Algebraic and combinatorial properties of binomial edge ideals
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Resumo


**Palavras-chave:** Álgebra Comutativa Combinatória, Grafos simples, Ideais binomiais de arestas, Condição de Cohen-Macaulay, Regularidade de Castelnuovo-Mumford.
Abstract

This dissertation is an introduction to Combinatorial Commutative Algebra and more precisely to a topic of great interest since its introduction in 2009: binomial edge ideals. Starting with the study of monomial ideals and Gröbner bases (a fundamental pre-requisite for anyone who wants to study Combinatorial Commutative Algebra), this dissertation consists mainly in the study of the algebraic properties of a binomial edge ideal $J_G$ in terms of the combinatorial properties of the simple graph $G$. More precisely, for any simple graph $G$ the minimal prime ideals of $J_G$ will be determined and, for some classes of graphs, we will study whether $J_G$ satisfies the Cohen-Macaulay property. Finally, we will study the Castelnuovo-Mumford regularity of $J_G$.

Keywords: Combinatorial Commutative Algebra, Simple graphs, Binomial edge ideals, Cohen-Macaulay property, Castelnuovo-Mumford regularity.
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Commutative algebra was built in step with algebraic geometry and played an essential role in its development. In the 1950’s, homological aspects of modern commutative algebra became a new and important focus of research inspired by the work of Melvin Hochster. In 1975, Richard Stanley proved affirmatively the upper bound conjecture for spheres by using the theory of Cohen-Macaulay rings. This created another new trend of commutative algebra, as it turned out that commutative algebra supplies basic methods in the algebraic study of combinatorics on convex polytopes and simplicial complexes. Stanley was the first to use concepts and techniques from commutative algebra in a systematic way to study simplicial complexes by considering the Hilbert function of Stanley-Reisner rings, whose defining ideals are generated by square-free monomials. Since then, the study of square-free monomial ideals from both the algebraic and combinatorial points of view has become a very active area of research in commutative algebra.

Finite simple graphs are just a special class of simplicial complexes and so the results on Stanley-Reisner ideals can be used to study monomial edge ideals. The study of these ideals was started by Rafael Villarreal in the 1990’s and, since then, a lot of mathematicians studied their algebraic properties in terms of the combinatorial properties of the underlying graph. In 2003, in [10], Jrgen Herzog and Takayuki Hibi classified Cohen-Macaulay bipartite graphs and in 2010, in [14], Russ Woodroofe showed that the regularity of the monomial edge ideal of a weakly chordal graph is given by its induced matching number. These two results are particularly important in studying binomial edge ideals of closed graphs.

On the other hand, in the late 1980’s, the theory of Gröbner bases and initial ideals came into fashion in many branches of mathematics, since it provided new methods. They have been used not only for computational purposes but also to deduce theoretical results in commutative algebra and combinatorics. For example, based on the fundamental work of Gelfand, Kapranov, Zelevinsky and Sturmfels, the study of regular triangulations of a convex polytope by using suitable initial ideals (far beyond the classical techniques in combinatorics) turned out to be a very successful approach, and, after the pioneering work of Sturmfels, the algebraic properties of determinantal ideals have been explored by considering their initial ideal, which for a suitable monomial order is a square-free monomial ideal and hence is accessible to powerful techniques.

At about the same time, Galligo, Bayer and Stillman observed that generic initial ideals have particularly nice combinatorial structures and provide a basic tool for the combinatorial and computational study of the minimal free resolution of a graded ideal of the polynomial ring. Algebraic shifting, which was introduced by
Kalai and which contributed to the modern development of enumerative combinatorics on simplicial complexes, can be discussed in the frame of generic initial ideals.

Chapter 0 is basically a list of definitions and results in Commutative Algebra and Graph Theory which will be used throughout this dissertation. Consulting this chapter is recommendable for any reader who is not familiarized with these definitions and results.

