Symmetries and parametrizations of the Quark & Lepton Sectors

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(June 2015)

Abstract

It is studied the quark and lepton sectors separately. We try to explain the current phenomenological results by extending the Standard Model (SM).

For the Yukawa sector of quarks, we propose a $Z_3 \otimes S(2)_L \otimes S(2)_R$ symmetry, in a minimal extension of the SM with vector-like quarks (VLQ), one for each up- and down-sector. It was possible to obtain the correct pattern of masses and mixing, including naturally suppressed Z mediated Flavour-Changing Neutral Currents (FCNC), by means of a partial breaking of the symmetry down to $S(2)_L \otimes S(2)_R$ in the SM-part of the model.

As for the lepton sector, we focus on the hypothesis of quasi-degeneracy of three Majorana neutrinos. The smallness of the reactor angle may just reflect the lifting of this mass degeneracy. We propose a new parametrization for the large leptonic mixing [1] that reveals aspects that are less clear in other parametrizations. Several scenario-cases are presented with mixing angles in agreement with the current phenomenology and leading to large CP violation.

Keywords: Yukawa sector, Vector-like quarks, CP violation, Flavour-Changing Neutral Currents, Quasi-degeneracy, Large leptonic mixing

I. Introduction

In the last decades, the Standard Model of particle physics has been one of the triumphs of modern physics. However, the SM does not have any symmetry that constrains the Yukawa sector, which has too much free parameters. Most of the attempts to understand the observed pattern of the fermionic masses and mixings are based in constraining the Yukawa structures with abelian or non-abelian symmetries. As far as we know, the fundamental mechanism responsible for the current flavor structure and mass spectra has not been discovered yet.

A close inspection of the quark mass spectra tells us that the mass of the third generation dominates over the other two generations, with the first family mass being lighter in comparison to the corresponding mass of the second family. This can be roughly explained by a limit situation where all mass matrices are proportional to the democratic matrix, which all entries are equal to one, or in a Heavy Basis where the only non zero entry is the element $(3, 3)$ [2]. The mass spectra consists of two levels: a nondegenerate and massive eigenvalue; the other corresponds to a two-fold degeneracy with vanishing mass eigenvalues. We can find symmetries that establish this feature: a non-abelian symmetry like $S(3)$, applied separately to each left- and right-handed quark fields, $S(3)_L \otimes S(3)_R$ [3]; a discrete symmetry Z_3 could also be behind of the flavour Democracy [4].

One hypothesis to describe the nature correctly is to consider a realistic quark mass matrix as a perturbation of the Democratic/Heavy situation. In order to generate mass for the second family, one needs to break the flavour permutational symmetry $S_L(3) \otimes S_R(3)$ [3], i.e. down to $S_L(2) \otimes S_R(2)$ and successively break the $S_L(2) \otimes S_R(2)$ flavour permutation symmetry in order to generate mass for the remaining families of quarks.

A possible minimal extension for the SM is to add exotic fermions, like vector-like quarks [5]. Vector-like quarks may provide a larger framework where difficulties such as the USY problem of not accounting for the observed strength of CP violation are solved [6]. There is also another interesting feature with regard to vector-like quarks. By imposing a flavor democracy in the SM quarks couplings within a framework containing a isosinglet vector-like quark in each the up- and down-sector, it is possible to generate mass for the first quark families by softening the steps in the breaking, keeping the SM part of the mass matrix invariant under a $S(2)_L \otimes S(2)_R$ symmetry.

The observation of neutrino oscillations established that neutrinos are massive, which consequently leads to the presence of mixing in the leptonic sector. The leptonic mixing is quite different from the quark mixing. The current experimental results are unclear about the nature of neutrinos, its mass specturm and the nature of leptonic CP violation. The case where neutrinos are quasi-degenerate and Majorana is very appealing. The observed large leptonic mixing may reflect a quasi-degeneracy of three Majorana neutrinos [7]. This hypothesis is very interesting since in the limit of exact degeneracy, the lepton mixing matrix necessarily has a vanishing element and the smallness of $|U_{13}|$ may just reflect the lifting of the mass degeneracy, i.e. the breaking of an exact family symmetry, as it was studied in detail for the quark-sector. The physics yielding the neutrino mass matrix, as a mere theoretical hypothesis, could unveil the presence of a dominant S_3 symmetry responsible for the neutrinos' degeneracy, and a sub-dominant term invariant under a S_2 $\mu-\tau$ symmetry that breaks the degeneracy to a quasi-degenerate neutrino mass spectrum [8]. The nice feature of this model pattern is the possibility of generating a mixing matrix close to the TBM, with a non-zero reactor angle, by allowing a slight breaking of the $\mu - \tau$ symmetry. Other attempts have been made in order to obtain a quasi-degenerate mass spectrum: abelian family symmetry like the ones used to generate the hierarchical character of the quarks [9]; a low energy mass matrix conserving $L_\mu - L_\tau$ [10]; symmetries like $SO(3)$ [11]. In [12], it was developed the formalism of the symmetry building in such a way that partially and complete degenerate neutrino spectra is connected with the mixing angles and CP-phases.

Some underlying aspects, such as symmetries that the experimental data may suggest, can be understood by means of parametrizations. Although parametrizations are equivalent among themselves, some of them may be much appropriate to describe symmetries and patterns. To study large leptonic CP violation from a new perspective, we propose a new parametrization for the leptonic mixing [1], that reveals interesting aspects that are less clear in other parametrizations with regard to the quasi-degenerate Majorana neutrinos. Several scenario-cases are presented with mixing angles in agreement with the current experimental results and leading to large CP violation.

The extended abstract is organized as follows. First, in section II, we deduce the most general effective mass matrix capable of generating mass for a SM Yukawa structure with a massless first family in a extended version of the SM with a isosinglet VLQ in each sector. Then, we present the $Z_3 \otimes S(2)_L \otimes S(2)_R$ CP invariant model with VLQ in both the up- and down-quark sector and a numerical example of how this model can fit the current experimental data and suppress the FCNC mediated by the Z boson. In section III, for leptonic sector, we deduce the new parametrization and discuss its usefulness when used for quasi-degenerate Majorana neutrinos. It is also discussed the existence of large CP violation by identifying several scenario-cases. In section IV, we present our conclusions.

II. Quark sector

II.A Generalized effective mass matrix

Consider the SM, with one isosinglet vector-like quark D with charge $-1/3$. Let us assume that $\bar{Q}_{L,i}\Phi D_R$ exists and, after the SSB, the Higgs $SU(2)_L$ doublet acquires a VEV such that the quark mass matrix is:

$$\mathcal{M}_d = \left(\begin{array}{ccc|c} & m_d & & Y \\ & & & \\ - & - & - & - \\ & X^\dagger & & M \end{array}\right). \qquad (1)$$

The couplings that generate Y have the same order of magnitude as m_d, i.e. they are in the electroweak scale. This is a crucial ingredient when computing the new generalized effective matrix. We diagonalize $\mathcal{M}_d \cdot \mathcal{M}_d^\dagger$ with the following unitary matrix:

$$\mathcal{W} = \begin{pmatrix} K & R \\ S & T \end{pmatrix},$$

$$\mathcal{W}^\dagger \cdot \mathcal{M}_d \cdot \mathcal{M}_d^\dagger \cdot \mathcal{W} = \left(\begin{array}{ccc|c} & H_\lambda & & \\ & & & \\ - & - & - & - \\ & & & H_D \end{array}\right). \qquad (2)$$

The block H_λ is a 3×3 real and diagonal matrix, and H_D is a 1×1 real element. From (2), one obtains the following 4 equations :

$$\begin{aligned} &\left(m_d \cdot m_d^\dagger + Y \cdot Y^\dagger\right) \cdot K + (m_d \cdot X + MY) \cdot S = K \cdot H_\lambda, \\ &\left(m_d \cdot m_d^\dagger + Y \cdot Y^\dagger\right) \cdot R + (m_d \cdot X + MY) \cdot T = H_D R, \\ &\left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right) \cdot K + \left(X^\dagger \cdot X + M^2\right) \cdot S = S \cdot H_\lambda, \\ &\left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right) \cdot R + \left(X^\dagger \cdot X + M^2\right) \cdot T = TH_D \end{aligned} \qquad (3)$$

The third equation of (3) can be rewritten as:

$$S = -\frac{\left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right) \cdot K - S \cdot H_\lambda}{X^\dagger \cdot X + M^2}. \qquad (4)$$

One has $K \sim \mathcal{O}(1)$, $X \sim M \sim \mathcal{O}(V)$ and $Y \sim m_d \sim \mathcal{O}(v)$. Remember that in our assumption v is the electroweak scale where occurs the SSB of the SM and V corresponds to a larger scale of the couplings invariant under the gauge symmetry, i.e. $V >> v$.

