Comparison between equilibrium finite elements for the Reissner-Mindlin and Kirchhoff slab theories

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Abstract: In this dissertation, hybrid equilibrium finite element formulations, which provide equilibrated solutions, are used for the analysis of linear elastic Reissner-Mindlin and Kirchhoff slabs. The conditions for each theory and formulation are presented, as well as the distinction between the classical formulation of the finite element method and its hybrid equilibrium counterpart. The test problems studied aim to explore the interaction of different slab geometries, boundary conditions and thicknesses. Effects such as the strain energy of the solutions, their convergence based on the refinement used and the “boundary layer” effect are evaluated. A comparative study of the bending and shear stress fields, as well as the deformed slab configurations corresponding to the different theories, are also presented.

1. Introduction

The dissertation summarised herein presents a comparative study of the behaviour of Reissner-Mindlin and Kirchhoff slab theories through three examples. These examples intend to characterize the influence of shear deformation, quantifying the energy and effects that cause different behaviours of the slab; check how the different boundary conditions interact; analyse effects that emerge due to the imposition of the conditions of the different models, an example of which is the “boundary layer” effect in thin Reissner-Mindlin plates; and to qualify the solutions by convergence studies.

2. Reissner-Mindlin and Kirchhoff slab theories

In the Reissner-Mindlin theory, it is assumed that straight fibres initially perpendicular to the middle plane of the slab remain straight after the deformation of the structural element and are inextensible [1]. This theory applies to thick slabs, taking into consideration the shear deformation, with the result that rotations are independent of the transverse displacements, [2]. The displacements and generalised strains are given by:

\[ u = \begin{bmatrix} w \\ \theta_x \\ \theta_y \end{bmatrix}, \quad \varepsilon = \begin{bmatrix} \chi_{xx} \\ \chi_{yy} \\ 2\chi_{xy} \\ \gamma_{xx} \\ \gamma_{yy} \end{bmatrix}, \quad \text{(2.1)} \]

where \( w \) is the transverse displacement, \( \theta_i \) are the rotations, while the curvatures and the shear distortions are \( \chi_{ij} \) and \( \gamma_{ij} \) respectively. The compatibility condition between strains and displacements is expressed as follows:

\[ \varepsilon = d u \]

\[
\begin{bmatrix}
0 & \frac{d}{dx} & 0 \\
0 & 0 & \frac{d}{dy} \\
0 & \frac{d}{dy} & \frac{d}{dx} \\
\frac{d}{dx} & 1 & 0 \\
\frac{d}{dy} & 0 & 1
\end{bmatrix}, \quad \text{(2.2)}
\]
where $d$ is the compatibility differential operator. The body forces and generalised stresses are given by

$$ q = \begin{bmatrix} \bar{q} \\ m_x \\ m_y \end{bmatrix} \quad \quad \sigma = \begin{bmatrix} m_{xx} \\ m_{yy} \\ m_{xy} \\ v_x \\ v_y \end{bmatrix} $$  \hfill (2.3)

where $\bar{q}$ is the transverse load, $\bar{m}_{ij}$ are moments prescribed in the domain, $m_{ij}$ and $v_i$ represent the internal moments and shear forces. The equilibrium of these fields is given by the condition:

$$ d^* \sigma + q = 0 $$  \hfill (2.4)

The constitutive relation is a material law that relates the stresses and the strains; it is written in terms of flexibility as:

$$ \varepsilon = f \sigma + \varepsilon_{\theta} \quad f = \frac{12}{Eh} \begin{bmatrix} 1 \h^2 & -\frac{\theta}{h^2} & 0 & 0 & 0 \\ -\frac{\theta}{h^2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\theta) & 0 & 0 \\ 0 & 0 & 0 & (1+\theta) & 0 \\ 0 & 0 & 0 & 0 & (1+\theta) \end{bmatrix} $$  \hfill (2.5)

Parameter $E$ corresponds to Young’s modulus and $\theta$ corresponds to the Poisson’s coefficient; $f$ represents the flexibility matrix and $\varepsilon_{\theta}$ refers to the thermal strains.

