and by its circular orbit. In the former case, the source terms are given by

$$\begin{aligned} &S_{pol,K}^{l=1}(\omega, r) = \\ &= \frac{8\sqrt{6}\gamma m_0\left(\mu^2 r^2 + 2ir\omega + 2\right)e^{i\omega T(r)}}{(4\mu^2 M r^2 - 8M - \mu^4 r^5 - 6\mu^2 r^3 - 4r^3\omega^2)}\,, \end{aligned} \tag{8}$$

$$S_{pol,\eta}^{l=1}(\omega, r) = \frac{F(r)}{r}S_{pol,K}^{l=1}\,, \tag{9}$$

while in the latter they are of the form

$$\begin{aligned} S_{pol,\eta}^{l=1}(\omega, r) &= A_\eta(\omega, r)\delta(r - R_c) \\ &+ B_\eta(\omega, r)\delta'(r - R_c)\,, \end{aligned} \tag{10a}$$

$$\begin{aligned} S_{pol,K}^{l=1}(\omega, r) &= A_K(\omega, r)\delta(r - R_c) \\ &+ B_K(\omega, r)\delta'(r - R_c)\,, \end{aligned} \tag{10b}$$

where $\delta'(r - R_c)$ is the derivative of the Dirac delta and R_c is the radius of the orbit of the point particle. The numerical solutions to these equations were obtained in [1].

2.3. Axial sector

Out of the three spherical harmonics corresponding to the axial sector, none of them is defined for $l = 0$. We obtained the equations for $l = 1$, which in GR is also nonradiative, corresponding to a contribution to the angular momentum of the originally static black hole. Instead, in dRGT massive gravity these equations are wavelike, meaning that one of the new states of the massive graviton may be excited by a dipolar axial perturbation. We also obtained the $l \geq 2$ equations, which merely represent a small correction to the GR Regge-Wheeler equations, dependent on μ.

2.3.1 $l = 1$

For $l = 1$ only the spherical harmonics corresponding to h_0 and h_1 are defined. Manipulated the field, trace and gauge equations we obtain a single master equation,

$$\frac{d^2\mathbf{\Psi}}{dr_*^2} + [\omega^2 - \mathbf{V}_{ax}(r)]\mathbf{\Psi} = \mathbf{S}_{\rm ax}^{l=1}\,, \tag{11}$$

where

$$Q(r) = \frac{h_1(r)}{F(r)}\,, \tag{12}$$

with $F(r) = \left(1 - \frac{2M}{r}\right)$ and

$$V_{\rm ax}^{l=1}(r) = F(r)\left(\mu^2 + \frac{6}{r^2} - \frac{16M}{r^3}\right)\,. \tag{13}$$

However, the source term $\mathbf{S}_{\rm ax}^{l=1}$ is zero for both types of motion of the point particle that we studied. This means these perturbations do not excite any extra mode of the system.

2.3.2 $l \geq 2$

For $l \geq 2$ we now have all three axial spherical harmonics defined, together with three perturbation functions. We defined them through a system of two coupled wavelike equations,

$$\frac{d^2\mathbf{\Psi}}{dr_*^2} + [\omega^2 - \mathbf{V}_{ax}(r)]\mathbf{\Psi} = \mathbf{S}_{ax}^{l\geq 2}\,, \tag{14}$$

where $\mathbf{\Psi} = \begin{pmatrix} Q & Z \end{pmatrix}^T$ and, bearing in mind that $\lambda = l(l+1)$:

$$Q(r) = \frac{h_1(r)}{F(r)} \quad , \quad Z(r) = h_2(r) \, , \tag{15}$$

and

$$\mathbf{V}_{ax} = F(r) \begin{pmatrix} \mu^2 + 2\frac{\lambda+3}{r^2} - \frac{16M}{r^3} & 2i\lambda\frac{3M-r}{r^3} \\ \frac{4i}{r^3} & \mu^2 + \frac{2\lambda}{r^2} + \frac{2M}{r^3} \end{pmatrix} . \tag{16}$$

While the source term is zero for the radial infall, it is not for a circular orbit:

$$\mathbf{S}_{ax}^{l\geq 2} = \begin{pmatrix} 0 \\ -\frac{8\sqrt{2}\pi F(r) r}{\sqrt{\lambda(\lambda+1)}} D_{lm}(\omega, r) \end{pmatrix} , \tag{17}$$

where $D_{lm}(\omega, r)$ is one of the functions in the spherical harmonics decomposition of the stress-energy tensor.

