Cyclic Cosmologies

Inês Terrucha*
Centro de Astrofísica e Gravitação - CENTRA, Departamento de Física, Instituto Superior Técnico - IST, Universidade de Lisboa - UL, Avenida Rovisco Pais 1, 1049-001, Portugal

The big bang initial singularity is one of the outstanding problems in current theoretical cosmology. Universes that collapse with a bounce to expand back again, possibly in self-sustaining cycles, can avoid the initial singularity altogether and thus solve this problem. There are several proposals for bouncing universes, a common feature between them is that at such early times and so at such high energies, Planck scale quantum gravity phenomena must take hold of the physical processes and thus general relativity ceases to be valid there. In this thesis, to find a universe with a bouncing solution, we propose a theory that extends general relativity, by adding to the usual Einstein-Hilbert Ricci scalar term R in the gravitational action, an $f(G)$ term, where G is the Gauss-Bonnet invariant, and f some function of it. In general, the modified gravity theories contain higher order differential equations in the metric gravitational potentials making their solutions difficult to interpret. To bypass this complication, resort to a technique that reduces the order of the $f(G)$ theory to second order differential equations is necessary. This order reduction technique enables one to find solutions which are perturbatively close to general relativity. In building the covariant gravitational action it is found that $f(G)$ has to include the terms $G \ln G$ and $\sqrt[6]{G}$ to have a theory that yields a universe with a bounce at some critical density ρ_c.

Keywords: general relativity; cyclic cosmology; bouncing solution; modified Gauss-Bonnet; $f(G)$ gravity

I. INTRODUCTION

The physics of the big bang initial singularity is still an open problem in cosmology and fundamental physics. A complete description of the universe must avoid at all costs spacetime singularities as their existence makes the future physically unpredictable. It is supposed that in a quantum gravity regime new physics sets in and spacetimes singularities get a proper description. In such a regime there are several possibilities, but an intriguing one is to suppose that at some tiny scale, of the order of few Planck units, the universe undergoes a bounce, such that a previously collapsing universe expands back again originating our own visible universe, yet to possibly collapse again in a cyclic way, giving rise to a cyclic universe.

Sotiriou [1] analyzed $f(R)$ gravity with the aim of finding bouncing cosmological solutions. These $f(R)$ theories are extensions of General Relativity in the sense that the Einstein-Hilbert action that does depend only on the Ricci scalar R, and possibly on the cosmological constant Λ, can have a more general form substituting R by a generic function $f(R)$ [2]. The $f(R)$ theories possess, in general, spacetime derivatives of order higher than two, which in turn can generate non physical solutions. One way to deal with these solutions is to exclude them whenever they appear. A more interesting way is to devise a method in which these solutions do not appear at all. One such method states that the interesting physical solutions in $f(R)$ theories are the ones that can be found from General Relativity in a perturbative way, the other solutions being considered artificial. This is the technique of order reduction [3, 4] which builds solutions perturbatively close to General Relativity which are then second order differential equations by construction. Relying on this method it was possible to construct a Friedmann equation with a bounce [1], namely, $H^2 = \frac{8\pi}{3} G_N \rho \left(1 - \frac{\rho}{\rho_c}\right)$, where $H = H(t)$ is the Hubble function defined in terms of the scale factor $a(t)$ by $H \equiv \dot{a}/a$, with t being the time parameter and a dot denotes differentiation with respect to t, G_N is Newton's constant, $\rho = \rho(t)$ is the energy density, ρ_c is a critical energy density, and the velocity of light is put equal to one. This modified Friedmann equation yields a bounce at $\rho(t) = \rho_c$. Bouncing cosmological solutions of this type have been found in Ref. [5] where modifications of General Relativity were used, and in Refs. [6–8] in which an effective action from loop quantum gravity has a bounce with $\rho_c = \frac{\sqrt{3}}{16\pi^2\gamma^3} \rho_{Pl}$, where $\rho_{Pl} = \frac{1}{G_N^2 \hbar}$ is the Planck density, $\hbar$ being the Planck constant, γ is the Barbero-Immirzi parameter, and $\frac{\sqrt{3}}{16\pi^2\gamma^3}$ is a number of order one, see also [9] for cosmologies with a bounce using the ADM formalism in loop quantum cosmology. In addition, bouncing cosmologies are an integral part of ekpyrotic scenarios [10].