In the kinematic boundary $\Gamma_k$, and in the static boundary $\Gamma_s$, displacement and forces are imposed, respectively:

$$ u = u_k, \quad N\sigma = t_k $$  \hfill (2.6)

where matrix $N$ is formed by the components of the normal external versor of the boundary $\Gamma_k$, $n_x$ and $n_y$, given by [1]:

$$ N = \begin{bmatrix} 0 & 0 & n_x & n_y \\ n_x & 0 & 0 & 0 \\ 0 & n_y & n_x & 0 \end{bmatrix} \quad t_k = \begin{bmatrix} v_x \\ m_{xx} \\ m_{xy} \end{bmatrix} $$  \hfill (2.7)

The Kirchhoff theory assumes the Reissner-Mindlin theory hypotheses. Furthermore, it also admits that the fibres initially perpendicular to the middle plane of the slab remain perpendicular to the deformed middle surface [3]. The effect of the shear deformation is neglected and consequently the rotation field, $\theta$, is dependent of the transverse displacement, $w$. Unlike what happens with Reissner-Mindlin slabs, for Kirchhoff slabs only two boundary conditions are imposed on each boundary.
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The moments and effective shear forces are obtained by applying the following tensor transformation rules:

\[ m_{nn} = m_x n_x^2 + m_y n_y^2 + 2m_{xy} n_x n_y; \]  
\[ m_{nt} = (m_y - m_x)n_x n_y + m_{xy}(n_x^2 - n_y^2); \]  

\[
\begin{align*}
\tau_n &= \frac{\partial m_x}{\partial x}(n_x + n_x n_x^2) + \frac{\partial m_y}{\partial y}(-n_x^2 n_y) + \frac{\partial m_z}{\partial x}(n_y + n_x^2 n_y) + \frac{\partial m_x}{\partial x}(n_y - n_x^2 n_y) + \\
&\quad + \frac{\partial m_z}{\partial y}(n_x - n_x n_x^2 + n_y^2).
\end{align*}
\]  

(2.10)

A corner force, which is equal to the sum of the twisting moments at the endpoints of the sides, must also be considered:

\[ \sum m_{nt} + F^v = 0. \]  

(2.11)

3. Hybrid Equilibrium Formulation

In the Reissner-Mindlin theory, the field of generalised stresses in finite element \( i \) is approximated by:

\[ \sigma_{e(i)} = S_{(i)} \hat{\sigma}_{(i)} + \sigma_{0(i)}. \]  

(3.1)

To guarantee equilibrium inside the elements, the approximation functions are chosen so that they are self-equilibrated,

\[ d^* S_{(i)} = 0 \]  

(3.2)

And the particular solution is chosen in order to verify

\[ d^* \sigma_{0(i)} + q = 0. \]  

(3.3)

The procedure to generate a set of approximated functions of polynomial forces with degree \( \leq p \) is described in [4].

For \( p = 2 \), these approximation functions are:

\[
S_{(i)} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & x & 0 & 2y & 0 & 0 & 0 & x^2 & 0 & 2xy & 0 & 3y^2 \\
1 & 0 & 0 & 2x & 0 & y & 0 & 0 & 0 & 3x^2 & 0 & 2xy & 0 & y^2 & 0 & 0 & 0 \\
0 & -1/2 & 0 & 0 & -x & -x/2 & -y/2 & -y & 0 & 0 & -3x^2/2 & -x^2/2 & -xy & -xy & y^2/2 & 3y^2/2 & 0 \\
0 & 0 & 0 & 0 & 0 & 1/2 & -1 & 0 & 0 & 0 & 0 & x & -x & y & -3y & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 1/2 & 0 & 0 & 0 & 0 & -3x & x & -y & y & 0 & 0 & 0 & 0 & 0
\end{bmatrix}.
\]  

(3.4)

In this case, the moments are functions of degree \( p \), while the shear forces are degree of \( p - 1 \).