3. Monopolar perturbation

For this thesis we studied more in depth the $l = 0$ perturbation in the polar sector, solving numerically equation (4), whose notation we simplify to

$$\frac{d^2\varphi_0}{dr_*^2} + (\omega^2 - V(\omega, r))\varphi_0(r) = S(\omega, r) \, . \tag{18}$$

Here we present the methods used to do it and what information we were able to extract for such solution.

3.1. Numerical methods

The method used to solve equation (18) was that of the variation of constants. This requires that we first solve the corresponding homogeneous (source-free) equation, afterwards using these to build the solution to the non-homogeneous one. To find them, we must analyse equation (18) at the boundaries, that is, at the horizon ($r \to r_s \Leftrightarrow r_* \to -\infty$) and at infinity ($r \to +\infty \Leftrightarrow r_* \to +\infty$):

$$\frac{d^2\psi_h}{dr_*^2} + \omega^2\psi_h = 0 \, , \tag{19}$$

$$\frac{d^2\psi_h}{dr_*^2} + (\omega^2 - \mu^2)\psi_h = 0 \, . \tag{20}$$

These equations show us what are the admissible solutions at the boundaries: ingoing and outgoing waves of frequency ω, in one case, and of frequency $\sqrt{\omega^2 - \mu^2}$, in the other. Based on this, we can finally state the boundary conditions of our homogeneous solutions. We impose that one of the solutions, from hereon named ψ_h, must be composed only by ingoing waves at the horizon, while the other, ψ_∞, must be composed only by outgoing waves at infinity. Having such conditions, we computed these two solutions through the *NDSolve* function of the software *Mathematica*.

With the homogeneous solutions, we can pose the ansatz for the solutions of the non-homogeneous solutions

$$\varphi_0(r) = C_h(r)\psi_h(r) + C_\infty(r)\psi_\infty(r) \, , \tag{21}$$

where C_h and C_∞ must be such that

$$C_h'(r)\psi_h(r) + C_\infty'(r)\psi_\infty(r) = 0 \, . \tag{22}$$

As our purpose was to study the gravitational waves we may detect coming from the perturbed black hole, we focused on the outgoing part of the solution, $\Psi_{\text{out}} = C_\infty(r)\psi_\infty(r)$. This function can be shown to be given by

$$\begin{aligned} \Psi_{\text{out}}(\omega, r_*) &= \psi_\infty(\omega, r_*) \int_{-\infty}^{r_*} dr'_* \, \frac{\psi_h(\omega, r')S(\omega, r')}{W(\omega, r')} \\ &= e^{ik_\omega r_*} \int_{-\infty}^{+\infty} dr'_* \, \frac{\psi_h(\omega, r'_*)S(\omega, r'_*)}{W(\omega, r'_*)} \, , \end{aligned} \tag{23}$$

where $W(\omega, r') = \psi_h\psi_\infty' - \psi_h'\psi_\infty$ is the Wronskian and the last equality was obtained in the large radius limit. This was the expression used in our computations of the non-homogeneous solutions. To finally obtain the solution in the time domain, we merely have to take the Fourier transform of (23),

$$\Psi_{\text{out}}(t, r_u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \, \Psi_{\text{out}}(\omega, r_u) e^{-i\omega t} \, , \tag{24}$$

which we did for specific radii r_u, defined through the quantity we called the extraction radius $R = \mu r_u$.

3.2. Power spectra

With expression (23), we numerically computed the master function φ_0 of the polar $l = 0$ mode for a radially infalling highly relativistic particle. We did so taking $\gamma = 1$, $m_0 = 1$ and $M = 1$. From it, we were able to obtain the corresponding energy spectrum in the frequency domain, through the definition of $\frac{dE}{d\omega}$ in expression (D.14). We calculated it for several values of the graviton mass μ, and present the plots in figure 1.

As can be directly seen, all the spectra for this mode peak quite close to the frequency corresponding to the graviton mass μ, rapidaly decaying after this peak. On the other hand, for $\omega < \mu$ the spectra are null. This happens because for these frequencies the outgoing modes are in fact exponentially dampened, due to the factor of $e^{ik_\omega r_*}$ in expression (23). While unimportant for the emission of gravitational radiation at very long distances from the

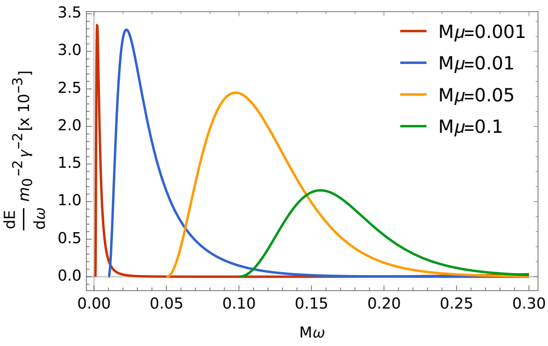

Figure 1: GW energy spectra for the $l = 0$ polar mode for a radially infalling particle and $M\mu = \{0.001, 0.01, 0.05, 0.1\}$.