Chapter 1 is just a first course in monomial ideals, simplicial complexes and Gröbner bases. Even if a reader does not want to study monomial edge ideals or binomial edge ideals, this chapter is meant to be a first introduction to combinatorial Commutative Algebra. To complete one’s learning, we recommend, for example, the first two chapters of [5] or the first three chapters of [6].

In chapter 2, binomials edge ideals are studied. These ideals were introduced in [11], in 2009, where its minimal primes ideals were studied only in terms of the combinatorial properties of their underlying graphs. The results on minimal prime ideals of a binomial edge ideal apply for the class of conditional independence ideals where a fixed binary variable is independent of a collection of other variables, given the remaining ones. In this case the primary decomposition has a natural statistical interpretation.

Similar to what happens with monomial edge ideals, a general classification of Cohen-Macaulay binomial edge ideals seems to be hopeless. However, in [12], in 2010, the Cohen-Macaulayness of binomial edge ideals was studied for two special classes of graphs: closed graphs and chordal graphs such that any two maximal cliques intersect in at most one vertex.

In chapter 3, the regularity of binomial edge ideals is studied. Woodroofe studied the regularity of monomial edge ideals. Recently some mathematicians started studying the regularity of binomial edge ideals. In [17], in 2013, Viviana Ene and Andrei Zarojanu showed that the regularity of the binomial edge ideal of a closed graph is given by the lengths of its induced paths. With respect to the class of chordal graphs such that any two maximal cliques intersect in at most one vertex, Ene and Zarojanu showed that this class of graphs satisfy both the Madani-Kiani and the Matsuda-Murai conjectures.

While in this dissertation the Matsuda-Murai conjecture was meant to be presented as a conjecture, it happened to be fully shown this year, on April 6th, in [18] by Madani and Kiani. Since the partial proof for chordal graphs such that any two maximal cliques intersect in at most one vertex is an interesting proof which uses some nice results on the combinatorial data of these graphs, I decided to keep it in the dissertation, not forgetting to indicate a reference for the full proof of the Matsuda-Murai conjecture.
Chapter 0

Preliminaries

0.1 Basic notations

- If \( A \) and \( B \) are two sets, we write \( A \subset B \) when \( A \) is a subset of \( B \) and we write \( A \subsetneq B \) when \( A \) is a proper subset of \( B \).
- The set of integers is denoted by \( \mathbb{Z} \).
- The set of non-negative integers is denoted by \( \mathbb{N} \).
- Given a positive integer \( n \), we denote \( [n] = \{1, \cdots, n\} \).
- Given \( a, b \in \mathbb{Z} \), we denote \( [a, b] = \{x \in \mathbb{Z} : a \leq x \leq b\} \).
- All rings will be commutative and will have a unit.

0.2 Basic commutative algebra

In this section \( R \) is a ring and \( M \) is an \( R \)-module.

**Proposition 0.2.1.** Let \( I, J \subset R \) be two ideals. Then there exists an exact sequence

\[
0 \rightarrow \frac{R}{I \cap J} \rightarrow \frac{R}{I} \oplus \frac{R}{J} \rightarrow \frac{R}{I+J} \rightarrow 0.
\]

**Proof.** Consider the \( R \)-homomorphisms \( f : R/(I \cap J) \rightarrow R/I \oplus R/J \) and \( g : R/I \oplus R/J \rightarrow R/(I + J) \) given by \( f(r + I \cap J) = (r + I, r + J), \forall r \in R \) and \( g(r + I, s + J) = r - s + I + J, \forall r, s \in R \). These \( R \)-homomorphisms are well defined, \( f \) is injective, \( g \) is surjective and \( g \circ f = 0 \) holds. It remains to check that \( \ker(g) \subset \text{im}(f) \).

Let \( (r + I, s + J) \in \ker(g) \). Then \( r - s \in I + J \). Pick \( r' \in I \) and \( s' \in J \) such that \( r - s = r' + s' \). Then \( r - r' = s + s' \), and since \( r + I = r - r' + I \) and \( s + J = s + s' + J \), it follows that \( (r + I, s + J) = (r - r' + I, s + s' + J) = (r - r' + I, r - r' + J) = f(r - r' + I \cap J) \). \( \square \)
Definition 0.2.2. Let $I, J \subset R$ be two ideals. The ideal $I : J = \{ f \in R : f g \in I, \forall g \in J \}$ is called the colon ideal of $I$ with respect to $J$.