The following particular solution was used for an arbitrary polynomial transverse load:
\[ \sigma_{0(i)} = \begin{bmatrix} \frac{1}{2} p_x^{(2)}(q) \\ \frac{1}{2} p_y^{(2)}(q) \\ -\frac{1}{2} p_x(q) \\ -\frac{1}{2} p_y(q) \end{bmatrix}. \] (3.5)

The displacements of a particular point on one side are arranged in a vector with three components, the transverse displacement and two rotations.

The displacements are separately approximated in each side \( j \), by:

\[ \nu_{e(j)} = V^{(j)} \tilde{e}(j) + \bar{\nu}(j). \] (3.6)

In this equation, if \( \Gamma_{e(j)} \subset \Gamma_u \), \( \nu_{e(j)} = \tilde{e}(j) = u_l \); if \( \Gamma_{e(j)} \not\subset \Gamma_u \), \( \nu_{e(j)} = 0 \).

The formulation of this hybrid finite elements is described in [1].

Equilibrium between elements on side \( j \), is imposed in a weak form using the displacement approximation functions as weight functions by:

\[ \sum_{i} \left( \int_{\Gamma_{i(j)}} V^{(j)}^T N_{(j),i} S_{(0)} d\Gamma \right) \tilde{e}_{(i)} = \int_{\Gamma_{i(j)}} V^{(j)}^T e_{(j)} d\Gamma - \sum_{i} \left( \int_{\Gamma_{i(j)}} V^{(j)}^T N_{(j),i} \sigma_{0,i} d\Gamma \right). \] (3.7)

The displacement approximation function of the sides are also polynomial.

In this equation, if \( \Gamma_{i(j)} \subset \Gamma_t \), \( \tilde{e}_{(i)} = t_l \); if \( \Gamma_{i(j)} \not\subset \Gamma_t \), \( \tilde{e}_{(i)} = 0 \).

In each element \( i \), compatibility is imposed in a weighted residual form, resulting in:

\[ -\left( \int_{\Gamma_{i(j)}} S^{(j)} T f S_{(0)} d\Omega \right) \tilde{e}_{(i)} = \int_{\Gamma_{i(j)}} S^{(j)} T e_{(j)} d\Gamma - \sum_{i} \left( \int_{\Gamma_{i(j)}} S^{(j)} T N_{(j),i} \sigma_{0,i} d\Gamma \right). \] (3.8)

The global algebraic system is obtained by assembling the compatibility equation of all the finite elements and the equilibrium equations of all the sides that don’t belong to the kinematic boundary. Assembling the matrices and vectors of the elements into global ones, the algebraic system can be written as:

\[ \begin{bmatrix} -F & D^T \\ D & 0 \end{bmatrix} \begin{bmatrix} \tilde{\xi} \\ \bar{\xi} \end{bmatrix} = \begin{bmatrix} \tilde{\xi}_0 - \bar{\xi} \\ \tilde{\xi} - \tilde{\xi}_0 \end{bmatrix}. \] (3.9)

Considering that the degree of the transverse load cannot exceed \( p-2 \) and the sides are straight, to impose shear force equilibrium the transversal displacement is approximated by monomials \( r_i \), with \( i \leq p-1 \); to impose the local moment equilibrium, the rotations are approximated by monomials \( r_i \), with \( i \leq p \). The number of displacement approximation functions on each side is \( 3p+2 \).
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When the hybrid formulation is applied to slabs modelled by the Kirchhoff theory, it is necessary to consider the modifications of the equilibrium conditions, which involve the bending moment, the effective shear forces and the corner forces, it is also necessary to consider the modifications of the compatibility conditions.