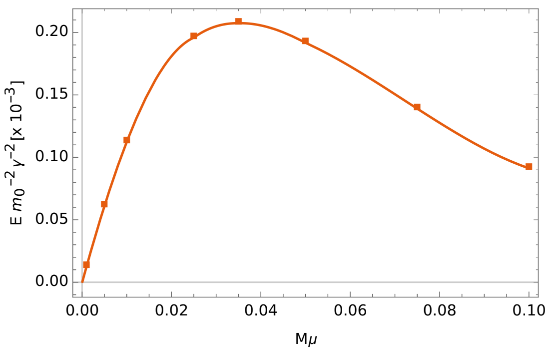

Figure 2: Total energy of the polar $l = 0$ mode. The presented curve is an interpolation of the calculated points, represented by the squares.

black hole, on which we are focusing, these low-frequency modes may prove important for the evolution of a Schwarzschild black hole after it is perturbed, meriting some further study.

Integrating the spectra, we also obtained the energy of the waves in terms of μ, shown in figure 2. For a low graviton mass, which we know must be our case, the energy seems to decrease to zero. This is in accordance with what we would expect from a working theory of a massive graviton: as μ decreases we gradually approach the massless graviton, that is, GR, in which the polar $l = 0$ mode does not radiate any energy away.

3.3. Waveforms

Applying the inverse Fourier transform to the frequency domain solution we obtained the signal, in function of retarded time, of the master function, which we plotted for different values of μ and extraction radius R.

We sweeped a larger range of values of μ and R than the ones presented, respectively, $\{0.01, 0.025, 0.05, 0.075, 0.1\}$ and $\{10, 25, 50, 75, 100, 150, 200, 250\}$. For each of these waveforms we took the peak value of the master function, φ_0^{peak}, and the delay between the beginning of the signal and its peak, $(t-r)^{\text{peak}}$, and fitted this data to powers of $M\mu$ and R, obtaining the following expressions:

$$
\begin{aligned}
K^{\text{peak}} &\sim 0.068 \frac{0.01 m_0 \gamma}{0.01 M} \frac{M^{0.60}}{\mu^{0.84} {r_u}^{1.44}} \\
&\sim 8.1 \times 10^{-17} \frac{m_0 \gamma}{0.01 M} \left(\frac{10^{-23}\text{eV}}{\mu}\right)^{0.84} \left(\frac{M}{M_\odot}\right)^{0.60} \left(\frac{8\text{kpc}}{r}\right)^{1.44} \\
&\sim 3.7 \times 10^{-23} \frac{m_0 \gamma}{0.01 M} \left(\frac{10^{-23}\text{eV}}{\mu}\right)^{0.84} \left(\frac{M}{M_\odot}\right)^{0.60} \left(\frac{1\text{Gpc}}{r}\right)^{1.44} ,
\end{aligned} \tag{25}
$$

$$
(t-r)^{\text{peak}} \sim 1.976 (M\mu)^{-0.285} R^{0.895} M \sim 1.976 M^{0.715} \mu^{0.610} {r_u}^{0.895} \tag{26}
$$

$$
\sim 133 \left(\frac{\mu}{10^{-23}\text{eV}}\right)^{0.610} \left(\frac{M}{M_\odot}\right)^{0.715} \left(\frac{r}{8\text{kpc}}\right)^{0.895} s \tag{27}
$$

$$
\sim 4.85 \times 10^{6} \left(\frac{\mu}{10^{-23}\text{eV}}\right)^{0.610} \left(\frac{M}{M_\odot}\right)^{0.715} \left(\frac{r}{1\text{Gpc}}\right)^{0.895} s , \tag{28}
$$

where the M dependency was adjusted with dimensional analysis considerations and the distance of 8kpc was picked to correspond to Saggitarius A*, thought to be a supermassive black hole at the centre of our galaxy.