Definition 0.2.3. The ideal $\sqrt{I} = \{ f \in R : \exists k > 0 : f^k \in I \}$ is called the radical ideal of $I$.

Definition 0.2.4. The ideal $I$ is called a radical ideal if $I = \sqrt{I}$.

Proposition 0.2.5. The radical ideal of a given ideal is the intersection of the prime ideals containing it.

Proof. See [2, 1].

Corollary 0.2.6. An ideal is a radical ideal if and only if it is the intersection of prime ideals.

Definition 0.2.7. If $I$ is an ideal of $R$, then $I[x]$ is the ideal of the polynomials in $R[x]$ whose coefficients lie in $I$. Equivalently, considering the inclusion $I \subset R \subset R[x]$, $I[x]$ is the ideal of $R[x]$ generated by the elements of $I$.

Notation 0.2.8. Similarly $I[x_1, \cdots, x_n]$ is the ideal of the polynomials in $R[x_1, \cdots, x_n]$ whose coefficients lie in $I$.

Proposition 0.2.9. Let $P$ be an ideal of $R$. Then $P$ is a prime ideal of $R$ if and only if $P[x]$ is a prime ideal of $R[x]$.

Proof. Since $P = P[x] \cap R$, $P[x]$ being a prime ideal of $R[x]$ implies $P$ is a prime ideal of $R$.

Conversely, let $f, g \in R[x] \setminus P[x]$. Let $f = \sum f_k x^k$ and $g = \sum g_k x^k$. Let $m$ and $n$ be the smallest integers such that $f_m \notin P$ and $g_n \notin P$, respectively. Then the coefficient of degree $m + n$ of $fg$ is

$$(f_m g_0 + \cdots + f_m g_{n-1}) + f_m g_n + (f_m g_n + \cdots + f_m g_{m+n}).$$

Since $f_0, \cdots, f_m-1 \in P$, it follows that $f_m g_{n-1} + \cdots + f_m g_{m+n} \in P$. Similarly, $g_0, \cdots, g_{n-1} \in P$ implies $f_m g_0 + \cdots + f_m g_{n-1} \in P$. But $f_m, g_n \notin P$ implies $f_m g_n \notin P$, hence $fg \notin P[x]$. 

Corollary 0.2.10. Let $P$ be an ideal of $R$. Then $P$ is a prime ideal of $R[x]$ if and only if $P[x_1, \cdots, x_n]$ is a prime ideal of $R[x_1, \cdots, x_n]$.

Sometimes, when there is no ambiguity, $I$ may denote any of the ideals $I$, $I[x]$ or $I[x_1, \cdots, x_n]$.

Definition 0.2.11. A presentation of an ideal $I$ as an intersection $I = Q_1 \cap \cdots \cap Q_m$ of ideals is called irredundant if none of the ideals $Q_i$ can be omitted in this presentation.

Notation 0.2.12. The set of minimal prime ideals of an ideal $I$ is $\text{Min}(I)$.

Lemma 0.2.13. Suppose $I$ has irredundant presentation $I = P_1 \cap \cdots \cap P_m$ as an intersection of prime ideals. Then $\text{Min}(I) = \{ P_1, \cdots, P_m \}$.  

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Proof. Suppose without loss of generality that $P_1 \not\in \text{Min}(I)$. Then there exists a prime ideal $Q$ such that $I \subset Q \subset P_1$. Since $I = P_1 \cap \cdots \cap P_m \subset Q$ and $Q$ is a prime ideal, then some $P_i$ is contained in $Q$, thus $P_i \subset Q \subset P_1$, therefore the presentation $I = P_1 \cap \cdots \cap P_m$ is not irredundant. Hence $\{P_1, \ldots, P_m\} \subset \text{Min}(I)$.