The immediate consequence of the scale definition of each term is that $S \cdot H_\lambda$ is of $\mathcal{O}(v^2)$ at most, which makes it negligible when compared with $\left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right) \cdot K \sim \mathcal{O}(vV)$. Thus, eq. (4) can be reduced to:

$$S = -\frac{\left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right) \cdot K}{X^\dagger \cdot X + M^2}. \qquad (5)$$

Substituting the last equation in the first equation of (3), one arrives at:

$$\begin{aligned} H_{eff} &= K \cdot H_\lambda \cdot K^\dagger \\ &= m_d \cdot m_d^\dagger + Y \cdot Y^\dagger - \frac{(m_d \cdot X + MY) \cdot \left(X^\dagger \cdot m_d^\dagger + MY^\dagger\right)}{X^\dagger \cdot X + M^2}. \end{aligned} \qquad (6)$$

The matrix H_{eff} is of $\mathcal{O}(v^2)$, as expected. The matrix K that diagonalizes H_{eff} plays the role of the CKM matrix 3×3 involving only the SM quarks.

II.B Breaking the $\mathbf{Z_3 \otimes S(2)_L \otimes S(2)_R}$ to $\mathbf{S(2)_L \otimes S(2)_R}$ symmetry model

II.B.1 The model

We presented an entirely new model. The model consists of an extension of the SM with one vector-like quark and three extra Higgs fields (apart from the SM-Higgs), for each sector. At the level of the SM, after a symmetry breaking, the quark mass matrices exhibit a $S(2)_L \otimes S(2)_R$ permutation symmetry. However, at a higher level, i.e. when considering the whole 4×4 quark matrix with the inclusion of the VLQ, this symmetry is broken. Thus, in the effective quark mass matrix, the would-be massless quark, from the 3×3 SM quark mass matrix with the $S(2)_L \otimes S(2)_R$ symmetry, gets a contribution from the terms which involve the VLQ and acquires a mass. Moreover, if we embedded the whole theory in a $Z_3 \otimes S(2)_L \otimes S(2)_R$ symmetry, we are able, not only to obtain the correct hierarchy for the quarks of each sector but also to generate a realistic CKM matrix, where all mixing angles, I_{CP} and angles of the unitarity triangle are within $1\,\sigma$ of the experimental results.

First, we construct an invariant Lagrangian under $Z_3 \otimes S(2)_L \otimes S(2)_R$ flavour symmetry, in the Heavy Basis.

One defines a generator P of Z_3 and a generator S' of S_2[1]:

$$P = \begin{pmatrix} w & & \\ & w & \\ & & 1 \end{pmatrix}, \quad S' = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \tag{7}$$

where $w = e^{i\frac{2\pi}{3}}$. This is a Z_3 symmetry since $P^3 = I_{3\times3}$ and $P^2 = P^\dagger$. It can also be readily verified that $S'^2 = I_{3\times3}$.

Under this Z_3 discrete symmetry and also imposing $S(2)_L \otimes S(2)_R$ to both left- and right-handed fields, the standard quarks transform in the following way:

$$\begin{aligned} &Q_{L,i} \to P^\dagger_{ij}\, Q_{L,j}, \quad u_{R,i} \to P_{ij}\, u_{R,j}, \quad d_{R,i} \to P_{ij}\, d_{R,j}, \\ &Q_{L,i} \to S'_{ij}\, Q_{L,i}, \quad u_{R,i} \to S'_{ij}\, u_{R,j}, \quad d_{R,i} \to S'_{ij}\, d_{R,j}. \end{aligned} \tag{8}$$

As for the interaction terms between the SM quarks and one $SU(2)_L$ scalar Higgs doublet, Φ_{SM}, the only invariant interaction terms are $\bar{Q}_{L,3}\Phi_{SM} d_{R,3}$ and $\bar{Q}_{L,3}\tilde{\Phi}_{SM} u_{R,3}$.

In order to achieve the desirable structure of (1), after the SSB, it is convenient to introduce three more $SU(2)_L$ scalar doublets Φ^u_i, three Φ^d_i $SU(2)_L$ scalar doublets and three complex scalar Higgs singlets S_i, such that the following transformation properties under Z_3 hold:

$$\Phi^d_i \to P^\dagger_{ij}\, \Phi^d_j, \quad \Phi^u_i \to P_{ij}\, \Phi^u_j, \quad S_i \to P^\dagger_{ij}\, S_j. \tag{9}$$

The previous fields must also be constrained to the $S(2)$ symmetry generated by S,

$$\Phi^d_i \to S'_{ij}\, \Phi^d_j, \quad \Phi^u_i \to S'_{ij}\, \Phi^u_j, \quad S_i \to S'_{ij}\, S_j. \tag{10}$$

The main purpose of the additional constraint is to impose the interaction terms of the type $\bar{Q}_{L,i}\Phi^d_i D_R$ and to forbid combinations $\bar{Q}_{L,i}\Phi^d_j D_R$, with $i \neq j$. The same applies for the up-type quarks. One only has to substitute D_R for U_R and Φ^d_i for $\tilde{\Phi}^u_i$.

Additionally, we impose a Z_4 symmetry under which all the SM quark fields transform trivially and the remaining fields transform as:

$$\begin{aligned} &\Phi^u_j \to -i\,\Phi^u_j, \quad &&\Phi^d_j \to i\,\Phi^d_j, \quad &&S_j \to -i\,S_j, \\ &D_L \to -i\,D_L, \quad &&D_R \to -i\,D_R, \quad &&U_L \to -i\,U_L, \\ &U_R \to -i\,U_R. \end{aligned} \tag{11}$$

This allow us to forbid the interaction terms of the type $\bar{Q}_{L,i}\Phi^{u,d}_k d_{R,j}$ or $\bar{Q}_{L,i}\tilde{\Phi}^{u,d}_k u_{R,j}$, as well as the interaction terms of the type $\bar{Q}_{L,i}\Phi_{SM} D_R$ or $\bar{Q}_{L,i}\tilde{\Phi}_{SM} U_R$. At first sight, this could also be accomplished with a Z_2 discrete symmetry, however one wants to avoid terms such as $\bar{Q}_{L,3}\Phi^u_3 D_R$ or $\bar{Q}_{L,3}\tilde{\Phi}^d_3 U_R$ that are allowed by (8) and (10) [2].

The most general Yukawa Lagrangian invariant under $Z_3 \otimes S(2)_L \otimes S(2)_R$ reads:

$$\begin{aligned} -\mathcal{L}_Y =& Y^d_{3,3}\bar{Q}_{L,3}\Phi_{SM} d_{R,3} + Y^u_{3,3}\bar{Q}_{L,3}\tilde{\Phi}_{SM} u_{R,3} \\ &+ Y^d_1\bar{Q}_{L,1}\Phi^d_1 D_R + Y^u_1\bar{Q}_{L,1}\tilde{\Phi}^u_1 U_R + Y^d_2\bar{Q}_{L,2}\Phi^d_2 D_R \\ &+ Y^u_2\bar{Q}_{L,2}\tilde{\Phi}^u_2 U_R + Y^d_1\bar{Q}_{L,3}\Phi^d_3 D_R + Y^u_3\bar{Q}_{L,3}\tilde{\Phi}^u_3 U_R \\ &+ X^d_1\bar{D}_L S_1 d_{R,1} + X^u_1\bar{U}_L S_1 u_{R,1} + X^d_2\bar{D}_L S_2 d_{R,2} \\ &+ X^u_2\bar{U}_L S_1 u_{R,2} + \bar{D}_L X^d_3 S_3 d_{R,3} + \bar{U}_L X^u_3 S_3 u_{R,3} \\ &+ \mu^u\bar{U}_L U_R + \mu^d\bar{D}_L D_R + H.c.\,. \end{aligned} \tag{12}$$

After the SSB, the generic mass matrices for the up- and down-type quark would be:

$$\mathcal{M}_q^{Z_3\otimes S(2)_L\otimes S(2)_R} = c_q \begin{pmatrix} 0 & 0 & 0 & r_{1,q} \\ 0 & 0 & 0 & r_{2,q} \\ 0 & 0 & 1 & r_{3,q} \\ k_{1,q} & k_{2,q} & k_{3,q} & t_q \end{pmatrix}. \tag{13}$$

where the label q corresponds to u for up-type and to d for the down-type. As before, in the last step one can avoid the existence of one massless quark generation by breaking the symmetry $Z_3 \otimes S(2)_L \otimes S(2)_R$ to $S(2)_L \otimes S(2)_R$, simply adding a perturbative term in (12) to the light quarks. A mass matrix invariant under $S(2)_L \otimes S(2)_R$ is:

$$\mathcal{M}_q^{S(2)_L\otimes S(2)_R} = \epsilon_q c_q \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma_q & \rho_q \\ 0 & \chi_q & 1 \end{pmatrix}, \tag{14}$$

where $\epsilon_{u,d}$ is a small parameter. The final mass matrix, after a successive breaking and including the $S(2)_L \otimes S(2)_R$ contribution of eq. (14), is:

$$\mathcal{M}_q = c_q(1+\epsilon_q) \begin{pmatrix} 0 & 0 & 0 & r_{1,q} \\ 0 & \gamma_q & \rho_q & r_{2,q} \\ 0 & \chi_q & 1 & r_{3,q} \\ k_{1,q} & k_{2,q} & k_{3,q} & t_q \end{pmatrix}. \tag{15}$$