These results indicate that, for realistic values of μ, the peak of the amplitude of the GWs we are studying has a large enough value to have a chance of being detected by either LIGO or, in the future, by LISA, while reaching us within a reasonable time of the beginning of the signal. On reaching the detectors, these would feel an identi-

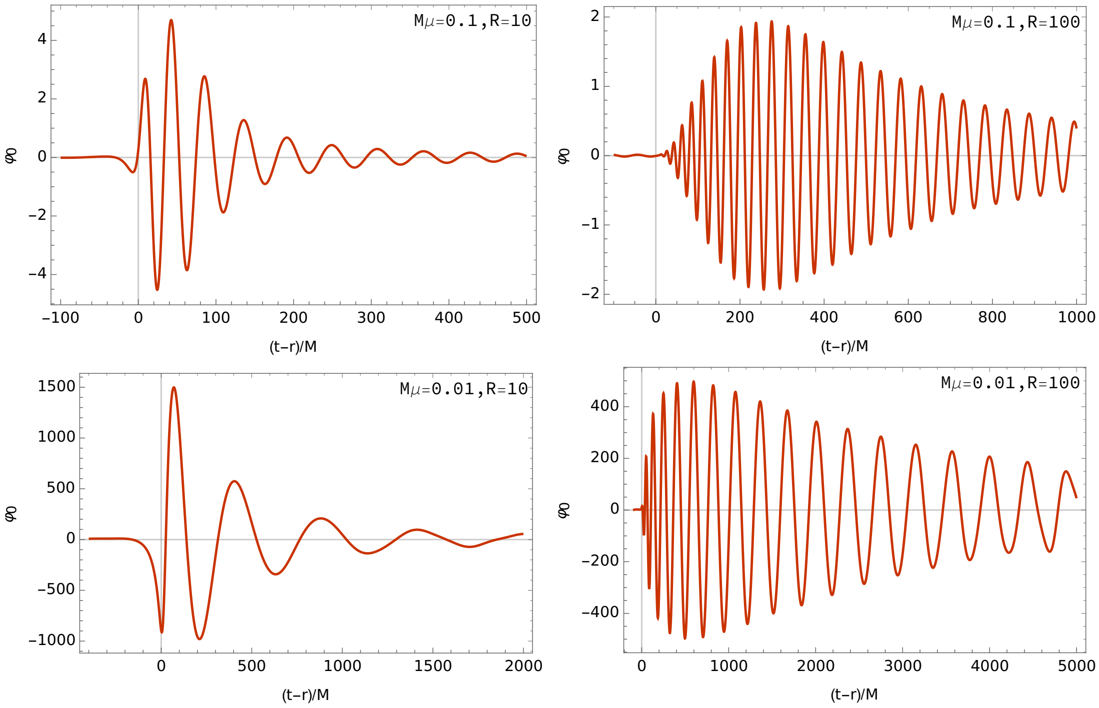

Figure 3: Waveforms of φ_0 for $M\mu = \{0.1, 0.01\}$ and $R = \{10, 100\}$

cal perturbation, differing only in the amplitude, depending on how far each of the detectors would be from the source.

However, considerations from the previous section extinguish our hopes of detecting a signal with a contribution from the polar $l = 0$ mode. We have seen that the frequency of oscillation for which the energy of the wave peaks is $\omega \sim \mu < 1.0 \times 10^{-23}\text{eV}/c^2 = 2.42 \times 10^{-9}\text{Hz}$, which is manifestly out of the range of both current and near future detectors.

4. Conclusions

In the dRGT massive gravity theory, the idea of a massive graviton has matured to a point where it can not only rival GR in what was its original motivation but also be compared with it in several other sectors. Our work of the perturbation of a Schwarzschild black hole metric by a point particle was a step in this direction, as it had, to our knowledge, never before been broached in this particular theory. This system proved to be a fertile ground for comparisons between the two theories and also opens the way for further studies of dRGT massive gravity.

We had two main results. First of all, after obtaining the master equations for each multipolar order and parity sector, we concluded that the monopolar and dipolar perturbations led to oscillating solutions, unlike in GR. These correspond to the excitation of modes the massless graviton does not possess, being the vector and scalar modes of the massive graviton. Such modes would lead to waveforms different than the ones we have so far detected, their detection (or absence) possibly providing a smoking gun as to the validity of dRHT massive gravity.

Secondly, we studied in detail the polar $l = 0$ mode for the radial infall of a point particle. We concluded that the signal it produces was not only of a reasonable enough amplitude to be measured but also that it should reach such amplitude in a fairly short time interval. However, we also found that the energy of this signal is mainly carried by oscillations of frequency $\omega \sim \mu < 10^{-9}\text{Hz}$, which is blatantly out of the range of our current and planned detectors. Another conclusion of this analysis was that the total energy carried by this mode goes to zero with the mass of the graviton, as one would expect: the zero mass limit of the theory would give us GR, for which the polar $l = 0$ mode is not excited.