On the other hand, let $Q \in \text{Min}(I)$. Again, since $I = P_1 \cap \cdots \cap P_m \subset Q$ and $Q$ is a prime ideal, some $P_i$ is contained in $Q$, and since both $P_i$ and $Q$ are minimal prime ideals of $I$, $Q = P_i$. Hence, $\text{Min}(I) = \{P_1, \ldots, P_m\}$, as desired.

Lemma 0.2.14 (Prime avoidance). Let $I$ be an ideal and let $P_1, \ldots, P_n$ be prime ideals such that $I \subset P_1 \cup \cdots \cup P_n$. Then $I \subset P_k$ for some $1 \leq k \leq n$.

Proof. See [2 1].

Definition 0.2.15. A prime ideal $P$ is called an associated prime ideal of $M$ if there exists an element $m \in M \setminus \{0\}$ such that $P = \text{Ann}(m)$.

Notation 0.2.16. The set of associated prime ideals of $M$ is denoted $\text{Ass}(M)$.

Proposition 0.2.17. Every maximal ideal of the set $\Sigma = \{\text{Ann}(m) : m \in M \setminus \{0\}\}$ is a prime ideal.

Proof. See [1 2:6].

Corollary 0.2.18. If $R$ is Noetherian, then $\text{Ass}(M) \neq \emptyset$.

Proposition 0.2.19. If $R$ is Noetherian and $M$ is finitely generated, then $\text{Ass}(M)$ is a finite non-empty set.

Proof. See [1 2:6].

As a corollary of the Hilbert's basis theorem (see [2 7]), the polynomial rings $K[x_1, \ldots, x_n]$, where $K$ is a field, are Noetherian. Since most rings on this dissertation will be quotients of these rings (and hence Noetherian), from now on all rings are Noetherian and all modules are finitely generated (hence Noetherian).

Proposition 0.2.20. For every prime ideal $P$ one has $P \in \text{Ass}(M)$ if and only if $P_P \in \text{Ass}(M_P)$.

Proof. See [1 2:6]

Proposition 0.2.21. For every ideal $I$ one has $\text{Min}(I) \subset \text{Ass}(R/I)$.

Proof. Let $P \in \text{Min}(I)$. Then $P_P \in \text{Min}(I_P)$, thus $P_P$ is the only prime ideal containing $I_P$. Since all associated prime ideals of $R_P/I_P$ must contain $I_P$, it follows that $\text{Ass}(R_P/I_P) \subset \{P_P\}$, and since $\text{Ass}(R_P/I_P) \neq \emptyset$, then $\text{Ass}(R_P/I_P) = \{P_P\}$. On the other hand, $R_P/I_P$ and $(R/I)_P$ are isomorphic $R_P$-modules, and so $\text{Ass}((R/I)_P) = \{P_P\}$. Hence $P \in \text{Ass}(R/I)$.

Definition 0.2.22. An ideal $I$ is a primary ideal if for every $x, y \in R$ such that $xy \in I$ one has $x \in I$ or $y \in \sqrt{I}$. 
Proposition 0.2.23. If $I$ is a primary ideal, then $\sqrt{I}$ is a prime ideal.

Definition 0.2.24. An ideal $I$ is a $P$-primary ideal if it is a primary ideal such that $\sqrt{I} = P$.

Proposition 0.2.25. An ideal $I$ is a $P$-primary ideal if and only if $\text{Ass}(R/I) = \{P\}$.


Definition 0.2.26. A primary irredundant decomposition of an ideal $I$ is an irredundant decomposition $I = Q_1 \cap \cdots \cap Q_m$ such that all ideals $Q_i$ are primary ideals.

Proposition 0.2.27. If $I \neq R$, then $I$ admits a primary irredundant decomposition. Moreover, if $I = Q_1 \cap \cdots \cap Q_m$ is such a decomposition with $P_i = \sqrt{Q_i}$ for $i \in [m]$, then $\text{Ass}(R/I) = \{P_1, \cdots, P_m\}$.


