On Pure Vertex Decomposable Graphs and their Dominating Shedding Vertices

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Combinatorial commutative algebra is a relatively recent area of mathematics which uses methods of commutative algebra to solve combinatorial problems. It lies at the intersection between these two theories. In 1975 Richard Stanley used the theory of Cohen-Macaulay rings to prove affirmatively the upper bound conjecture for spheres, observing that commutative algebra could supply basic methods in the algebraic study of convex polytopes and simplicial complexes. Later, in 1983, he published a book that would become a reference for the development of combinatorial commutative algebra ([14]).

Besides the contribution of Richard Stanley, we must also mention the importance of Rafael Villarreal's work on the subject. He studied Cohen-Macaulay graphs (see [18]). More specifically, given a (finite simple) graph G with vertex set $V = \{1, \dots, n\}$ and edge set E, one considers the ring $R = K[x_1, \dots, x_n]/I$, where $I = (\{x_i x_j \mid \{i,j\} \in E\})$ and K is a field, and we set that G is Cohen-Macaulay if and only if R is a Cohen-Macaulay ring. With this, combinatorial properties of a graph can be inferred using notions of commutative algebra.

In addition to the Cohen-Macaulay condition, there are other properties of graphs, monomial edge ideals and simplicial complexes that can illustrate some interactions between commutative algebra and combinatorics. Examples of those properties are: shellability, (pure and non-pure) vertex decomposability and well-coveredness. These properties have been studied in ([1],[2], [3], [4],[5], [6], [7], [8], [9], [10], [11]), and the following implications hold (see [1],[7], [10], [11])

pure vertex decomposable $\Rightarrow$ pure shellable $\Rightarrow$ Cohen-Macaulay $\Rightarrow$ well-covered

Generally, the implications above are strict; however, there are families of graphs for which the pure vertex decomposable property is equivalent to the Cohen-Macaulay property. Examples of such families are bipartite graphs (see [3] and [4]), very-well covered graphs (see [6] and [12]), and graphs without 4-cycles and 5-cycles (see [13]).

From all the properties mentioned above, the ones that will be more important for us are the pure and non-pure vertex decomposable properties. Pure vertex decomposability was first introduced by Provan and Billera ([15]) for simplicial complexes. They studied the notion of

k-decomposability for simplicial complexes, and the most restrictive case, $k = 0$, is what we call pure vertex decomposability. The non-pure version of vertex decomposability (what we will simply call vertex decomposability) was introduced by Björner and Wachs ([16]).

This thesis is based on a recent paper by Adam Van Tuyl, Jonathan Baker and Kevin Vander Meulen ([17]). Its goal is to review and establish bridges between the existing literature on vertex decomposability. In particular, we prove that if a graph is well-covered and vertex decomposable, then it is pure vertex decomposable.
Moreover, the Cohen-Macaulay property is also introduced in this thesis, since it is necessary to understand the motivation that led Adam Van Tuyl and the other authors to create the paper that served as base for this study. Their paper was initially motivated by a conjecture of Villarreal ([18]) on Cohen-Macaulay graphs. Based upon computer experiments on all graphs on six or less vertices, Villarreal proposed the following:

Conjecture ([18], Conjectures 1 and 2)**.** *Let G be a Cohen-Macaulay graph with vertex set V and let*

$$D = \{x \in V \mid G \setminus x \text{ is a Cohen-Macaulay graph}\}.$$

Then $D \neq \emptyset$ and D is a dominating set of G.

A dominating set D of a graph G with vertex set V is a subset of V such that every vertex of $V \setminus D$ is adjacent to a vertex of D. It is already known that the above conjecture is false. Earl, Kevin Vander Meulen and Adam Van Tuyl ([19]) found an example of a circulant graph G on 16 vertices such that G is Cohen-Macaulay, but there is no vertex x such that $G \setminus x$ is Cohen-Macaulay.
Although the conjecture above is false in general, Villarreal's work suggests that there may exist some nice subset of Cohen-Macaulay graphs for which the conjecture still holds. Since pure vertex decomposable graphs are Cohen-Macaulay, Jonathan Baker, Kevin Vander Meulen and Adam Van Tuyl ([17]) considered the following variation of conjecture above:

Question. *Let G be a pure vertex decomposable graph with vertex set V. Let*

$$\mathrm{Shed}(G) = \{x \in V \mid G \setminus x \text{ is a pure vertex decomposable graph}\}.$$

Is $\mathrm{Shed}(G)$ a dominating set of G?

The set $\mathrm{Shed}(G)$ denotes the set of shedding vertices of a pure vertex decomposable graph G. The precise definition of the set of shedding vertices is

$$\mathrm{Shed}(G) = \{x \in V \mid G \setminus x \text{ and } G \setminus N[x] \text{ are pure vertex decomposable graphs}\},$$

but, as we will see, for any pure vertex decomposable graph G, one has that $G \setminus N[x]$ is pure vertex decomposable for any vertex x of G. Moreover, it will follow from the definition of pure vertex decomposable graphs that $\mathrm{Shed}(G) \neq \emptyset$, and that is the reason for not including that condition in the question. In the last chapter of this thesis, among other things, we are going to explore some families of pure vertex decomposable graphs for which the question above is answered positively and negatively.