However, the dipolar perturbations can still reveal some unexpected features of dRGT massive gravity. The work [1] extends this thesis in this direction.

Besides this, our work still has several threads from which to pull in the future. One such thread corresponds to the low-frequency modes that are not emitted by the perturbed black hole. These will certainly trigger the evolution of the Schwarszchild solution into something else entirely, possibly revealing new black hole solutions in dRGT massive gravity.

Another topic that might be of interest is the study of the perturbations not of a static black hole but of a rotating one, starting from the Kerr solution. This would be a more realistic model of a black hole, leading to waveforms that would resemble more closely what our detectors would hypothetically measure. This would be a major step in the process of settling whether dRGT massive gravity is a valid theory and, if not, where it does fail.

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A. Field equations of dRGT massive gravity

The theory of dRGT massive gravity is based on bimetric gravity, a theory in which we have not one but two metrics. The generic lagrangian of such a theory is given by

$$S=\int d^4x\sqrt{-g}\left[R_g+\frac{\sqrt{-f}}{\sqrt{-g}}R_f\right]. \tag{A.1}$$

where R_f is the Ricci scalar of the $\mathbf{f}$ metric and likewise for $\mathbf{g}$. For dRGT massive gravity, we add to this action a mass term, which consists on a potential built on a particular combination of the two metrics, $\sqrt{g^{-1}f}$, defined by $\sqrt{g^{-1}f}\sqrt{g^{-1}f}=g^{-1}f$. For our purposes, we can consider the $\mathbf{f}$ metric to be non-dynamical, being fixed to some metric, while $\mathbf{g}$ is a perturbation of said metric. This leads to the action

$$S=\int d^4x\sqrt{-g}\left[M_g^2R_g-2M_v^4\sum_{n=0}^{4}\beta_nV_n(\sqrt{g^{-1}f})\right]+S_m\,, \tag{A.2}$$

where S_m is the matter action, M_g^2 and M_v^2 are coupling constants and:

$$V_0(\Pi)=1\,, \tag{A.3}$$
$$V_1(\Pi)=[\Pi]\,, \tag{A.4}$$
$$V_2(\Pi)=\frac{1}{2}\left([\Pi]^2-[\Pi^2]\right), \tag{A.5}$$
$$V_3(\Pi)=\frac{1}{6}\left([\Pi]^3-3[\Pi][\Pi^2]+2[\Pi^3]\right), \tag{A.6}$$
$$V_4(\Pi)=\det(\Pi)\,, \tag{A.7}$$

$[\Pi]$ being the trace of the tensor Π.

The field equations of this theory are

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\frac{M_v^4}{M_g^2}\sum_{n=0}^{3}(-1)^n\beta_ng_{\mu\rho}Y^{\rho}{}_{(n)\nu}(\sqrt{g^{-1}f})=\frac{1}{2M_g^2}T_{\mu\nu}\,, \tag{A.8}$$

where $Y^{\rho}_{(n)\nu}(\Pi)=\sum_{r=0}^{n}(-1)^r(\Pi^{n-r})^{\rho}{}_{\nu}V_r(\Pi)$. To obtain the linearised field equations of this theory we pose that $g_{\mu\nu}=f_{\mu\nu}+h_{\mu\nu}$, $\mathbf{h}$ being the perturbation metric. From this comes the relation $(\sqrt{g^{-1}f})^{\rho}{}_{\nu}\simeq\delta^{\rho}_{\nu}-\frac{1}{2}f^{\rho\mu}h_{\mu\nu}$, which, after lengthy substitution, gives us

$$\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma}+\frac{\mu^2}{2}(h_{\mu\nu}-f_{\mu\nu}h)=\frac{1}{2M_g^2}T^{(1)}_{\mu\nu}\,, \tag{A.9}$$

where $\mu^2=\frac{M_v^4}{M_g^2}(\beta_1+2\beta_2+\beta_3)$ the mass of the graviton, $T^{(1)}_{\mu\nu}$ is the first order part of the stress energy tensor and $\overline{\mathcal{E}}^{\rho\sigma}{}_{\mu\nu}h_{\rho\sigma}$ is the usual linearised Einstein tensor. Taking the covariant derivative $\overline{\nabla}^{\mu}$ and the trace of equation (A.9) we obtain, respectively,

$$\overline{\nabla}^{\mu}h_{\mu\nu}=\alpha\overline{\nabla}_{\nu}T\,, \tag{A.10a}$$
$$h=\alpha T\,, \tag{A.10b}$$

with $\alpha=-\frac{2M_g^2}{3\mu^2}$.