It is certainly of interest to see whether modified Gauss-Bonnet $f(G)$ gravity theories, where G is the Gauss-Bonnet term, can also yield non-singular bouncing solutions using order reduction technique. The Gauss-Bonnet term G is given by $G = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}$, where R_{abcd}, R_{ab}, and R are the Riemann tensor, the Ricci tensor, and the Ricci scalar, respectively. Although in four spacetime dimensions G is a topological invariant and does not contribute to the dynamics, $f(G)$ is non trivial even in four dimensions and

*Electronic address: ines.terrucha@tecnico.ulisboa.pt

the theory gets new degrees of freedom. Interesting cosmological bouncing solutions in $f(G)$ gravity have been found in Refs. [11–13], see [14] for a review on bouncing cosmologies in $f(G)$ gravity and other theories. These $f(G)$ theories had been earlier proposed [15] not in the context of bouncing solutions, but rather to overcome the dark energy problem and providing a solution for the large structure of the universe. In this paper we extend the approach and the results for $f(R)$ theories given in Ref. [1] into $f(G)$ modified gravity theories and use order reduction technique to derive a Friedmann type equation with a bounce and to build an effective action invariant under diffeomorphisms.

The paper is organized as follows. In Sec. II we present the $f(G)$ theory and the order reduced field equations, i.e., the second order field equations that are perturbatively close to General Relativity. In Sec. III we choose the ansatz of a Friedmann-Lemaître-Robertson-Walker (FLRW) line element and deduce a Friedmann equation for the evolution of the universe. In Sec. IV we assume the simplest possible model, i.e., a universe with zero cosmological constant, zero spatial curvature, and composed of a stiff fluid and derive the Lagrangian and the conditions obeyed by the $f(G)$ Gauss-Bonnet modified gravity to have a universe with a bounce and thus a cyclic universe. In Sec. V we conclude.

II. MODIFIED GAUSS-BONNET $f(G)$ GRAVITY THEORY AND ORDER REDUCED $f(G)$ FIELD EQUATIONS

The action for the modified Gauss-Bonnet $f(G)$ gravity in four dimensions reads as

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,\mathcal{L}_{\rm grav} + S_{\rm matter}(g_{\mu\nu},\psi)\,, \tag{1}$$

where $\kappa = 8\pi G_{_N}$, g is the determinant of the metric g_{ab}, a, b are spacetime indices, $\mathcal{L}_{\rm grav}$ is the Lagrangian density for the gravity sector, $S_{\rm matter}$ is the matter action defined as a function of g_{ab} and ψ stands for matter fields. Further, we assume that the Lagrangian density for the gravity sector is given by

$$\mathcal{L}_{\rm grav} = R + f(G)\,, \tag{2}$$

where R is the Ricci scalar and $f(G)$ is a function of the Gauss-Bonnet term G defined as

$$G = R^2 - 4R^{ab}R_{ab} + R^{abcd}R_{abcd}\,, \tag{3}$$

with R_{abcd} being the Riemann tensor and R_{ab} the Ricci tensor. The Riemann tensor can be defined in terms of the Weyl tensor C_{abcd}, the Ricci tensor, the Ricci scalar, and the metric g_{ab} as

$$\begin{aligned} R_{abcd} \;=\; & C_{abcd} - \frac{1}{2}(g_{ad}R_{cb} + g_{bc}R_{da} - g_{ac}R_{db} \\ & - g_{bd}R_{ca}) - \frac{1}{6}(g_{ac}g_{db} - g_{ad}g_{cb})R. \end{aligned} \tag{4}$$