We did not specify if the Yukawa couplings are all complex numbers. Generically, one can have all Yukawa couplings to be arbitrary complex numbers. This is important in order to generate sufficient CP violation through the Kobayashi-Maskawa mechanism. However, there is an interesting possibility to be explored. If all Yukawa couplings are real, e.g. if one imposes CP invariance on the Lagrangian and if the extra $SU(2)_L$ doublets are allowed to acquire a VEV with a phase, then it is possible to generate sufficient CP violation through these phases [13]. We only need to prove that the minimization conditions of the scalar potential accepts VEVs with phases. Let us suppose that the VEVs of all scalar fields of the model are:

$$\begin{aligned} &\left\langle\Phi^d_1\right\rangle = \begin{pmatrix} 0 \\ v_d e^{i\delta_1}\end{pmatrix}, \quad &&\left\langle\Phi^d_2\right\rangle = \begin{pmatrix} 0 \\ v_d\end{pmatrix}, \quad &&\left\langle\Phi^d_3\right\rangle = \begin{pmatrix} 0 \\ 0\end{pmatrix}, \\ &\langle\Phi^u_1\rangle = \begin{pmatrix} 0 \\ v_u e^{i\delta_1}\end{pmatrix}, \quad &&\langle\Phi^u_2\rangle = \begin{pmatrix} 0 \\ v_u\end{pmatrix}, \quad &&\langle\Phi^u_3\rangle = \begin{pmatrix} 0 \\ 0\end{pmatrix}, \\ &\langle\Phi_{SM}\rangle = \begin{pmatrix} 0 \\ v\end{pmatrix} \quad &&\langle S_1\rangle = s_1, \quad &&\langle S_2\rangle = s_2, \\ &\langle S_3\rangle = s_3. \end{aligned} \tag{16}$$

Then, the obtained mass matrices, with the redefinition of the previous parameters and by making transforma-

[1] The permutation group S_2 is isomorphic to Z_2

[2] We use $\tilde{\Phi}^u_i$ for the Higgs doublets of the up-sector instead of Φ^u_i, because we will later define the VEV of all Higgs doublets as being $\Phi_i = \begin{pmatrix} 0 \\ v \end{pmatrix}$, which means that the up quarks will only acquire mass by coupling with $\tilde{\Phi}^u_i$.

tions of the right-handed quarks, are:

$$\mathcal{M}_d = \lambda'_d \begin{pmatrix} 0 & 0 & 0 & y_{1,d}e^{i\delta_1} \\ 0 & c'_d & d'_d & y_{2,d} \\ 0 & 0 & 1 & 0 \\ x_{1,d} & x'_{2,d} & x'_{3,d} & M_d \end{pmatrix} \cdot O^d_{(23)},$$
$$\mathcal{M}_u = \lambda'_u \begin{pmatrix} 0 & 0 & 0 & y_{1,u}e^{-i\delta_1} \\ 0 & c'_u & d'_u & y_{2,u} \\ 0 & 0 & 1 & 0 \\ x_{1,u} & x'_{2,u} & x'_{3,u} & M_u \end{pmatrix} \cdot O^u_{(23)}. \tag{17}$$

Under the symmetry $Z_3 \otimes S(2)$ generated by (7) and also contemplating the Z_4 discrete symmetry imposed to the fields, expressed in the multiple transformations (11), the most invariant scalar potential is given in Appendix A. The terms of the type $\left(S_2^\dagger S_1 \Phi_1^{d\dagger}\Phi_2^d\right)$ in (43) were crucial in order to obtain the desired result. Indeed, it is possible to have a phase for the VEV that allows the generation of CP violation (44). Consequently, besides generating CP violation without introducing another complex perturbation term, one can generate sufficient CP violation. This will be shown in the next subsection by performing a numerical analysis.

II.B.2 Numerical analysis

Next we present a suitable point in the allowed parameter space, for the spontaneous CP violation of (17). Consider the following input:

$$\begin{array}{lll} c'_u = 0.0004, & d'_u = 0.0017, & c'_d = 0.0243, \\ d'_d = 0.0439, & y_{1u} = 0.00045, & y_{2u} = 0.0057, \\ y_{1d} = 0.008, & y_{2d} = 0.0055, & x_{1u} = 1.5, \\ x'_{2u} = 6.1, & x'_{3u} = 0.05, & M_u = 6.9 \\ x_{1d} = 30.6, & x'_{2d} = 162, & x'_{3d} = 4.5, \\ M_d = 288, & \delta_1 = 0.747, & \end{array} \tag{18}$$

which leads to the following 3×3 CKM matrix, using the general effective matrix for both up- and down-type mass matrices:

$$V_{CKM} = \begin{pmatrix} 0.97421 & 0.22560 & 0.00357 \\ 0.22547 & 0.97335 & 0.04182 \\ 0.00848 & 0.04111 & 0.99912 \end{pmatrix} \tag{19}$$

with the following relevant physical quantities:

$$\begin{array}{lll} m_u = 0.0014\,\text{GeV}, & m_c = 0.612\,\text{GeV}, & m_t = 171.7\,\text{GeV}, \\ m_d = 0.00273\,\text{GeV}, & m_s = 0.0546\,\text{GeV}, & m_b = 2.89\,\text{GeV}, \\ |I_{CP}| = 2.94 \times 10^{-5}, & \sin 2\beta = 0.684, & \gamma = 63.5^\circ. \end{array} \tag{20}$$

On the other hand, by using the total down-type 4×4 quark mass matrix, one obtains the full CKM matrix V^G_{CKM}:

$$V^G_{CKM} = \begin{pmatrix} 0.97421 & 0.22560 & 0.00357 & 0.00002 \\ 0.22547 & 0.97335 & 0.04182 & 0.00043 \\ 0.00847 & 0.04111 & 0.99911 & 0.00054 \\ 0.00007 & 0.00040 & 0.00055 & 1.00000 \end{pmatrix} \tag{21}$$

with the following relevant physical quantities:

$$\begin{array}{lll} m_d = 0.00273\,\text{GeV}, & m_s = 0.0546\,\text{GeV}, & m_b = 2.89\,\text{GeV}, \\ m_D = 958\,\text{GeV}, & m_u = 0.0014\,\text{GeV}, & m_c = 0.612\,\text{GeV}, \\ m_t = 171.7\,\text{GeV}, & m_U = 1602\,\text{GeV}, & \gamma = 63.5^\circ, \\ \sin 2\beta = 0.684, & |I_{CP}| = 2.93 \times 10^{-5}. & \end{array} \tag{22}$$

As for the FCNC, these are highly suppressed. As an example, we show $V_N^d \equiv V^{G\,\dagger}_{CKM} \cdot V^G_{CKM}$:

$$V_N^d = \begin{pmatrix} 1 & 7.76 \times 10^{-17} & 5.61 \times 10^{-17} & 1.51 \times 10^{-20} \\ 7.76 \times 10^{-17} & 1 & 6.69 \times 10^{-17} & 2.17 \times 10^{-19} \\ 5.61 \times 10^{-17} & 6.69 \times 10^{-17} & 1 & 3.25 \times 10^{-19} \\ 1.51 \times 10^{-20} & 2.17 \times 10^{-19} & 3.25 \times 10^{-19} & 1 \end{pmatrix} \tag{23}$$

There exists an almost perfect correspondence between the output from the quark effective mass matrix and the output from the 4×4 quark mass matrix. If one compares the CKM matrix computed using the effective matrix and the 3×3 block from the generalized CKM matrix, it is straightforward to conclude that all approximations that we have made to reach to the effective matrix are correct. This model produces an output where all physical quantities are in agreement with the experimental data up to $1\,\sigma$. In this example, the predicted masses of the vector-like quarks are of the order of TeV, $m_U = 1602$ GeV and $m_D = 958$ GeV, which allows us to control the suppression of the FCNC.

With these results, we have shown that, indeed, it is possible to generate a realistic pattern for mass matrices with vector-like quarks, by a simple breaking of the initial symmetry. Moreover, with two isosinglet vector-like quarks, each in a different sector, and with a symmetry breaking pattern from $Z_3 \otimes S(2)_L \otimes S(2)_R$ to $S(2)_L \otimes S(2)_R$ in the 3×3 mass matrix of the SM quarks, it is possible to generate a mass spectrum without any massless quark family, contrary to the situation in the SM context where a similar symmetry needed to be broken two times in order to produce the same physical result.