B. Decomposition in tensor spherical harmonics

To simplify the field equations we made use of the spherical symmetry of the system we studied and decomposed the field equations in tensor spherical harmonics, as done by Regge and Wheeler in [12] and Zerilli in [13]. These spherical harmonics, defined correctly in [15], are tensors that must be orthonormal to each other

$$\left(\mathbf{a}^{(\mathrm{sph})}_{lm},\mathbf{a}^{(\mathrm{sph})}_{l'm'}\right)=\delta_{ll'}\delta_{mm'}\,. \tag{B.1}$$

with respect to the inner product

$$\left(\mathbf{A},\mathbf{B}\right)=\int d\Omega\,A^*_{\mu\nu}\eta^{\mu\alpha}\eta^{\nu\beta}B_{\alpha\beta}\,, \tag{B.2}$$

η being the Minkowski metric and $\mathbf{A}$ and $\mathbf{B}$ being any two 2^{nd} order tensors.

The spherical harmonics can be further divided betwene the axial ones, ($\mathbf{c^{(0)}}_{lm}$,$\mathbf{c}_{lm}$,$\mathbf{d}_{lm}$) and the polar ones, ($\mathbf{a}_{lm}$,$\mathbf{a^{(0)}}_{lm}$,$\mathbf{a^{(1)}}_{lm}$,$\mathbf{b^{(0)}}_{lm}$,$\mathbf{b}_{lm}$,$\mathbf{g}_{lm}$,$\mathbf{f}_{lm}$). These two sectors are completely independent, further simplifying the field equations.

Knowing this, we can pose an ansatz for the perturbation metric,

$$\mathbf{h}=\mathbf{h^{pol}}+\mathbf{h^{ax}}\,, \tag{B.3}$$

$$
\begin{aligned}
\mathbf{h^{pol}} =& \sum_{lm} \Bigg[F(r) H_{0lm}(t,r)\mathbf{a}^{(0)}_{lm} - i\sqrt{2} H_{1lm}(t,r)\mathbf{a}^{(1)}_{lm} + \frac{1}{F(r)} H_{2lm}(t,r)\mathbf{a}_{lm} \\
& - \frac{i}{r}\sqrt{2l(l+1)}\eta_{0lm}(t,r)\mathbf{b}^{(0)}_{lm} + \frac{1}{r}\sqrt{2l(l+1)}\eta_{1lm}(t,r)\mathbf{b}_{lm} \\
& + \sqrt{\frac{1}{2}l(l+1)(l(l+1)-2)} G_{lm}(t,r)\mathbf{f}_{lm} + \Bigg(\sqrt{2}K_{lm}(t,r) - \frac{l(l+1)}{\sqrt{2}} G_{lm}(t,r) \Bigg)\mathbf{g}_{lm} ,
\end{aligned} \tag{B.4}
$$

$$
\mathbf{h^{ax}} = -\frac{\sqrt{2l(l+1)}}{r} h_{0lm}(t,r)\mathbf{c}^{(0)}_{lm} + \frac{i\sqrt{2l(l+1)}}{r} h_{1lm}(t,r)\mathbf{c}_{lm} + \frac{\sqrt{2l(l+1)(l(l+1)-2)}}{2r} h_{2lm}(t,r)\mathbf{d}_{lm} \Bigg] . \tag{B.5}
$$

As for the stress-energy tensor, we can decompose it similarly,

$$
\begin{aligned}
\mathbf{T} =& \sum_{lm} \big[A^{(0)}_{lm}(t,r)\mathbf{a}^{(0)}_{lm} + A^{(1)}_{lm}(t,r)\mathbf{a}^{(1)}_{lm} + A_{lm}(t,r)\mathbf{a}_{lm} + B^{(0)}_{lm}(t,r)\mathbf{b}^{(0)}_{lm} + B_{lm}(t,r)\mathbf{b}_{lm} + G^{(s)}_{lm}(t,r)\mathbf{g}_{lm} \\
& + F_{lm}(t,r)\mathbf{f}_{lm} + Q^{(0)}_{lm}(t,r)\mathbf{c}^{(0)}_{lm} + Q_{lm}(t,r)\mathbf{c}_{lm} + D_{lm}(t,r)\mathbf{d}_{lm} \big] ,
\end{aligned} \tag{B.6}
$$

where the perturbation functions ($A^{(0)}_{lm}(t,r)$,$A^{(1)}_{lm}(t,r)$,...,$D_{lm}(t,r)$) are given by the inner product between the stress-energy tensor and the corresponding spherical harmonic. For instance:

$$
A^{(0)}_{lm}(t,r) = \big(\mathbf{a}^{(0)}_{lm}, \mathbf{T}\big) . \tag{B.7}
$$