In $f(G)$ gravity, in contrast to $f(R)$ gravity, the equations of motion not only depend on R and R_{ab}, but also on R_{abcd} and on G. Now we make the simplifying assumption that the spacetime has zero Weyl tensor, $C_{abcd} = 0$. This is in line with what we will do next, when working with a FLRW line element for which the Weyl tensor vanishes. For $C_{abcd} = 0$, we have that G and R_{abcd} can be written in terms of R and R_{ab}, see Eqs. (3) and (4). Then Eq. (3) simplifies to

$$G = \frac{2}{3}R^2 - 2R_{ab}R^{ab}. \tag{5}$$

The principle of the least action, $\delta S = 0$, applied to Eqs. (1) and (2) yields

$$\begin{aligned} & R_{ab} - \frac{1}{2}g_{ab}R - \frac{1}{2}g_{ab}f(G) + f'(G)[2RR_{ab} \\ & +2R_{acde}R_b{}^{cde} - 4R_{cb}R^c{}_a - 4g^{ce}g^{df}R_{acbd}R_{ef}] \\ & -4g_{ab}R^{cd}\nabla_c\nabla_d f'(G) + 4R^c{}_a\nabla_c\nabla_b f'(G) \\ & +4R^c{}_b\nabla_c\nabla_a f'(G) + 2g_{ab}R\Box f'(G) - 2R\nabla_a\nabla_b f'(G) \\ & +4g^{ec}g^{fd}R_{acbd}\nabla_f\nabla_e f'(G) - 4R_{ab}\Box f'(G) = \kappa T_{ab}, \end{aligned} \tag{6}$$

where T_{ab}, defined by $T_{ab} = \frac{-2}{\sqrt{-g}}\frac{\delta S_{\rm matter}}{\delta g^{ab}}$, is the stress-energy tensor. In Eq. (6) a prime designates derivation with respect to G, ∇_a is the metric covariant derivative and we define the d'Alembertian as $\Box = g^{ab}\nabla_a\nabla_b$.

Without loss of generality we may parametrize $f(G)$ as

$$f(G) = 2\Lambda + \epsilon\varphi(G)\,, \tag{7}$$

where Λ stands for the cosmological constant, ϵ is a dimensionless parameter that when set to zero gives simple General Relativity plus a cosmological constant, and $\varphi(G)$ is a function of the invariant G. In the following we work at lowest order in ϵ. We will then find Eq. (6) in this order.

To proceed we substitute $f(G)$ and $f'(G)$ with $\epsilon = 0$ in Eq. (6), and express R and R_{ab} at the lowest order. This process yields,

$$R^T = -4\Lambda - \kappa T \tag{8}$$

$$R^T_{ab} = -\frac{\kappa}{2}g_{ab}T - \Lambda g_{ab} + \kappa T_{ab}, \tag{9}$$

where we denote lowest order values by using the superscript T. At lowest order, $\epsilon = 0$, and using Eqs. (8) and (9), we obtain for Eqs. (4) and (5) the following expressions

$$\begin{aligned} R^T_{abcd} = & -\frac{\kappa}{2}(g_{ad}T_{cb} + g_{bc}T_{da} - g_{ac}T_{db} - g_{bd}T_{ca}) \\ & -\frac{1}{3}(g_{ac}g_{bd} - g_{ad}g_{cb})(\Lambda + \kappa T)\,, \end{aligned} \tag{10}$$

$$G^T = \frac{2}{3}\kappa^2T^2 - 2\kappa^2T_{ab}T^{ab} + \frac{8}{3}\Lambda^2 + \frac{4}{3}\Lambda\kappa T. \tag{11}$$

The application of order reduction is equivalent to replacing R, R_{ab}, R_{abcd}, and G by, respectively, R^T, R^T_{ab},

R^T_{abcd}, and G^T in Eq. (6). This procedure brings us to the expression

$$R_{ab}-\frac{1}{2}g_{ab}R+\epsilon\bigg[-\frac{1}{2}g_{ab}\varphi^T +\varphi'^T\Big(2R^TR^T_{ab}-4R^T_{cb}g^{cd}R^T_{ad} +2R^T_{acde}g^{cf}g^{dg}g^{eh}R^T_{bfgh}-4g^{cd}g^{ef}R^T_{acbe}R^T_{df}\Big) -4R^T_{ab}\Box\varphi'^T-4g_{ab}g^{ce}g^{df}R^T_{ef}\nabla_c\nabla_d\varphi'^T +4g^{cd}R^T_{da}\nabla_c\nabla_b\varphi'^T+4g^{cd}R^T_{db}\nabla_c\nabla_a\varphi'^T +2g_{ab}R^T\Box\varphi'^T-2R^T\nabla_a\nabla_b\varphi'^T +4g^{ce}g^{df}R^T_{aebf}\nabla_d\nabla_c\varphi'^T\bigg]=\kappa T_{ab}\,, \quad (12)$$