III. Leptonic sector

III.A New parametrization for leptonic mixing

A general unitary matrix V can be parametrized by the standard form:

$$V^{SP} = K \cdot O_{23} \cdot K_D \cdot O_{13} \cdot O_{12} \cdot K_M, \tag{24}$$

where O_{ij} are elementary orthogonal relations in the (ij) plane, $K_D = \text{Diag}\left(1\ ,\ 1\ ,\ e^{i\alpha_D}\right)$ has one phase, $K_M = \text{Diag}\left(1\ ,\ e^{i\alpha_1^M}\ ,\ e^{i\alpha_2^M}\right)$ has two phases and $K = \text{Diag}\left(e^{i\phi_1}\ ,\ e^{i\phi_2}\ ,\ e^{i\phi_3}\right)$ is some general pure phase unitary matrix with three phases.

We work in the Weak Basis where the charged leptons mass matrix is diagonal, the Majorana mass matrix is diagonalized by V^{SP} and eq. (24) corresponds to the PMNS matrix. In this basis, the diagonal phase matrix K can be absorbed by the neutrino fields.

This parametrization is analogous to the one that is used for the quark sector (if we consider only the moduli of its elements, both parametrizations are exactly the same). However, the quark mixing is quite different from the leptonic mixing. Given the existence of large leptonic mixing and a possible different nature of the neutrinos, i.e. a quasi-degenerate spectrum instead of a hierarchical one, we argue that it may be useful to consider a different parametrization for the leptonic sector.

We propose a new parametrization for the leptonic mixing matrix of the form:

$$V = O_{23} \cdot O_{12} \cdot K^i_\alpha \cdot O, \tag{25}$$

where O_{ij} are elementary orthogonal relations in the (ij) plane, $K = \mathrm{Diag}\left(1\ ,\ i\ ,\ e^{i\alpha}\right)$ has two complex phases, i and α, and O is a general orthogonal matrix with three angles. We shall prove that any unitary matrix can be written in this form.

Proof:

Consider a general unitary matrix V and compute the following symmetric unitary matrix:

$$S = V^* \cdot V^\dagger \tag{26}$$

If we take the complex phases of the first column and first row of S such that these become real, in a similar way of [14], the matrix S can be rewritten as:

$$S = K_S \cdot U_0^* \cdot U_0^\dagger \cdot K_S \tag{27}$$

with $K_S = \mathrm{Diag}\left(e^{i\alpha_1}\ ,\ e^{i\alpha_2}\ ,\ e^{i\alpha_3}\right)$. The diagonal matrix K_S has no physical meaning, since it only rephases the PMNS matrix V on the left. In a Weak Basis where the charged lepton mass matrix is diagonal, the phases of K_S can be absorbed by a redefinition of the right-handed charged lepton fields. Given V, we can compute explicitly K_S, O_{23} and K_α^i of U_0. We now define the following combination of unitary matrices,

$$W_0 \equiv \left(U_0^T \cdot K_S^*\right) \cdot V^*. \tag{28}$$

The matrix W_0 is real and orthogonal:

$$\begin{aligned} W_0 \cdot W_0^T &= \left(U_0^T \cdot K_S^*\right) \cdot V^* \cdot V^\dagger \cdot \left(K_S^* \cdot U_0\right) \\ &\overbrace{=}^{\text{inserting eq. (26)}} \left(U_0^T \cdot K_S^*\right) \cdot S \cdot \left(K_S^* \cdot U_0\right) \\ &\overbrace{=}^{\text{inserting eq. (27)}} \mathrm{I}_{3\times 3}. \end{aligned} \tag{29}$$

We constructed W_0 as a combination of unitary matrices. From now on, we rewrite W_0 explicitly as $W_0 \equiv O$, where the last one stands for the fact that it is a real-orthogonal matrix with three angles. Finally, from eq. (28), we find for the general unitary matrix V:

$$V = K_S^* \cdot U_0 \cdot O \to V = O_{23} \cdot O_{12} \cdot K_\alpha^i \cdot O \tag{30}$$

where we have discarded the unphysical pure phase matrix K_S^* that can be absorbed by the fields. With this new parametrization, we continue to have 6 physical parameters: 2 angles in O_{23} and O_{12}; 3 angles in O; one complex phase α besides i in K_α^i. From now on, we use explicitly the following full notation:

$$V = O_{23}^L \cdot O_{12}^L \cdot K_\alpha^i \cdot O_{23}^R \cdot O_{13}^R \cdot O_{12}^R \tag{31}$$

where we have identified each of the elementary orthogonal rotations, either on the left of the CP violating pure phase matrix $K_\alpha^i = \mathrm{Diag}\left(1\ ,\ i\ ,\ e^{i\alpha}\right)$ or on the right, with a superscript L, R.

Other parametrizations:

Using the same procedure, we could also obtain another forms of parametrization for the lepton mixing matrix [15], e.g. one can have a parametrization where $V = O_{23} \cdot O_{13} \cdot K_i^\alpha \cdot O$ with $K_i^\alpha = \mathrm{Diag}\left(1\ ,\ e^{i\alpha}\ ,\ i\right)$, or even other variations such as $V = O_{12} \cdot O_{23} \cdot \tilde{K}_i^\alpha \cdot O$ with $\tilde{K}_i^\alpha = \mathrm{Diag}\left(e^{i\alpha}\ ,\ i\ ,\ 1\right)$. Here, we concentrate on the parametrization of eq. (31) and discuss its usefulness.

Usefulness

We still do not know the exact nature of neutrinos: if they are Majorana or Dirac, as well if neutrinos mass spectrum is hierarchical or quasi-degenerate. It turns out that if the mass spectrum is quasi-degenerate, the new parametrization will be very useful and may reflect some specific nature of neutrinos, i.e. if they are Majorana and quasi-degenerate. The left part $O_{23}^L \cdot O_{12}^L \cdot K_\alpha^i$ in eq. (31) suggests some major intrinsic Majorana character of neutrino mixing and CP violation, while the right part $O_{23}^R \cdot O_{13}^R \cdot O_{12}^R$, with three angles, reflects that there are 3 neutrino families and results in small mixing, comparable to the quark sector, of the order of the Cabibbo angle. Thus, from this point of view, the dominant contribution for large neutrino mixing must come from the Majorana character of neutrinos.

Other interesting aspect that arrives with this new parametrization is the fact that the Dirac and Majorana CP violation quantities are related to just one complex phase α present in K_α^i.

III.B Degenerate and quasi-degenerate Majorana neutrinos

If neutrinos are quasi-degenerate, this new parametrization shows that we do not need some unitary complex matrix W in $V = U_0 \cdot W$, as in [14], to describe the contributions arising from the perturbation. All perturbations of an exact degenerate limit result in a leptonic mixing matrix $V = U_0 \cdot O$, where U_0 is somehow related to some degenerate limit and O is some general orthogonal matrix that arises from the perturbation.

The matrix U_0 that diagonalizes the degenerate mass matrix is not exactly the same present in the mixing matrix of the quasi-degenerate neutrinos. Suppose that the full mass matrix for quasi-degenerate neutrinos is:

$$M = \mu\left(S_0 + \epsilon^2 Q\right). \tag{32}$$

Then, the full lepton mixing matrix for the quasi-degenerate case is $V = U_0' \cdot O$, where U_0' is the same form of U_0. As a consequence of the perturbation, the U_0' matrix will differ slightly from the unitary matrix U_0 that diagonalizes the degenerate limit part S_0, when $\epsilon^2 \to 0$. Next, we shall analyze this issue more carefully using numerical simulations.