The equation of interior equilibrium is the same as for the Reissner-Mindlin theory, so the approximation of the generalised stresses can be the same as well.

The displacements of a particular point on one side are arranged in a vector with two components, the normal rotation and the transverse displacement,

\[ \bar{V}^\tau = [\bar{\theta}^\tau, w^\tau] = \begin{bmatrix} V^\tau \phi_n^\tau \\ 0 \\ V^\tau w \end{bmatrix} \left[ \bar{\phi}^\tau \phi_n^\tau \right] = V^\tau \bar{\phi}^\tau. \]

(3.10)

This model uses monomials to construct a basis for this approximation. Since these functions are used in a weak form of equilibrium, which intends to impose a strong form equilibrium, the displacement approximation degree must be at least as high as the degree of the projection on the side of the generalized stresses.

The displacement in a vertex is approximated in an independent way from the displacement in the sides that are connected to the vertex,

\[ \bar{V}^\tau = [w^\tau] = [1][\bar{\phi}^\tau] = V^\tau \bar{\phi}^\tau. \]

(3.11)

To express the equilibrium of discrete generalized forces in the boundary, \( \tau \), it is possible to use the approximated displacement variables, \( \bar{\phi} \), obtaining the additional equation,

\[ \bar{\phi}^\tau \bar{t} = \int_r V^\tau \tau \, d\Gamma = \bar{\phi}^\tau \int_r V^\tau \tau \, d\Gamma = \int_r V^\tau \tau \, d\Gamma = \{\bar{m}^\tau, \bar{\phi}^\tau, \bar{\phi}^\tau \}. \]

(3.12)

On the sides, it is necessary to integrate the projection, but in the case of vertices, the forces are directly the dual variables, as expressed by the identity matrix \( V^\tau \). Replacing the moments by their approximations, the following equation is valid for sides and vertices:

\[ 0 = \int_r V^\tau \tau \, d\Gamma = \int_r V^\tau \tau^0 \, d\Gamma + \sum_e \int_r \int_r V^\tau B_e m^0 \, d\Gamma + \sum_e \int_r V^\tau B_e S_e d\Gamma = \bar{t}^0 + \sum_e \bar{t}^0_e + \sum_e D_e \bar{S}_e, \]

(3.13)

where the term \( \tau^0 \) takes into account the contributions of \( \bar{t} \).

Compatibility is, once again, imposed in a weak form, considering the strains, weighted by the same functions that are used to approximate the moments

\[ \int_\Omega S^T f \, S \, d\Omega + \int_\Omega S^T m \, d\Omega = - \int_\Omega S^T d\Omega. \]

(3.14)

The terms on the left of the equation correspond to the flexibility matrix, \( f \), and to the initial stress, \( e^0 \), introduced by the particular solution.

The terms on the right-hand side of the equation correspond to the primary step of the hybrid equilibrium formulation, which integrates by parts the domain integral, in a way that \( \int S^T D \, d\Omega \) is transformed into \( \int (D^T S) w \, d\Omega \), which is zero. Thus the compatibility is imposed in a weak form, requiring only the approximation of the displacements on the boundary.

Since \( D \) is an second order operator, the integration at the boundary must be decomposed into two steps. The details of this process can be found in [5], leading to the expression:
Joining the matrixes of the sides and vertices with their respective variables, [5]:

\[ D = \begin{bmatrix} D^{x} \\ D^{y} \end{bmatrix} \]  

\[ v = \begin{bmatrix} v^{x} \\ v^{y} \end{bmatrix}. \]  

The governing system can be written a form similar to the one obtained for the Mindlin-Reissner theory, (3.9).

4. Mesh Refinement and Strain Energy

To reduce the error of the finite element solutions, it is necessary to perform a refinement of the approximation used. There are two ways to achieve this: increasing the number of elements of the finite element mesh, thus decreasing their size, \( h \)-refinement, or increasing the degree of the functions that constitute the basis of the approximation, \( p \)-refinement.