where $\varphi^T=\varphi(G^T)$ and $\varphi'^T=\varphi'(G^T)$, with the prime denoting differentiation with respect to the Gauss-Bonnet invariant G^T. Eq. (12) is the order reduced field equation that we wanted.

III. MODIFIED FIRST FRIEDMANN EQUATION

Our next step is to derive a modified first Friedmann equation from the order reduced field equation, Eq. (12). To do so we choose a FLRW line element of the type

$$ds^2=-dt^2+a^2\left[\frac{dr^2}{1-kr^2}+r^2d\theta^2+r^2\sin^2\theta d\phi^2\right] \quad (13)$$

where, t is the time coordinate, (r,θ,ϕ) are the spatial coordinates, $a=a(t)$ is the cosmological scale factor, and $k=-1,0,1$ yields hyperbolic, flat, and spherical spaces, respectively. We assume a perfect fluid description, so the stress-energy tensor is

$$T_{ab}=(\rho+p)\,u_au_b+pg_{ab}\,, \quad (14)$$

where $\rho=\rho(t)$ and $p=p(t)$ are the energy density and pressure of the fluid, respectively, with

$$p=w\rho\,, \quad (15)$$

and where u^μ is the fluid's 4-velocity. Defining the Hubble function $H=H(t)$ by

$$H=\frac{\dot a}{a}\,, \quad (16)$$

where a dot means derivative with respect to time t, the zero-zero component of Eq. (12) is

$$\frac{6k}{a^2}+6H^2+2\Lambda-2\kappa\rho+\epsilon\varphi^T -\frac{4}{3}\epsilon(\Lambda-\kappa\rho)\Big[6H\dot\varphi'^T+\varphi'^T\big(2\Lambda+\kappa\rho(1+3w)\big)\Big]=0\,. \quad (17)$$

An independent equation can be taken by noting that, assuming a FLRW metric, i.e., Eq. (13), one has $\nabla^aT_{ab}=0$ in Eq. (12). This is equivalent to energy conservation which reads

$$\dot\rho=-3H(1+w)\rho\,. \quad (18)$$

Now, using the chain rule, one has $\dot\varphi'^T=\frac{\partial\varphi'^T}{\partial G^T}\frac{\partial G^T}{\partial\rho}\dot\rho$. Substituting $\dot\rho$ from Eq. (18) one has

$$\dot\varphi'^T=\varphi''^T4H\kappa\rho(1+w)(\Lambda+2\rho(\kappa+3\kappa w)-3\Lambda w)\,. \quad (19)$$

After placing Eq. (19) into Eq. (17), we get

$$H^2=-\frac{k}{a^2}-\frac{\Lambda}{3}+\frac{1}{3}\kappa\rho+\frac{\epsilon}{18}\bigg[-3\varphi^T+\frac{96\kappa k\rho\varphi''^T(1+w)(\kappa\rho-\Lambda)(\Lambda+2\kappa\rho(1+3w)-3\Lambda w)}{a^2} +4(\Lambda-\kappa\rho)\Big(8\kappa\rho\varphi''^T(1+w)(\kappa\rho-\Lambda)(\Lambda+2\kappa\rho(1+3w) -3\Lambda w)+\varphi'^T(2\Lambda+\kappa\rho(1+3w))\Big)\bigg]\,. \quad (20)$$

When substituting Eq. (19) into Eq. (17), we have used the lowest order value for H^2, namely, $H^2=-\frac{k}{a^2}-\frac{\Lambda}{3}+\frac{1}{3}\kappa\rho$. Eq. (20) is the modified Friedmann equation for $f(G)$ gravity after a process of order reduction.