Corollary 0.2.28. If $I \neq R$ is a radical ideal, then $\text{Min}(I) = \text{Ass}(R/I)$.

Proof. Just recall that $I$ is the intersection of its minimal prime ideals and such intersection is a primary irredundant decomposition.

0.3 Height, dimension and grade

Recall that all rings considered are Noetherian and that all modules considered are finitely generated.

Definition 0.3.1. The height of a prime ideal $P$ is

$$\text{ht}(P) = \sup\{n \in \mathbb{N} : \text{there exists an ascending chain of prime ideals } P_0 \subseteq \cdots \subseteq P_n = P\}.$$ 

Theorem 0.3.2. If $P$ is a minimal prime of $I = (r_1, \cdots, r_n)$, then $\text{ht}(P) \leq n$.


Corollary 0.3.3. If $P$ is a prime ideal, then $\text{ht}(P) < \infty$.

Definition 0.3.4. The height of an ideal $I$ is

$$\text{ht}(I) = \min\{\text{ht}(P) : P \in \text{Min}(I)\}.$$ 

Definition 0.3.5. An ideal $I$ is unmixed if $\text{ht}(I) = \text{ht}(P)$ for every $P \in \text{Ass}(R/I)$.

In particular, if $I$ is unmixed, then $\text{Min}(I) = \text{Ass}(R/I)$.

Definition 0.3.6. The Krull dimension of a ring $R$ is

$$\dim(R) = \sup\{n \in \mathbb{N} : \text{there exists an ascending chain of prime ideals } P_0 \subseteq \cdots \subseteq P_n \subseteq R\}.$$
Though prime ideals in a Noetherian ring always have finite height, there are Noetherian rings with infinite Krull dimension. Nonetheless, since polynomial rings have finite Krull dimension and most rings considered in this dissertation are quotients of polynomial rings, we will assume that all rings considered are Noetherian rings with finite Krull dimension.

**Proposition 0.3.7.** For every prime ideal $P$ one has $\dim(R) \geq \text{ht}(P) + \dim(R/P)$.

**Corollary 0.3.8.** For every ideal $I$ (not necessarily prime) one has $\dim(R) \geq \text{ht}(I) + \dim(R/I)$.

**Proof.** Pick $P \in \text{Min}(I)$ such that $\dim(R/P) = \dim(R/I)$. Then $\dim(R) \geq \text{ht}(P) + \dim(R/P) = \text{ht}(P) + \dim(R/I) \geq \text{ht}(I) + \dim(R/I)$, as desired. $\square$

**Definition 0.3.9.** Let $r \in R$. We say that $r$ is regular on $M$, or $M$-regular, if $rm = 0$ implies $m = 0$ for every $m \in M$.

**Notation 0.3.10.** The set of elements of $R$ which are not $M$-regular is denoted by $Z(M)$.

**Proposition 0.3.11.** $Z(M)$ is the union of the associated primes of $M$.

**Proof.** Let $P \in \text{Ass}(M)$. Let $m \in M$ such that $P = \text{Ann}(m)$. Then $Pm = 0$, and since $m \neq 0$, $P \subset Z(M)$.

Let $r \in Z(M)$ and let $\Sigma = \{\text{Ann}(m) : m \in M \setminus \{0\}\}$. Pick $x \in M \setminus \{0\}$ such that $rx = 0$. Since $R$ is Noetherian, then there exists a maximal ideal in $\Sigma$ containing $\text{Ann}(x)$, say $\text{Ann}(y)$. By proposition 0.2.17, $\text{Ann}(y)$ is a prime ideal, and thus an associated prime, containing $r$. $\square$

**Corollary 0.3.12.** If $I$ is an ideal without $M$-regular elements, then $I$ is contained in some associated prime of $M$.