As for quantities that are sensitive to CP violation, the Weak Basis invariant $G_m \equiv \mathrm{Tr}\left[\left(M_\nu \cdot H_l \cdot M_\nu^*\right),\ H_l^*\right]^3$, which signals the presence of CP violation even in the neutrino degeneracy limit, is defined in [14]. With the new parametrization, this invariant takes a new form:

$$G \equiv \frac{G_m}{\Delta_m} = \frac{3}{4}\left|\sin 2\theta_L\ \sin 4\theta_L\ \sin^2 2\phi_L\ \sin 2\alpha\right|, \tag{33}$$

where $\Delta_m = \mu^6\left(m_\tau^2 - m_\mu^2\right)^2\left(m_\tau^2 - m_e^2\right)^2\left(m_\mu^2 - m_e^2\right)^2$ and θ_L, ϕ_L and α are the angles of O_{23}^L, O_{12}^L and K_α^i, respectively. It is interesting the fact that G depends only on the left part of eq. (31) and on $\sin\alpha$. All CP violation effects, i.e. the combined effects from Dirac- and Majorana-type CP violation, are present in α phase. Intuitively, we may question ourselves whether there exists a process in nature that could be directly related with

Case	O_{23}^L	O_{12}^L	O_{23}^R	O_{13}^R	O_{12}^R
I-A	$-\pi/4$	$\sin^{-1}(1/\sqrt{3})$	$\epsilon\, t_{23}$	0	$\epsilon\, t_{12}$
I-B	$-\pi/4$	$\epsilon\, t_{12}$	$\epsilon\, t_{23}$	$\epsilon\, t_{13}$	$\sin^{-1}(1/\sqrt{3})$
I-C	$-\pi/4$	$\sin^{-1}(1/2)$	$\epsilon\, t_{23}$	ϵt_{13}	$\sin^{-1}(1/\sqrt{6})$
II-A	$\epsilon\, t_{23}$	$\epsilon\, t_{12}$	$-\pi/4$	$\epsilon\, t_{13}$	$\sin^{-1}(1/\sqrt{3})$
II-B	$\sin^{-1}(1/\sqrt{3})$	$\epsilon\, t_{12}$	$-\pi/4$	$\epsilon\, t_{13}$	$\sin^{-1}(1/\sqrt{3})$

Table 1: Values of the parameters for each case.

Case	I_{CP}	α_1^M	α_2^M
I-A	$\frac{\sqrt{2}}{6\sqrt{3}}\,\lvert\epsilon\, t_{23}\cos\alpha\rvert$	$\tan^{-1}\left(\frac{\sqrt{2}}{\epsilon\, t_{12}}\right)$	$\pi/2$
I-B	$\frac{\epsilon}{3\sqrt{2}}\,\lvert t_{13}\cos\alpha\rvert$	$\tan^{-1}\left(\frac{3}{\sqrt{2}}\epsilon\, t_{12}\right)$	$\tan^{-1}\left(\frac{\epsilon t_{12}}{\sqrt{2}}\left(1+\frac{\sqrt{2}t_{23}}{t_{13}}\right)\right)$
I-C	$\frac{\epsilon}{8}\left\lvert t_{13}\left(-\sin\alpha+\sqrt{\frac{5}{3}}\cos\alpha\right)+\frac{1}{\sqrt{3}}t_{23}\left(\sqrt{\frac{5}{3}}\sin\alpha+\cos\alpha\right)\right\rvert$	$\tan^{-1}\left(3\sqrt{\frac{3}{5}}\right)$	$\tan^{-1}\left(\frac{\sqrt{3}t_{13}+\sqrt{15}t_{23}}{3\sqrt{5}t_{13}-t_{23}}\right)$
II-A	$\frac{\epsilon}{6}\,\lvert t_{12}\rvert$	$\tan^{-1}\left(\frac{3}{2}\epsilon\, t_{12}\right)$	$\tan^{-1}\left(\frac{t_{12}}{\sqrt{2}t_{13}}\right)$
II-B	$\frac{\epsilon}{18}\,\lvert 4t_{13}\cos\alpha+t_{12}\rvert$	$\tan^{-1}\left(\frac{3}{2}\epsilon\, t_{12}\right)$	$\tan^{-1}\left(\frac{t_{12}}{\sqrt{2}t_{13}}\right)$

Table 2: The I_{CP} and Majorana phases as functions of the perturbed parameters in leading order.

this combined CP violating quantity, instead of searching processes that are related separately either with the Dirac or Majorana effects of CP violation.

If neutrinos are Majorana, we should observe the existence of processes in nature such as neutrinoless double beta decay. With this new parametrization, the parameter $m_{\beta\beta}$ takes the following form:

$$|m_{\beta\beta}| = \left|\mu\,(S_0)_{11}\right| = \left|\mu\cos 2\phi_{12}^L\right| \tag{34}$$

in zeroth order in ϵ. Using the standard parametrization for $|m_{\beta\beta}|$, we obtain the following approximation:

$$|m_{\beta\beta}| = \left|\mu\left(\cos^2(\phi_{sol}) + e^{2i\alpha_1^M}\sin^2(\phi_{sol})\right)\right| \tag{35}$$

neglecting the terms with V_{13}^2.

If somehow we could measure separately the values for μ and $m_{\beta\beta}$, it will be possible analyze the influence of the Majorana phase α_1^M, i.e. if this phase has a significant contribution. We should also note that, by comparing eq. (34) with eq. (35), we may identify the solar mixing angle (ϕ_{sol}) with ϕ_{12}. However, if this is not the case, then automatically we know that the perturbation in eq. (31), represented by the right part, has a significant influence in the solar mixing angle. E.g., suppose that inserting (future) experimental results in eq. (35) yields $\alpha_1^M = 0$, which from eq. (34) results in $\theta_{12} = 0$. Thus, a large solar angle must come mainly from O, i.e., the right part of eq. (31).

III.C Leptonic CP violation from a new perspective

In the case of the Standard Parametrization in eq. (24), if neutrinos are of the Dirac type, maximum CP violation occurs by choosing $\alpha_D = \pi/2$. This is the only phase responsible for generating large CP violation. On the other-hand, if neutrinos are Majorana there are two more CP violating phases α_1^M and α_2^M. Thus, one finds that large CP violation is limited to consider these two facts.

It turns out that if we switch to the new parametrization, one gets a much richer structure for large CP violation, particularly if neutrinos are quasi-degenerate. Next, we shall present different limit cases where it is possible to generate large CP violation. We explore all combinations of the O_{23}'s and O_{12}'s that result in both solar and atmospheric mixing angles near experimental result, while the other O_{ij}'s are kept small in order to obtain mixing matrices near to $V_{13} = 0$, i.e. with a small reactor angle. Our choice to fix some parameters is justified for the closeness of the TBM with the experimental data. Thus, in zeroth order, we are able to reproduce the TBM. The free parameters, which are treated as perturbation parameters, will be responsible for approximate our initial ansatz to the current mixing angles, always with large leptonic CP violation.

Limit case I

In this limit, we consider that the combination $O_{23}^R \cdot O_{13}^R$ has small rotation angles, depending on some small parameter ϵ typically of the order of Cabibbo angle or smaller:

$$V = O_{23}^L \cdot O_{12}^L \cdot K_\alpha^i \cdot O_{23}^{\epsilon R} \cdot O_{13}^{\epsilon R} \cdot O_{12}^R. \tag{36}$$

It is used a notation where the angle θ_L refers to O_{23}^L, ϕ_L and ϕ_R refer to O_{12}^L and O_{12}^R, respectively, while the small angles coming from $O_{23}^{\epsilon R}$ and $O_{13}^{\epsilon R}$ are denoted as $\epsilon\, t_{23}$ and $\epsilon\, t_{13}$, respectively. Within limit case I, we may distinguish two opposite scenarios and some extra intermediate scenario, denoted by I-A, I-B and I-C, respectively.

Limit case II

In this limit, we consider that the combination $O_{23}^L \cdot K_\alpha^i \cdot O_{23}^R \cdot O_{12}^R$ gives the large contribution whereas the other O_{12}^L and O_{13}^R have small angles:

$$V = O_{23}^L \cdot O_{12}^{\epsilon L} \cdot K_\alpha^i \cdot O_{23}^R \cdot O_{13}^{\epsilon R} \cdot O_{12}^R. \tag{37}$$

For this case we use the following notation: θ^L and θ_R are the angles of the orthogonal matrices O_{23}^L and O_{23}^R, respectively; ϕ_R refers to O_{12}^R; the smaller angles $\epsilon\, t_{12}$ and $\epsilon\, t_{13}$ are the rotation angles of $O_{12}^{\epsilon L}$ and $O_{13}^{\epsilon R}$, respectively. As in the limit case I, we may construct two

Case	$\sin^2(\theta_{atm})$	$\sin^2(\phi_{sol})$	$\|V_{13}\|^2$
I-A	$\frac{1}{2}+\sqrt{\frac{2}{3}}\epsilon\, t_{23}\sin\alpha$	$\frac{1}{3}+\frac{\epsilon^2}{9}\left(3t_{12}^2-2t_{23}^2\right)$	$\frac{\epsilon^2 t_{23}^2}{3}$
I-B	$\frac{1}{2}+\epsilon\, t_{23}\sin\alpha$	$\frac{1}{3}+\frac{t_{12}^2}{3}\epsilon^2$	$\epsilon^2 t_{13}^2$
I-C	$\frac{1}{2}\left(1-\epsilon\, t_{13}\cos\alpha-\sqrt{3}\,\epsilon\, t_{23}\sin\alpha\right)$	$\frac{1}{3}+\frac{\epsilon^2}{24}\left(3t_{13}^2-2\sqrt{5}t_{13}t_{23}-3t_{23}^2\right)$	$\frac{\epsilon^2}{4}\left(3t_{13}^2+t_{23}^2\right)$
II-A	$\frac{1}{2}+\epsilon\, t_{23}\sin\alpha$	$\frac{1}{3}+\frac{t_{12}^2}{6}\epsilon^2$	$\frac{\epsilon^2}{2}\left(2t_{13}^2+t_{12}^2\right)$
II-B	$\frac{1}{2}+\frac{\sqrt{2}}{3}\sin\alpha-\frac{\epsilon^2}{12}\left(t_{12}^2-8t_{12}t_{13}\cos\alpha\right)$	$\frac{1}{3}+\frac{t_{12}^2}{6}\epsilon^2$	$\frac{\epsilon^2}{2}\left(2t_{13}^2+t_{12}^2\right)$

Table 3: Mixing angles as functions of the perturbed parameters in leading order.

opposite scenarios: a scenario where O_{23}^L is large or a scenario where O_{23}^R is large. The scenario where O_{23}^L is large, whereas O_{23}^R is small, is already contained in the scenario I-A of limit case I (modulo some slight modifications which produce equivalent results). Therefore, it is sufficient to focus on a scenario where O_{23}^L is small and O_{23}^R is large, or exceptionally on a scenario between, where both are large.