It is also possible to improve the quality of the solution without increasing the dimension of the governing system by better distributing the degrees of freedom. The validation of the approximate solutions may be made from a visual assessment which considers an approximate solution fair or good when, by observation of the computational results, it agrees with the conditions of the theory. The deduction of the equations that characterize the behaviour of mechanical systems can be performed using variation calculus applied to energy functions, [1].

The formulation used, which is based on the approximation of the stress field, can be associated with the complementary potential energy of a mechanical system, which is given by:

\[ H_{c} = U_{c} - W_{c}, \]  

where \( W_{c} \) is the work of the applied forces, which is written as follows:

\[ W_{c} = \int_{R_{a}} t^{T} u_{r} \, d\Gamma, \]  

and \( U_{c} \) is the complementary strain energy, which is equal to the strain energy, defined as [6]

\[ U = \frac{1}{2} \iiint_{V} (\sigma_{xx}e_{xx} + \sigma_{yy}e_{yy} + \sigma_{zz}e_{zz} + 2\tau_{xy}e_{xy} + 2\tau_{xz}e_{xz} + 2\tau_{yz}e_{yz}) \, dV. \]  

To obtain a good approximation of strain energy of the "exact" solution, Richardson’s extrapolation is used, which considers a set of three solutions that have monotonic convergence.

This extrapolation is based on the law:

\[ U_{n} - U = KN_{n}^{-\beta} \]  

where \( N \) is the number of degrees of freedom of the mesh \( n \) and \( \beta > 0 \), [1].
5. Analysis of the test problems

In problem nº1 almost every type of support is considered, with built-in, simply supported and free boundaries, allowing to analyse the variations that result from their combination. It has four openings in its corners and a central opening, as represented in Figure 5.1.

![Figure 5.1 - Problem nº 1](image)

Characteristic values of the stress fields and normalized strain energy density fields, are represented for the Reissner-Mindlin theory, considering a thickness of 0.1, for the most refined mesh, which has 21138 elements, using approximation functions of degree 6.

![Figure 5.2 - Stresses Fields and Normalized Strain Energy Density Field for Reissner-Mindlin theory, thickness 0.1, Mesh 4, Approximation function degree 6.](image)

It should be noticed that the strain energy decreases with the number of elements when the kinematic boundaries conditions are homogeneous. Unlike what happens for compatible finite elements, equilibrated finite
elements are more flexible than the "exact" solution, thus leading to solutions that converge from above in terms of energy. It should also be noticed that the Reissner-Mindlin elements are more flexible than the Kirchhoff ones.

The lowest convergence rate was observed for the Kirchhoff elements. Convergence is slower for all degrees than for any other thickness in Reissner-Mindlin, because the singularities have a higher effect, and because the Reissner-Mindlin theory slabs have a convergence rate that is maximal for a particular thickness. This effect appears because of the lower effect of shear deformation, except between 0.05 and 0.1 transition where the convergence rate decreases due to the appearance of the "boundary layer" effect. With the increase of the numbers of degrees of freedom, the dispersion of the shear energy in absolute value and in percentage decreases, because of a better representation of the shear deformation effect, leading to the convergence to the "exact" value.

Another implication of interest is the fact that the shear energy increases with the decrease in thickness due to the increase of the slab deformability.

As it happens for the shear strain energy, the bending strain energy results dispersion also decreases for a larger number of either degrees of freedom or elements in the mesh.

Problem nº2 has the particularity of having a linear load applied from \((x, y) = (8/3, 0)\), to the coordinates \((x, y) = (8/3, 3)\), and a simple support, modeled in soft form, in the domain of the slab, from the coordinates \((x, y) = (4/3, 2)\), to the coordinates \((x, y) = (4, 2)\). The boundaries of the slab are modelled with built-in supports and with soft simple supports, as illustrated in Figure 5.3.