**Proof.** Just use prime avoidance and the previous proposition. $\square$

**Definition 0.3.13.** A sequence $r_1, \ldots, r_n \in R$ is called a regular sequence on $M$, or an $M$-regular sequence, if the following conditions hold:

- $(r_1, \ldots, r_n)M \neq M$.
- For every $1 \leq i \leq n$, $r_i$ is $M/(r_1, \ldots, r_{i-1})M$-regular.

**Theorem 0.3.14** (Rees). Let $I$ be an ideal such that $IM \neq M$. Then all maximal $M$-regular sequences in $I$ have the same length.

**Proof.** See [1.2]. $\square$

**Definition 0.3.15.** The common length of all maximal $M$-regular sequences in $I$ is called the grade of $I$ on $M$, denoted by $\text{grade}(I, M)$.

**Proposition 0.3.16.** Let $r \in I$ be $M$-regular. Then $\text{grade}(I, M/rM) = \text{grade}(I, M) - 1$. 

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Proof. Let \( r_1, \cdots, r_n \in I \) be a maximal \((M/rM)\)-regular sequence. Since \( r \in I \) is \( M \)-regular, it follows that \( r, r_1, \cdots, r_n \in I \) is a \( M \)-regular sequence, hence \( \text{grade}(I, M) \geq n + 1 = \text{grade}(I, M/rM) + 1 \).

Since \( r \in I \) is \( M \)-regular, by Rees theorem there exists a maximal \( M \)-regular sequence \( r, r_1, \cdots, r_n \in I \), therefore \( r_1, \cdots, r_n \in I \) is a \((M/rM)\)-regular sequence, hence \( \text{grade}(I, M/rM) \geq n = \text{grade}(I, M) - 1 \).

Combining these two inequalities one gets the desired result. \( \square \)

**Proposition 0.3.17.** Let \( P \) be a prime ideal and \( r \in P \) be regular. Then \( \text{ht}(P) \geq \text{ht}(P/(r)) + 1 \).

**Proof.** Let \( P_0 \subset \cdots \subset P_n \) be a maximal ascending chain of prime ideals such that \((r) \subset P_0 \subset \cdots \subset P_n = P\). Suppose \( P_0 \in \text{Min} (R) \). Then \( P_0 \in \text{Ass}(R) \) and so \( r \in \mathcal{Z}(R) \), a contradiction. Thus there exists a prime ideal \( P_{-1} \) strictly contained in \( P_0 \) and so \( P_{-1} \subset P_0 \subset \cdots \subset P_n = P \) is an ascending chain of prime ideals, hence \( \text{ht}(P) \geq n + 1 = \text{ht}(P/(r)) + 1 \). \( \square \)

**Corollary 0.3.18.** If \( r \in R \) is regular, then \( \dim(R) \geq \dim(R/(r)) + 1 \).

**Proof.** Let \( P \) be a prime ideal in \( R \) containing \( r \). Since \( r \in P \) is regular, then \( \dim(R) \geq \text{ht}(P) \geq \text{ht}(P/(r)) + 1 \). Since \( \dim(R) \geq \text{ht}(P/(r)) + 1 \) for any prime ideal \( P \) containing \( r \), it follows that \( \dim(R) \geq \dim(R/(r)) + 1 \). \( \square \)

**Proposition 0.3.19.** For every prime ideal \( P \) one has \( \text{grade}(P, R) \leq \text{ht}(P) \).

**Proof.** Induct on \( n = \text{grade}(P, R) \). The case \( n = 0 \) is trivial.

Suppose \( n > 0 \). Pick \( r \in P \) regular. Then \( P/(r) \) is a prime ideal of \( R/(r) \) such that \( \text{grade}(P/(r), R/(r)) = \text{grade}(P, R/(r)) = n - 1 \), thus \( \text{ht}(P/(r)) \geq n - 1 \) and hence \( \text{ht}(P) \geq n \), as desired. \( \square \)

**Corollary 0.3.20.** For every ideal \( I \) (not necessarily prime) one has \( \text{grade}(I, R) \leq \text{ht}(I) \).