We present three tables containing the angles and the physical relevant quantities as functions of the perturbed parameters in leading order.

All cases have large Dirac CP violation. Cases I-A and I-C produce large Majorana phases. Cases II-A and II-B produce one large Majorana phase and a smaller one. Case I-B produces two small Majorana phases.

For the cases I-B, II-A and II-B the value of $\sin^2(\phi_{sol})$ can not be lower than 1/3, which is not in agreement with the experimental results at $1\,\sigma$ level [16]. This results from our initial choice of beginning with an exact TBM for the mixing matrix.

Although, with regard to Scenario I-A, very similar results are obtained if one chooses as starting points the hexagonal mixing or golden ratio mixing 1 [7], some of our conclusions, with respect to other different scenarios, may depend significantly on the initial starting point.

III.D Standard Parametrization vs New Parametrization

To justify that the new parametrization is an added value to better understand neutrino physics, in particular quasi-degenerate Majorana neutrinos, we choose the scenario I-A and reproduce it in the standard parametrization. Scenario I-A seems to be the most appealing since we only need two extra parameters to fit the experimental results on lepton mixing. This scenario also provides large Dirac-CP violation and large Majorana phases.

In the standard parametrization, the scenario I-A is obtained with:

$$V_{SP}=O_{23}^{\pi/4}\cdot K_D\cdot O_{13}^{\theta}\cdot O_{12}^{\phi_0}, \qquad (38)$$

with $\sin\phi_0=\frac{1}{\sqrt{3}}$ and $\theta=0$. For simplicity, we leave out the Majorana phases. In order to have $|V_{13}|\neq 0$, we have to switch on the angle O_{13}, since $|V_{23}|=|V_{33}|$. However, irrespective to the value that we choose for θ, it is impossible to change the atmospheric angle. To avoid this problematic situation, we must choose from the start another angle for O_{23} different from $\pi/4$, or change the TBM limit with some additional contribution, afterwards. With this, we show that in the Standard Parametrization, it is impossible to adjust the TBM and correct the atmospheric mixing angle using the remaining parameters. This is in clear contrast with what one obtains in the context of the new parametrization. In scenario I-A, with the suitable choice for the parameter $\epsilon\, t_{23}$ in table 3, it is possible to adjust the atmospheric mixing angle and generate a small $|V_{13}|$, simultaneously, with large values for CP violation.

III.E Numerical analysis and stability

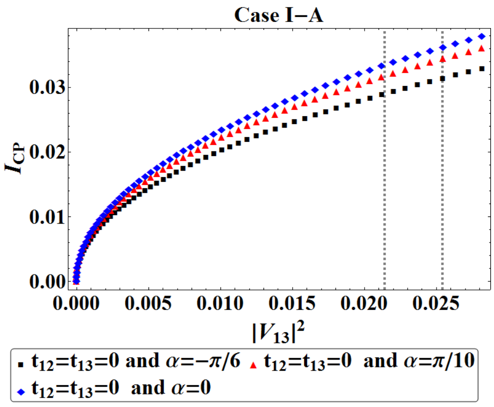
We shall give a numerical analysis of some scenarios described in the previous section:

$$\begin{aligned}
I-A:&\quad V_0=O_{23}^{\pi/4}\cdot O_{12}^{\phi_0}\cdot K_{\alpha_0}^i,\\
I-B:&\quad V_0=O_{23}^{\pi/4}\cdot K_{\alpha_0}^i\cdot O_{12}^{\phi_0},\\
I-C:&\quad V_0=O_{23}^{\pi/4}\cdot O_{12}^{\phi_1}\cdot K_{\alpha_0}^i\cdot O_{12}^{\phi_2},\\
II-A:&\quad V_0=O_{23}^{\pi/4}\cdot O_{12}^{\phi_0},\\
II-B:&\quad V_0=O_{23}^{\theta_0}\cdot K_{\alpha_0}^i\cdot O_{23}^{\pi/4}\cdot O_{12}^{\phi_0},
\end{aligned} \qquad (39)$$

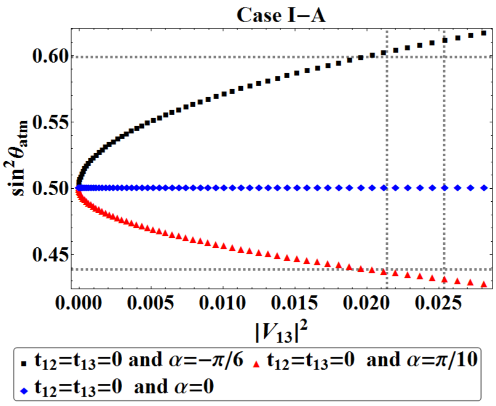
where $\sin\phi_0=\sin\theta_0=\frac{1}{\sqrt{3}}$, $\sin\phi_1=\frac{1}{2}$ and $\sin\phi_2=\frac{1}{\sqrt{6}}$.

In each scenario, we begin with a fixed scheme, the TBM scheme. We assume α with diverse fixed values for each scenario. As an example, we show the results for case I-A. From figure 1, we conclude that this case provides large Dirac-CP violation, in a region where the values for $|V_{13}|^2$ are in accordance with the experimental results. In addition, we also give a numerical analysis of the stability of the different scenarios. The full neutrino mass matrix is composed of an exact degenerate part in the form of a symmetric unitary matrix $S_0=U_0^*\cdot U_0^\dagger$, related to one of these TBM scenario schemes, and a part composed of a random perturbation Q proportional to a small parameter ϵ^2, eq. (32). The matrix U_0 corresponds to the left part of eq. (39), including the phase matrix K_α^i and will be different for each case, e.g. for case II-A, U_0 is the identity matrix. The right part of eq. (39) is denoted by O_0. In the case II-A, O_0 is the whole matrix V_0.

The matrix Q is some complex symmetric perturbation such that $\mathrm{Re}(Q_{ij})$ and $\mathrm{Im}(Q_{ij})$ are real numbers between -1 and 1. As for the ϵ^2, we take a fixed value, e.g. $\epsilon^2\lesssim 0.05$ and with a common neutrino mass $\mu\gtrsim 0.15$eV in order to guarantee that the numerical simulations are in a mass region where $\Delta m_{31}^2=\mathcal{O}(1)\times 10^{-3}\,\mathrm{eV}^2$. The parameter ϵ is of the order of the Cabibbo angle. We discard cases generated by the perturbation where $|\Delta m_{31}^2|<|\Delta m_{21}^2|$. For this analysis we do not impose any other restriction to mass differences, e.g. the experimental results for mass differences: further restrictions do not change significantly any of the plots. We are more concerned with the stability of each scenario when the degeneracy is lifted. Our numerical analysis consists in computing the full lepton mixing matrix V, for each different mixing scenario and random $Q's$ in M, such that

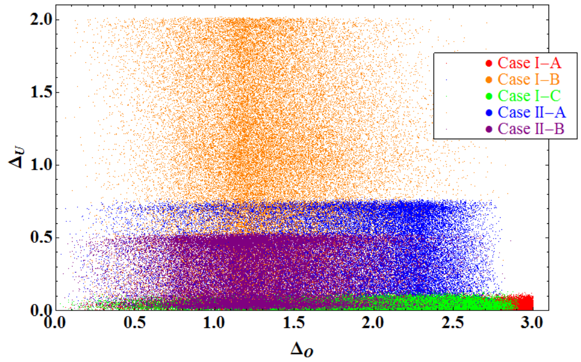


(a) Variation of I_{CP} with $|V_{13}|^2$ for case I-A.