![Figure 5.3 - Problem nº2](image)

Only approximation functions of degree six are presented for this problem, because the effect of modifying the approximation degree does not show significantly different conclusions from those obtained for the first problem.

From the results obtained we concluded that, except for the thickness 0.01, the convergence rate of Reissner-Mindlin elements is higher than for the Kirchhoff ones. As expected, the solution for the 0.01 thickness is the most approximated to the Kirchhoff theory.
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Greater differences appear at the simply supported boundaries due to the “boundary layer” effect.

The solutions for the 0.5 thickness have larger differences due the existence of larger shear deformations.

In problem nº3, Figure 5.5, the effect of considering a point support and a point load is studied.

The slab is loaded with a concentrated force of $1 \text{kN}$ or with an equivalent distributed load at coordinates $(x, y) = (3, 3)$.

Since point supports or point loads cannot be represented in Reissner-Mindlin equilibrium elements, we modelled the point supports as soft simple supports, with a square geometry, with a dimension size four hundred times smaller than the sides of the slab. The convergence occurs faster for the Kirchhoff theory than for the
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theory of Reissner-Mindlin, a result that may arise from the absence of the shear deformation influence. The characteristic values of the stress fields and strain energy field, for a thickness of 0.1, finer meshes, when the “filled” pillar is modelled for Reissner-Mindlin theory is presented in Figure 5.6. For a point support model using the Kirchhoff theory the same results are presented in Figure 5.7.

![Figure 5.6 - Stresses Fields and Normalized Strain Energy Density Field, Mesh Group nº1 thickness 0.1, “Filled” Pillar Model, Reissner-Mindlin theory](image)

![Figure 5.7 - Stresses Fields and Normalized Strain Energy Density Field, Mesh Group nº2, Point Support Model, Kirchhoff theory](image)
The evolution of the percentage of shear strain energy with the slab thickness is presented in Graphic 5.1.

![Graphic 5.1 - Evolution of the Percentage of Shear Strain Energy with the slab thickness, [7]](image)

It can be seen from the graph that the rate of shear strain energy decreases for the thicknesses between 0.5 to 0.1, in this transition the slab is thick, however, the transition 0.1 and 0.05 gives an increase the shear strain energy percentage, which, we were not expecting, this effect is probably related with the capacity of the finer meshes to represent the “boundary layer” effect.

The percentage decreases again in the 0.05 to 0.01 range, returning to the expectable evolution, [7].

6. Conclusions and Future Studies

The equilibrium hybrid formulation has the advantage over the classic formulation of finite element method of obtaining results that provide stress distributions that lead to safe designs from the viewpoint of limit analysis. It does not present the shear locking effect, although the processing times are higher, in theory, than the classic formulation.

Some of the factors that condition the convergence of a solution, when measured by the relative error in energy, are the singularities caused by the interaction of the boundary conditions, and, in the Reissner-Mindlin theory, the “boundary layer” effect and the variation of the shear strain energy with the variation of thickness.

In the case of the 1st Problem we can also conclude that the use of simply soft or hard support, does not lead to relevant differences in energy, stress fields or deformed slab configuration.

As for the second Problem, which contained a simple reentrant support and line load, none of the tests conducted produced results of great importance.

In the 3rd Problem, different forms of modelling the pillar were considered: "filled", "empty" and "point". There are large differences in terms of stress fields, strain energy or the deformed shape. The point support model of the Kirchhoff theory compared with the “filled” pillar support model of the Reissner-Mindlin slab theory, other than the expected differences in strain energy, associated to the stiffening of this model, also introduced the corner effect in the “filled” pillar, which explains the differences between both models in internal forces fields and strain energy.
It would be convenient to perform a numerical and eventually theoretical study in order to quantify the singularities and the boundary layer effect arising in the problems studied in this thesis.

Also relevant would be the study of the transition thickness between thin and thick slab, in order to assess which theory adapts best to problems of this kind.

7. References