In Chapter 0 we firstly introduce some basic commutative algebra, from Noetherian rings, passing through primary decomposition and then some necessary dimension theory in order to reach the notion of Cohen-Macaulay rings. Then we move our attention to monomial ideals, which take part in any introduction to combinatorial commutative algebra. Monomial ideals are relevant for us since both the Stanley-Reisner ideal of a simplicial complex and the monomial edge ideal of a graph are examples of monomial ideals. Those ideals establish the bridge between the theory of graphs or simplicial complexes and commutative algebra.

Chapter 1 is aimed to be a basic introduction to the theory of simplicial complexes and graph theory. Various different families of graphs are introduced, such as chordal graphs, complete graphs, simplicial graphs, bipartite graphs, Cameron-Walker graphs and well-covered graphs. Also, a short approach to the Cohen-Macaulayness of simplicial complexes and graphs is done. In particular, we introduce the Stanley-Reisner ideal of a simplicial complex and the monomial edge ideal of a graph.
The following is a list of definitions of families of graphs that will be particularly important for us.

Definition. *A graph G is bipartite if its vertex set $V(G)$ can be bipartitioned into two disjoint subsets V_1 and V_2 such that every edge of G has one vertex in V_1 and one vertex in V_2.*

Definition. *A graph G is called chordal if every cycle of G of length greater than 3 has a chord in G.*

Definition. *Let G be a graph. We say that G is well-covered or unmixed if all of its maximal independent sets have the same cardinality.*

A subset W of vertices of a graph G is called an independent set of G if no two vertices of W are adjacent.

Definition. *A very well-covered graph is a well-covered graph in which every maximal independent set has cardinality $\frac{|V|}{2}$.*

Definition. *Let $n \geq 1$ and $S \subseteq \{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. The circulant graph on S, denoted by $C_n(S)$, is the graph on the vertex set $\{0, \dots, n-1\}$ with all edges $\{a, b\}$ that satisfy $|a-b| \in S$ or $n-|a-b| \in S$.*

Definition. *A connected graph G is called a Cameron-Walker graph if $im(G) = m(G)$ and if G is neither a star nor a star triangle, where $m(G)$ is the number of edges of a matching of G of largest size and $im(G)$ is the number of edges of an induced matching of G of maximum size.*

Definition. *Let $G = (V, E)$ be a graph, where $V = \{x_1, \ldots, x_n\}$. A clique-whiskered graph G^π constructed from the graph G with clique vertex partition $\pi = \{W_1, \ldots, W_t\}$ is the graph with $V(G^\pi) = \{x_1, \ldots, x_n, w_1, \ldots, w_t\}$ and $E(G^\pi) = E \cup \{\{x, w_i\} \mid x \in W_i\}$. In order words, for each clique in the partition π, we add a new vertex w_i, and join w_i to all the vertices in the clique.*

In Chapter 2 we define (pure and non-pure) vertex decomposability separately for simplicial complexes and graphs, and derive some properties out of it. Moreover, we make a connection between (pure and non-pure) vertex decomposable simplicial complexes and (pure and non-pure) graphs via the independence complex. Then we study some families of pure vertex decomposable graphs for which the set of shedding vertices is a dominating set. Finally, we end up with the constructions of two families of pure vertex decomposable graphs for which the set of shedding vertices is not a dominating set. More specifically:

In section 2.1 we define vertex decomposability and pure vertex decomposability of simplicial complexes, and its main result is that every link of a face of a pure vertex decomposable simplicial complex is itself a pure vertex decomposable simplicial complex.

In section 2.2 we define vertex decomposability and pure vertex decomposability of graphs and relate these two definitions. We prove that a well-covered vertex decomposable graph is pure vertex decomposable. Moreover, we define the set of shedding vertices for pure vertex decomposable graphs and we prove that every neighbour of a simplicial vertex is a shedding vertex.

Definition. *Let G be a graph. We say that G is vertex decomposable if G is totally disconnected, or G contains a vertex x such that*

1. *both $G \setminus x$ and $G \setminus N[x]$ are vertex decomposable, and*
2. *no independent set of $G \setminus N[x]$ is a maximal independent set of $G \setminus x$.*

Definition. *Let G be a graph. We say that G is pure vertex decomposable if G is well-covered and either G is totally disconnected, or G contains a vertex x such that both $G \setminus x$ and $G \setminus N[x]$ are pure vertex decomposable.*

Proposition. *Let G be a well-covered graph. If G is vertex decomposable, then G is pure vertex decomposable.*

Definition. *Let G be a pure vertex decomposable graph. The set of shedding vertices of G is defined as follows:*

$$\mathrm{Shed}(G) := \{x \in V \mid G \setminus x \text{ and } G \setminus N[x] \text{ are pure vertex decomposable}\}.$$

Proposition. *Let G be a pure vertex decomposable graph. Then $G \setminus N[x]$ is pure vertex decomposable for every vertex x of G.*

Corollary. *Let G be a pure vertex decomposable graph. Then*

$$\mathrm{Shed}(G) = \{x \in V \mid G \setminus x \text{ is pure vertex decomposable}\}.$$

Theorem. *Let G be a pure vertex decomposable graph and x is a simplicial vertex of G. Then $N(x) \subseteq \mathrm{Shed}(G)$.*

In section 2.3 we study some families of pure vertex decomposable graphs for which the set of shedding vertices is a dominating set. The next result summarizes what is going to be presented.