C. Definition of the stress-energy tensor

We studied our field equations for two different types of motion of the point particle, highly relativistic radial infall and circular orbit. The former is defined by the quantities:

$$
\begin{aligned}
&\frac{dT}{d\tau} = \frac{\gamma}{\big(1-\frac{2M}{r}\big)} \quad , \quad \frac{dR}{dt} = -\Bigg(1 - \frac{2M}{r}\Bigg) , \\
&\frac{d\Theta}{dt} = 0 \quad , \quad \frac{d\Phi}{dt} = 0 \quad , \quad \frac{dT}{dr} = -\frac{1}{1-\frac{2M}{r}} ,
\end{aligned} \tag{C.1}
$$

where γ is the Lorentz factor of the particle. As for the circular orbit, it is defined by:

$$
\begin{aligned}
&\frac{dT}{d\tau} = \frac{1}{\sqrt{1-\frac{3M}{r}}} \quad , \quad \frac{dR}{dt} = 0 , \\
&\frac{d\Theta}{dt} = 0 \quad , \quad \frac{d\Phi}{dt} = \sqrt{\frac{M}{r^3}} = \omega_c .
\end{aligned} \tag{C.2}
$$

Substituting these quantities into the source functions defined in [15] gives us the source terms presented in section 2

D. Energy spectra

To calculate the energy spectrum $\frac{dE}{d\omega}$ we need to start from the stress-energy tensor of the field perturbation, obtained from Noether's theorem:

$$
T^{GW}_{\mu\nu} = \Bigg\langle \frac{\delta \mathcal{L}}{\delta(h^{\alpha\beta,\mu})} h^{\alpha\beta}{}_{,\nu} - \eta_{\mu\nu}\mathcal{L} \Bigg\rangle , \tag{D.1}
$$

where $\mathcal{L}$ is the lagrangian of the linearised dRGT massive gravity theory, which, at far field, is:

$$
\begin{aligned}
\mathcal{L} =& \frac{1}{64\pi}\Bigg[-\partial_\rho h_{\mu\nu}\partial^\rho h^{\mu\nu} + 2\partial_\rho h_{\mu\nu}\partial^\nu h^{\mu\rho} \\
& - 2\partial_\nu h^{\mu\nu}\partial_\mu h + \partial_\mu h \partial^\mu h + m_g^2\Bigg(h^2 - \frac{1}{2}h_{\mu\nu}h^{\mu\nu}\Bigg)\Bigg] .
\end{aligned} \tag{D.2}
$$

For this lagrangian, $T^{GW}_{\mu\nu}$ then reduces to:

$$
\Big\langle h_{\alpha\beta,\mu} h^{\alpha\beta}{}_{,\nu} \Big\rangle . \tag{D.3}
$$

This allows us to calculate the energy loss of the radiating source,

$$
\frac{dE}{dt} = \int dS^i\, T^{\rm GW}_{0i} = \int d\Omega\, T^{\rm GW}_{0i} n^i r^2 = \int d\Omega\, T^{\rm GW}_{0r} r^2 , \tag{D.4}
$$

where n^i is the normal of the surface element dS^i. Combining this with the expression for the gravitational stress-energy tensor, we get:

$$
\frac{dE}{dt} = \frac{r^2}{32\pi}\sum_{l,m}\sum_{l',m'} \int d\Omega \Big\langle h_{\alpha\beta,0} h^{\alpha\beta}{}_{,r} \Big\rangle . \tag{D.5}
$$