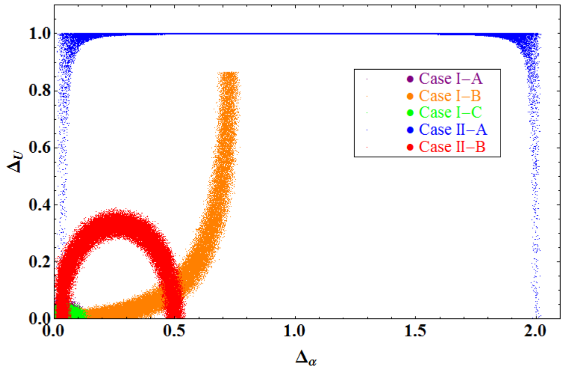


(b) Variation of $\sin^2(\theta_{atm})$ with $|V_{13}|^2$ for case I-A

Figure 1: Variation of I_{CP} and $\sin^2(\theta_{atm})$ with $|V_{13}|^2$ for case I-A. The dashed lines correspond to experimental limits of the physical quantities.



(a) Variation of Δ_U with Δ_O for all cases.



(b) Variation of Δ_U with Δ_α for all cases.

Figure 2: Numerical analysis of the stability for different scenarios. It was considered $\alpha_0 = \frac{\pi}{3}$ for the first three cases and $\alpha_0 = 0$ for the remaining ones.

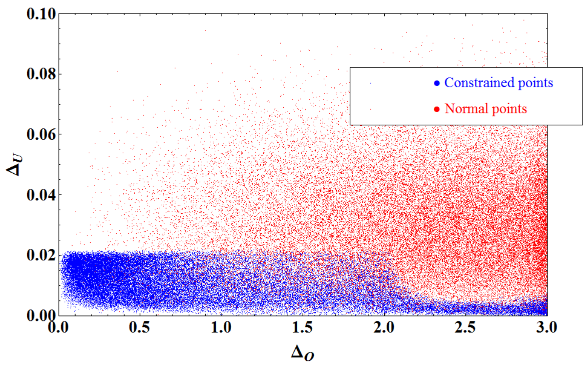


Figure 3: Numerical analysis of the stability for case I-A with $\alpha = \pi/9$, where it is compared the situation where Q is constrained by eq. (42) with the unconstrained Q.

$V^T \cdot M \cdot V = D$ is real and positive. The lepton mixing matrix can be decomposed in the new parametrization, which allows us to compare the new $U \equiv O_{23} \cdot O_{12} \cdot K^i_\alpha$ from the perturbation, with the original U_0 in the degeneracy limit, for each case in eq. (39). To measure how much U differ from U_0, as well the differences between O and O_0, we evaluate a quantity Δ_U and Δ_O defined by:

$$\begin{aligned} \Delta_U &= \frac{1}{2} \sum ||(U)_{ij}| - |(U_0)_{ij}||, \\ \Delta_O &= \frac{1}{2} \sum ||(O)_{ij}| - |(O_0)_{ij}||. \end{aligned} \tag{40}$$

The factor 1/2 is a suitable normalization factor chosen such that, e.g. in a case where the original $O_0 = I$ and the new O is such that $O = O_{12}$ with an angle $\sin\phi = 0.2$, then also $\Delta_0 \approx 0.2$, of the same order of Cabibbo angle. This definition is not sensitive to the phase factors of K^i_α. We compare the original phases α_0 present in U_0 with the ones after the perturbation, with any difference of π discarded:

$$\Delta_\alpha = ||\sin\alpha| - |\sin\alpha_0||\,. \tag{41}$$

From figure 1, we conclude that the Δ_U of scenarios I-A and I-C does not suffer any change with perturbations. From figure 2, scenario I-A is the one with less changes with respect to Δ_α. Thus, with regard to U, the scenario I-A is the most stable. However, the results for O are quite different. From figure 2, we see that the perturbations generate large Δ_0 contributions for all cases and in particular for scenario I-A. This situation may be improved by imposing certain restriction to Q, e.g. with some kind of symmetries. In figure 3, we give an example where the perturbations Q are restricted: certain elements are taken to be zero, while the imaginary part and the diagonal real part are taken to be 0.1 smaller than the others:

$$Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x_{23} \\ 0 & x_{23} & 0 \end{pmatrix} + 0.1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & x_{22} + i\, y_{22} & i\, y_{23} \\ 0 & i\, y_{23} & x_{33} + i\, y_{33} \end{pmatrix} \tag{42}$$

where $x's$ and $y's$ are real random numbers between -1 and 1. For the initial phase α_0 we take $\alpha_0 = \pi/9$. Most of the deviations Δ_0 are now around 0.2 of the order of the Cabibbo angle, and this does not affect having large values for Dirac-CP violation.

IV. Conclusions

For the quark sector, we presented an entirely new model. The model consisted of an extension of the SM with one vector-like quark and three extra Higgs fields (apart from the SM-Higgs), for each up- and down-type quark sector. At the level of the SM, the quark mass matrices exhibited $S(2)_L \otimes S(2)_R$ permutation symmetry. However, at a higher level, i.e. when considering the whole 4×4 quark matrix with the inclusion of the VLQ, this symmetry was broken. Thus, in the effective quark mass matrix, the would-be massless quark from 3×3 SM quark mass matrix with the $S(2)_L \otimes S(2)_R$ flavour symmetry got a contribution from the terms which involve the VLQ and acquired a mass. We also shaped the model in such a way that all Yukawa coupling were real, by requiring CP invariance of the Lagrangian, and we succeed to generate CP violation through the phases of the extra scalar fields. By minimizing the potential, we showed that, indeed, it was possible to have an extra scalar $SU(2)_L$ doublet with a phase that contributes to the CP violation in the VLQ framework. Moreover, by embedding the whole theory in a $Z_3 \otimes S(2)_L \otimes S(2)_R$ symmetry, we were able, not only to obtain the correct hierarchy for the quarks of each sector but also to generate a realistic CKM matrix, where all mixing angles, I_{CP} and the angles of the unitarity triangle are within $1\,\sigma$ of the experimental results. The FCNC are naturally suppressed with both VLQ having a mass of the order of 1 TeV. With this model, we clearly go a step further from the model in [6], where the USY proposal for the SM was extended with a VLQ to solve the problem of CP-violation in USY. Here, we introduce a symmetry principle to obtain the form of the quark mass matrices and keep the 3×3 SM quark mass matrix real, e.g. in the extended Democracy from [17], with one massless eigenvalue, while the whole 4×4 matrix effectively leads to a massive first family, solves the USY problem and leads to a correct mixing output.

With regard to the lepton sector, we focused in the case where neutrinos are assumed to be Majorana particles and quasi-degenerate. We proposed a new parametrization for leptonic mixing and we have identified several-limit cases with mixing angles in agreement with experimental results and leading to large CP violation. It turns out that if neutrinos are quasi-degenerate and Majorana, this parametrization is very useful. It may reflect some specific nature of neutrinos, suggesting that there is some major intrinsic Majorana character of neutrino mixing and CP violation in the left part of the parametrization, while the right part O may reflect that there are three neutrino families with small mass differences, resulting in small mixing, comparable to the quark sector, of the order of the Cabibbo angle. Thus, from this point of view, the dominant contribution for large neutrino mixing must come from the Majorana character of neutrinos.

This new parametrization enables an alternative perspective of large leptonic CP violation and shows interesting aspects that were less clear in the standard parametrization. From the limit cases studied, the scenario I-A was the most appealing. It only needs 2 extra parameters to fit the experimental results on lepton mixing and provides large Dirac-CP violation and large values for the Majorana-CP violating phases. These results are derived explicitly from the form of the new parametrization.

Furthermore, we also studied the stability of each scenario. We analyzed how much U_0 and O_0, in the limit of exact degeneracy, differ from U and O after introducing a random perturbation. We concluded that the left part of the parametrization behaves quite differently for each scenario. It turns out that, with regard to U_0, the scenario I-A is the most stable. As for the right part O of the parametrization, the perturbations generate large contributions for all cases. However, we have shown how to improve this situation, by imposing certain restrictions on the allowed perturbations.