Theorem. *Let G be a pure vertex decomposable graph. If G is*

1. *a bipartite graph, or*
2. *a chordal graph, or*
3. *a very well-covered graph, or*
4. *a circulant graph, or*
5. *a Cameron-Walker graph, or*
6. *a clique-whiskered graph.*

then Shed(G) is a dominating set.

Finally, in section 2.4 we present two constructions of families of pure vertex decomposable graphs for which the set of shedding vertices is not a dominating set.

The first construction is the following:

Let $k_1, \dots, k_m$ be m fixed integers such that $k_i \geq 2$ and $k_1 + \dots + k_m = n$. We define $D_n(k_1, \dots, k_m)$ to be the graph on the $5n$ vertices

$$V = X \cup Y \cup Z = \{x_1, \dots, x_{2n}\} \cup \{y_1, \dots, y_{2n}\} \cup \{z_1, \dots, z_n\}$$

with the edge set given by the following conditions:

1. the induced graph on Z is a complete graph on n vertices,
2. Y is an independent set,
3. the induced graph $G[X]$ is $K_{k_1,k_1} \sqcup \dots \sqcup K_{k_m,k_m}$ where the vertices of $G[X]$ are labeled so that the i-th complete bipartite graph has bipartition

   $$\{x_{2w+1}, x_{2w+3}, \dots, x_{2(w+k_i)-1}\} \cup \{x_{2w+2}, x_{2w+4}, \dots, x_{2(w+k_i)}\}$$

   with $w = \sum_{l=1}^{i-1} k_l$ where $w = 0$ if $i = 1$,

4. $\{x_j, y_j\}$ are edges for $1 \leq j \leq 2n$,
5. $\{z_j, y_{2j}\}$ and $\{z_j, y_{2j-1}\}$ are edges for $1 \leq j \leq n$,
6. $\{x_i, z_j\}$ is not an edge for every $i, j \in \{1, \ldots, 2n\}$,
7. $\{x_i, y_j\}$ is not an edge for every $i, j \in \{1, \ldots, 2n\}$,
8. $\{z_i, y_j\}$ is not an edge for i, j not as in 5.

The graph $D_n(k_1, \ldots, k_m)$ is formed by joining m complete bipartite graphs to a complete graph K_n by first passing through an independent set of vertices Y.

Theorem. *Let $G = D_n(k_1, \ldots, k_n)$ be constructed as above. Then G is pure vertex decomposable and $\mathrm{Shed}(G)$ is not a dominating set.*

The second construction is the following:
Let $n \geq 1$. We define the graph L_n to be the graph on $8n+1$ vertices with vertex set $V(L_n) = X \cup Y \cup Z \cup \{w\}$, where $X = \{x_{1,1}, x_{1,2}\} \cup \ldots \cup \{x_{n,1}, x_{n,2}\}$, $Y = \{y_{1,1}, y_{1,2}, y_{1,3}\} \cup \ldots \cup \{y_{n,1}, y_{n,2}, y_{n,3}\}$, $Z = \{z_{1,1}, z_{1,2}, z_{1,3}\} \cup \ldots \cup \{z_{n,1}, z_{n,2}, z_{n,3}\}$, and with edge set $E(L_n)$ satisfying the following conditions:

1. For each $i = 1, \ldots, n$ the induced graph on $\{x_{i,1}, x_{i,2}, y_{i,1}, y_{i,2}, y_{i,3}\}$ is a 5-cycle with edges $\{y_{i,1}, y_{i,2}\}$, $\{y_{i,2}, y_{i,3}\}$,$\{y_{i,3}, x_{i,2}\}$,$\{x_{i,2}, y_{i,1}\}$,$\{x_{i,1}, y_{i,1}\}$.
2. $\{z_{i,j}, y_{i,j}\} \in E(L_n)$ for $i = 1, \ldots, n$ and $j = 1, 2, 3$, and these edges form a matching between Y and Z.
3. The induced graph on $Z \cup \{w\}$ is the complete graph K_{3n+1}.

The graph L_n is formed by joining n 5-cycles to a complete graph K_{3n+1} in such a way that the edges $\{z_{i,j}, y_{i,j}\}$ form a matching between Y and Z.

Theorem. *Let $G = L_n$ be constructed as above. Then G is pure vertex decomposable and $\mathrm{Shed}(G)$ is not a dominating set.*

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