IV. BOUNCING COSMOLOGY IN GAUSS-BONNET MODIFIED GRAVITY

Let us now assume the simplest possible model, i.e., zero cosmological constant, zero spatial curvature, and a stiff fluid so that

$$\Lambda=0\,,\quad k=0\,,\quad w=1\,, \quad (21)$$

respectively. First, with these assumptions we have that Eqs. (11), (14), (15), and (21) yield

$$\rho^2=-\frac{3}{16\kappa^2}G^T\,. \quad (22)$$

Then, using Eq. (22) and the assumptions of Eq. (21), we see that Eq. (20) turns into

$$H^2=\frac{1}{3}\kappa\rho-\epsilon\left((G^T)^2\varphi''^T-\frac{1}{6}G^T\varphi'^T+\frac{1}{6}\varphi^T\right)\,. \quad (23)$$

We want to retrieve a cyclic universe dynamical equation with a bounce. Following e.g. [1], the appropriate Friedmann equation with a bounce can be written as

$$H^2=\frac{1}{3}\kappa\rho\left(1-\frac{\rho}{\rho_c}\right)\,, \quad (24)$$

where ρ_c is the critical energy density at which the bounce occurs.

Comparing Eqs. (23) with (24) we get $\epsilon\left((G^T)^2\varphi''^T - \frac{1}{6}G^T\varphi'^T + \frac{1}{6}\varphi^T\right) = \frac{\kappa\rho^2}{3\rho_c}$. Now, this equation upon using Eq. (22) turns into

$$\epsilon\left(-(G^T)^2\varphi''^T + \frac{1}{6}G^T\varphi'^T - \frac{1}{6}\varphi^T\right) = \frac{G^T}{16\kappa\rho_c}. \qquad (25)$$

This is a differential equation for φ^T. The solution for Eq. (25) reads as $\varphi(G) = \bar{c}_1\sqrt[6]{G} + \bar{c}_2 G - \frac{3G(5\ln(G)-6)}{200\kappa\rho_c\epsilon}$, with $\bar{c}_1$ and $\bar{c}_2$ being two arbitrary constants of integration. In this solution for $\varphi(G)$ we have dropped the superscript T since to ϵ order $f(G)$ and $f(G^T)$ are the same, see Eq. (7). Multiplying $\varphi(G)$ by ϵ we get an equation for $f(G)$, the quantity we have been after. It is given by

$$f(G) = c_1\sqrt[6]{G} + c_2 G - \frac{3G(5\ln(G)-6)}{200\kappa\rho_c}\,, \qquad (26)$$

where $c_1 = \epsilon\bar{c}_1$ and $c_2 = \epsilon\bar{c}_2$. So the theory has as Lagrangian for the gravity sector given by $\mathcal{L}_{\rm grav} = R + f(G)$, see Eq. (2), which after using Eq. (26) gives $\mathcal{L}_{\rm grav} = R + c_1\sqrt[6]{G} + c_2 G - \frac{3G(5\ln(G)-6)}{200\kappa\rho_c}$. Now, in four dimensions G is a topological invariant and so can be discarded. Then, the Lagrangian for the gravity sector of the modified Gauss-Bonnet $f(G)$ gravity is

$$\mathcal{L}_{\rm grav} = R + c_1\sqrt[6]{G} - \frac{3}{40\kappa\rho_c}\,G\ln G\,. \qquad (27)$$

Similar results were also found in Refs. [13, 16, 17] by using different approaches. Interesting to note that the Lagrangian we have found in Eq. (27) for the $f(G)$ theory, has the zeroth order Einstein-Hilbert term R plus the terms $\sqrt[6]{G}$ and $G\ln G$ as the first order corrections to General Relativity. This is in contrast with the $f(R)$ theory where it was found that the first order correction is an R^2 term, with $\mathcal{L}_{{\rm grav}\,f(R)} = R + \frac{1}{18\kappa\rho_c}R^2$ [1].

V. CONCLUSIONS

We have derived an effective Lagrangian, and so a covariant action, for a modified Gauss-Bonnet $f(G)$ gravity theory which yields a cosmological solution for a bouncing universe. There has been a growing interest in the understanding of the big bang initial singularity, and cyclic universes with a bounce of the sort we have presented here are viable candidates to finding a solution to this problem.