Substituting the metric (B.3) and integrating over time, we obtain the lost energy of the system:

$$
E = \frac{r^2}{32\pi}\int_{-\infty}^{\infty} dt\, [\mathcal{C}^{ax,lm}_{0r} + \mathcal{C}^{pol,lm}_{0r}] , \tag{D.6}
$$

$$\mathcal{C}^{ax,lm}_{0r} = -\frac{2l(l+1)}{r^2}h^*_{0,t}h_{0,r} + \frac{2l(l+1)}{r^2}h^*_{1,t}h_{1,r} + \frac{l(l+1)(l(l+1)-2)}{2r^2}h^*_{2,t}h_{2,r}\Big)\,, \tag{D.7}$$

$$\begin{aligned}\mathcal{C}^{ax,lm}_{0r} &= H^*_{0,t}H_{0,r} - 2H^*_{1,t}H_{1,r} + H^*_{2,t}H_{2,r} - \frac{2l(l+1)}{r^2}\eta^*_{0,t}\eta_{0,r} + \frac{2l(l+1)}{r^2}\eta^*_{1,t}\eta_{1,r} \\ &+ \frac{1}{2}l(l+1)(l(l+1)-2)G^*_{,t}G_{,r} + 2K^*_{,t}K_{,r} + \frac{l(l+1)}{2}G^*_{,t}G_{,r} - l(l+1)(K^*_{,t}G_{,r} + G^*_{,t}K_{,r})\,.\end{aligned} \tag{D.8}$$

We can simplify this further by analysing the perturbation functions. We can write any of them (we will represent them by a generic function A_{lm}) in terms of their Fourier transforms:

$$\begin{aligned}A_{lm}(t,r) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}A_{lm}(\omega,r) \\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}e^{i\sqrt{\omega^2-\mu^2}r}A_{lm}(\omega)\,,\end{aligned} \tag{D.9}$$

where the last equality is an approximation valid at long distances and α is the power of the radius with which the function decays. From this we can take the time and radial derivatives of the perturbation functions:

$$\frac{dA_{lm}}{dt} = \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}e^{i\sqrt{\omega^2-\mu^2}r}A_{lm}(\omega)(-i\omega)\,, \tag{D.10}$$

$$\frac{dA_{lm}}{dr} \simeq \frac{1}{\sqrt{2\pi}}\frac{1}{r^\alpha}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega t}e^{i\sqrt{\omega^2-\mu^2}r}A_{lm}(\omega)\left(i\sqrt{\omega^2-\mu^2}\right)\,, \tag{D.11}$$

with which we can obtain the relation

$$\begin{aligned}\int_{-\infty}^{\infty} dt\left\langle \frac{dA^*_{lm}}{dt}\frac{dA_{lm}}{dr}\right\rangle &= -\frac{1}{2\pi}\frac{1}{r^{2\alpha}}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}d\omega'\, e^{i(\omega'-\omega)t}e^{i\left(\sqrt{\omega^2-\mu^2}-\sqrt{\omega'^2-\mu^2}\right)}A^*_{lm}(\omega')A_{lm}(\omega)\omega'\sqrt{\omega^2-\mu^2} \\ &= -\frac{1}{r^{2\alpha}}\int_{-\infty}^{\infty}d\omega\,|A_{lm}(\omega)|^2\omega\sqrt{\omega^2-\mu^2}\,.\end{aligned} \tag{D.12}$$

This last relation allows us to write the total lost energy as an integral in $d\omega$ of some function. We define this integrand to be the energy spectrum $\frac{dE}{d\omega}$. Substituting the correct values of α for each perturbation function ($\alpha = 0$ for η_0, η_1, h_0, h_1 and h_2 and $\alpha = 1$ for H_0, H_1, H_2, K and G) we can finally write the energy spectrum,

$$\begin{aligned}\frac{dE}{d\omega} = -\frac{\omega\sqrt{\omega^2-\mu^2}}{32\pi}\Big(&|H_0(\omega)|^2 - 2|H_1(\omega)|^2 + |H_2(\omega)|^2 - 2\lambda|\eta_0(\omega)|^2 + 2\lambda|\eta_1(\omega)|^2 \\ &+ \frac{1}{2}\lambda(\lambda-1)|G(\omega)|^2 + 2|K(\omega)|^2 - \lambda(K^*(\omega)G(\omega) + G^*(\omega)K(\omega))\,. \\ &- 2\lambda|h_0(\omega)|^2 + 2\lambda|h_1(\omega)|^2 + \frac{1}{2}\lambda(\lambda-2)|h_2(\omega)|^2\Big)\end{aligned} \tag{D.13}$$

Specifying all these functions for the mode we solved numerically, the polar $l = 0$, we obtain a formula for the spectrum dependent only on the master function φ_0:

$$\left.\frac{dE}{d\omega}\right|_{\text{polar},l=0} = -\frac{3\mu^4}{8\pi}\omega\sqrt{\omega^2-\mu^2}|\varphi_0(\omega)|^2\,. \tag{D.14}$$

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