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Appendix A
Scalar Potential

The most general CP invariant scalar potential under the symmetries discussed in the section II.B is:

$$
\begin{aligned}
V =& \mu_1 \Phi_{SM}^{\dagger}\Phi_{SM} + \mu_2\left(\Phi_{SM}^{\dagger}\Phi_{SM}\right)^2 + \sum_{i=1}^{3}\lambda_{1,i}^{d}\left(\phi_i^{\dagger d}\phi_i^{d}\right) + \sum_{i=1}^{3}\lambda_{2,i}^{d}\left(\phi_i^{\dagger d}\phi_i^{d}\right)^2 + \sum_{i=1}^{3}\lambda_{1,i}^{u}\left(\phi_i^{\dagger u}\phi_i^{u}\right) \\
&+ \sum_{i=1}^{3}\lambda_{2,i}^{u}\left(\phi_i^{\dagger u}\phi_i^{u}\right)^2 + \sum_{i,j=1\ i<j}^{3}\lambda_{3,ij}^{u}\left(\phi_i^{\dagger u}\phi_j^{u}\right)\left(\phi_j^{\dagger u}\phi_i^{u}\right) + \sum_{i,j=1\ i<j}^{3}\lambda_{3,ij}^{d}\left(\phi_i^{\dagger d}\phi_j^{d}\right)\left(\phi_j^{\dagger d}\phi_i^{d}\right) \\
&+ \sum_{i,j=1\ i<j}^{3}\lambda_{4,ij}^{u}\left(\phi_i^{\dagger u}\phi_i^{u}\right)\left(\phi_j^{\dagger u}\phi_j^{u}\right) + \sum_{i,j=1\ i<j}^{3}\lambda_{4,ij}^{d}\left(\phi_i^{\dagger d}\phi_i^{d}\right)\left(\phi_j^{\dagger d}\phi_j^{d}\right) + \left(\Phi_{SM}^{\dagger}\Phi_{SM}\right)\sum_{i=1}^{3}\Big[\lambda_{5,i}^{d}\left(\phi_i^{\dagger d}\phi_i^{d}\right) \\
&+ \lambda_{5,i}^{u}\left(\phi_i^{\dagger u}\phi_i^{u}\right)\Big] + \sum_{i,j=1}^{3}\lambda_{6,ij}\left(\phi_i^{\dagger d}\phi_i^{d}\phi_j^{\dagger u}\phi_j^{u}\right) + \sum_{i,j=1}^{3}\lambda_{7,ij}\left(\phi_i^{\dagger u}\phi_j^{d}\phi_j^{\dagger d}\phi_i^{u}\right) + \sum_{i=1}^{3}\lambda_{8,i}\left(S_i^{*}S_i\right) + \sum_{i=1}^{3}\lambda_{9,i}\left(S_i^{*}S_i\right)^2 \\
&+ \left(\Phi_{SM}^{\dagger}\Phi_{SM}\right)\sum_{i}^{3}\left[\lambda_{10,i}\left(S_i^{*}S_i\right)\right] + \sum_{i,j=1}^{3}\lambda_{11,ij}^{d}\left(S_i^{*}S_i\right)\left(\phi_j^{\dagger d}\phi_j^{d}\right) + \sum_{i,j=1}^{3}\lambda_{11,ij}^{u}\left(S_i^{*}S_i\right)\left(\phi_j^{\dagger u}\phi_j^{u}\right) \\
&+ \sum_{i,j=1}^{3}\lambda_{12,ij}\left(S_i^{*}S_iS_j^{*}S_j\right) + \eta_1^{d}\left[\left(\Phi_2^{d\dagger}\Phi_1^{d}\right)^2 + H.c.\right] + \eta_1^{u}\left[\left(\Phi_2^{u\dagger}\Phi_1^{u}\right)^2 + H.c.\right] \\
&+ \eta_2\left[\left(\Phi_3^{u\dagger}\Phi_3^{d}\Phi_3^{u\dagger}\Phi_3^{d}\right) + H.c.\right] + \eta_3\left(\Phi_2^{\dagger d}\Phi_1^{d}\Phi_1^{u\dagger}\Phi_2^{u} + H.c.\right) + \eta_4\left(\Phi_2^{\dagger u}\Phi_1^{d}\Phi_2^{d\dagger}\Phi_1^{u} + H.c.\right) \\
&+ \eta_5\left(\Phi_3^{\dagger u}\Phi_1^{d}\Phi_3^{d\dagger}\Phi_1^{u} + H.c.\right) + \eta_6\left(\Phi_3^{\dagger u}\Phi_2^{d}\Phi_3^{d\dagger}\Phi_2^{u} + H.c.\right) + \eta_7\left(\Phi_1^{d\dagger}\Phi_2^{d}\Phi_3^{d\dagger}\Phi_3^{u} + H.c.\right) \\
&+ \eta_8\left(\Phi_1^{u\dagger}\Phi_2^{u}\Phi_3^{d\dagger}\Phi_3^{u} + H.c.\right) + \eta_8\left(\Phi_3^{\dagger d}\Phi_1^{d}\Phi_3^{u\dagger}\Phi_1^{u} + H.c.\right) + \eta_9\left(\Phi_3^{\dagger d}\Phi_2^{d}\Phi_3^{u\dagger}\Phi_2^{u} + H.c.\right) \\
&+ \eta_{10}\left(\Phi_2^{\dagger d}\Phi_1^{d}\Phi_2^{u\dagger}\Phi_1^{u} + H.c.\right) + \eta_{11}\left(\Phi_2^{\dagger u}\Phi_1^{d}\Phi_1^{u\dagger}\Phi_2^{d} + H.c.\right) + \eta_{12}\left(\Phi_3^{d\dagger}\Phi_1^{d}\Phi_1^{\dagger u}\Phi_3^{u} + H.c.\right) \\
&+ \eta_{13}\left(\Phi_3^{d\dagger}\Phi_2^{d}\Phi_2^{\dagger u}\Phi_3^{u} + H.c.\right) + \eta_{14}\left(\Phi_3^{\dagger u}\Phi_2^{d}\Phi_2^{u\dagger}\Phi_3^{d} + H.c.\right) + \eta_{15}\left(\Phi_3^{\dagger u}\Phi_1^{d}\Phi_1^{u\dagger}\Phi_3^{d} + H.c.\right) \\
&+ \eta_{16}\left(S_2^{*}S_1S_2^{*}S_1 + H.c.\right) + \eta_{17}^{d}\left(S_2^{*}S_1\Phi_1^{d\dagger}\Phi_2^{d} + H.c.\right) + \eta_{18}^{d}\left(S_2^{*}S_1\Phi_2^{d\dagger}\Phi_1^{d} + H.c\right) \\
&+ \eta_{17}^{u}\left(S_2^{*}S_1\Phi_1^{u\dagger}\Phi_2^{u} + H.c\right) + \eta_{18}^{u}\left(S_2^{*}S_1\Phi_2^{u\dagger}\Phi_1^{u} + H.c.\right) + \eta_{19}^{u}\left(S_1^{*}S_2\Phi_3^{u\dagger}\Phi_3^{u} + H.c.\right) \\
&+ \eta_{19}^{d}\left(S_1^{*}S_2\Phi_3^{d\dagger}\Phi_3^{d} + H.c.\right) + \eta_{20}^{d}\left(S_1^{*}S_3\Phi_3^{d\dagger}\Phi_1^{d} + H.c.\right) + \eta_{20}^{u}\left(S_1^{*}S_3\Phi_1^{u\dagger}\Phi_3^{u} + H.c.\right) + \eta_{21}\left[\left(S_1^{*}S_2\right)^2 + H.c.\right] \\
&+ \eta_{22}S_3^4 + \eta_{23}S_3^{*4} + \eta_{24}(S_1^3S_3^{*} + H.c.) + \eta_{25}(S_2^3S_3^{*} + H.c.) + \eta_{26}(S_1S_2^2S_3^{*} + H.c.) + \eta_{27}(S_1^2S_2S_3 + H.c.) \\
&+ \eta_{28}S_3^4 + \eta_{29}S_3^{*4} + \eta_{30}S_3^{*2}S_3^2 + \text{soft breaking terms}
\end{aligned} \tag{43}
$$

We are not interested in performing an extensive analysis of the parameters when calculating the conditions of stability of the vacuum. However, we prove that there exists possibility of having at least a phase δ_1 responsible for breaking the CP symmetry, in e.g. the VEV of $\Phi_1^{u,d}$. Moreover there exists a minimum with $v_u \neq 0$, $v_d \neq 0$, $s_1 \neq 0$ and $s_2 \neq 0$.
Thus, calculating $\frac{\partial V_0}{\partial \delta_1} = 0$, one obtains:

$$
\begin{aligned}
&\sin\delta_1\left[4\cos(\delta_1)v_uv_d\left(\eta_1^{d} + \eta_1^{u} + \eta_4 + \eta_{10}\right) + \left(\eta_{17}^{d} + \eta_{18}^{d} + \eta_{17}^{u} + \eta_{18}^{u}\right)s_1s_2\right] = 0, \\
&\Rightarrow \sin\delta_1 = 0 \quad \vee \quad \cos\delta_1 = -\frac{s_1s_2}{4v_uv_d}\frac{\eta_{17}^{d} + \eta_{18}^{d} + \eta_{17}^{u} + \eta_{18}^{u}}{\eta_1^{d} + \eta_1^{u} + \eta_4 + \eta_{10}}.
\end{aligned} \tag{44}
$$

We assumed that the soft breaking terms will not depend on the phases. An example of a soft breaking term would be μS_1S_1. This corresponds to the case where the phase appears in the VEV of the scalar doublet fields and not in the scalar singlet fields.