Acknowledgments

I would like to thank my supervisors, Prof. José Sande Lemos and Dr. Daniele Vernieri, for the patient guidance, encouragement and advice they have provided throughout my time as their student. I have been extremely lucky to have two supervisors who cared about my success as a student and in finishing this great work, and who responded to my questions and queries so promptly.

[1] T. P. Sotiriou, "Covariant effective action for loop quantum cosmology from order reduction", Phys. Rev. D **79**, 044035 (2009); arXiv:0811.1799 [gr-qc].

[2] T. P. Sotiriou and V. Faraoni, "$f(R)$ theories of gravity", Rev. Mod. Phys. **82**, 451 (2010); arXiv:0805.1726 [gr-qc].

[3] L. Bel and H. S. Zia, "Regular reduction of relativistic theories of gravitation with a quadratic Lagrangian", Phys. Rev. D **32**, 3128 (1985).

[4] J. Z. Simon, "Higher derivative Lagrangians, nonlocality, problems and solutions", Phys. Rev. D **41** 3720 (1990).

[5] A. H. Chamseddine and V. Mukhanov, "Resolving Cosmological Singularities", JCAP **1703**, no. 03, 009 (2017) [arXiv:1612.05860 [gr-qc]].

[6] M. Bojowald, "Absence of singularity in loop quantum cosmology", Phys. Rev. Lett. **86**, 5227 (2001); arXiv:gr-qc/0102069.

[7] A. Ashtekar, T. Pawlowski, and P. Singh, "Quantum nature of the big bang: An analytical and numerical investigation. I.", Phys. Rev. D **73**, 124038 (2006); arXiv:gr-qc/0604013.

[8] A. Ashtekar, T. Pawlowski, and P. Singh, "Quantum Nature of the Big Bang: Improved dynamics", Phys. Rev. D **74**, 084003 (2006); arXiv:gr-qc/0607039.

[9] J. Haro and J. Amoros, "Bouncing cosmologies via modified gravity in the ADM formalism: Application to loop quantum cosmology", arXiv:1712.08399 [gr-qc] (2017).

[10] P. J. Steinhardt and N. Turok, "Cosmic evolution in a cyclic universe", Phys. Rev. D **65**, 126003 (2002); arXiv:hep-th/0111098.

[11] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini, "Dark energy in modified Gauss-Bonnet gravity: Late-time acceleration and the hierarchy problem", Phys. Rev. D **73**, 084007 (2006); arXiv:hep-th/0601008.

[12] A. Escofet and E. Elizalde, "Gauss-Bonnet modified gravity models with bouncing behavior", Mod. Phys. Lett. A **31**, no. 17, 1650108 (2016); arXiv:1510.05848 [gr-qc].

[13] J. Haro, A. N. Makarenko, A. N. Myagky, S. D. Odintsov, and V. K. Oikonomou, "Bouncing loop quantum cosmology in Gauss-Bonnet gravity", Phys. Rev. D **92**, 124026 (2015); arXiv:1506.08273 [gr-qc].

[14] S. Nojiri, S. D. Odintsov, and V. K. Oikonomou, "Modified gravity theories on a nutshell: Inflation, bounce and late-time evolution", Phys. Rep. **692**, 1 (2017); arXiv:1705.11098 [gr-qc].

[15] S. Nojiri and S. D. Odintsov, "Modified Gauss-Bonnet theory as gravitational alternative for dark energy", Phys. Lett. B **631**, 1 (2005); arXiv:hep-th/0508049.

[16] K. Bamba, A. N. Makarenko, A. N. Myagky and S. D. Odintsov, "Bouncing cosmology in modified

Gauss-Bonnet gravity", Phys. Lett. B **732**, 349 (2014); [arXiv:1403.3242 [hep-th]].

[17] A. N. Makarenko and A. N. Myagky, "The asymptotic behavior of bouncing cosmological models in $F(\mathcal{G})$ gravity theory", Int. J. Geom. Meth. Mod. Phys. **14**, no. 10, 1750148 (2017); [arXiv:1708.03592 [gr-qc]].