**Proof.** Pick \( P \in \text{Min}(I) \) such that \( \text{ht}(P) = \text{ht}(I) \). Then \( \text{grade}(I, R) \leq \text{grade}(P, R) \leq \text{ht}(P) = \text{ht}(I) \). \( \square \)

**Definition 0.3.21.** If \( I \subset R \) is an ideal generated by a regular sequence, then \( I \) is called a complete intersection.

**Proposition 0.3.22.** If \( I \) is a complete intersection, then \( \text{grade}(I, R) = \text{ht}(I) \).

**Proof.** Let \( r_1, \cdots, r_n \) be the regular sequence that generates \( I \). Then \( \text{grade}(I, R) = n \). Since \( I = (r_1, \cdots, r_n) \), \( \text{ht}(P) \leq n \) for every \( P \in \text{Min}(I) \), thus by definition \( \text{ht}(I) \leq n \), hence \( n = \text{grade}(I, R) \leq \text{ht}(I) \leq n \). \( \square \)

**Proposition 0.3.23.** If \( I \) is an ideal of \( R \) and \( 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \) is a short exact sequence of finitely generated \( R \)-modules, then:

1. \( \text{grade}(I, A) \geq \min\{\text{grade}(I, B), \text{grade}(I, C) + 1\} \);
2. \( \text{grade}(I, B) \geq \min\{\text{grade}(I, A), \text{grade}(I, C)\} \);
3. \( \text{grade}(I, C) \geq \min\{\text{grade}(I, A) - 1, \text{grade}(I, B)\} \).

**Proof.** See [4] 1.2. \( \square \)
0.4 Graded rings and modules

Recall that all rings considered are Noetherian and that all modules considered are finitely generated.

Definition 0.4.1. A graded ring is a ring $R$ together with a decomposition $R = \bigoplus_{i=0}^{\infty} R_i$ as an abelian group such that $R_i R_j \subset R_{i+j}$, $\forall i, j \in \mathbb{N}$.

Example 0.4.2. The polynomial ring $R = K[x_1, \ldots, x_n]$ is a graded ring, where, for each $i \in \mathbb{N}$, $R_i$ is the $K$-vector subspace of homogeneous polynomials of degree $i$.

If $f \in R_i \setminus \{0\}$, we say that $f$ is homogeneous of degree $i$. Any element $f \in R$ can be written uniquely as $f = \sum_{i=0}^{\infty} f_i$, with $f_i \in R_i$ and only finitely many $f_i$ are non-zero. Such $f_i$ are called the homogeneous components of $f$.

Definition 0.4.3. Let $R$ be a graded ring. We say that $I$ is a graded ideal of $R$ if one of the following equivalent conditions hold:

- $I = \bigoplus_{i=0}^{\infty} (R_i \cap I)$.
- $I$ is generated by homogeneous elements of $R$.
- For every $f \in I$, all the homogeneous components of $f$ also belong to $I$.

Proposition 0.4.4. If $I$ is a graded ideal of $R$, then $R/I$ is a graded ring with $(R/I)_i = (R_i + I)/I$ for every $i \in \mathbb{N}$.

Proof. Since $R = \bigoplus_{i=0}^{\infty} R_i$, then $R = \sum_{i=0}^{\infty} (R_i + I)$, hence $R/I = \sum_{i=0}^{\infty} (R_i + I)/I$. Now we show that this sum is in fact a direct sum.

Let $r_i \in R_i$ such that $r_i + I \in \sum_{j \neq i} (R_j + I)$. Then $r_i + s = \sum_{j \neq i} r_j$, with $s \in I$ and $r_j \in R_j$, thus $s = \sum_{j \neq i} r_j - r_i$, and since $s \in I$ and $I$ is homogeneous, $r_i \in I$, therefore $r_i + I = 0$. Hence $R/I = \bigoplus_{i=0}^{\infty} (R_i + I)/I$.

By last, if $r_i \in R_i$ and $r_j \in R_j$, then $r_i r_j \in R_{i+j}$, therefore $(r_i + I)(r_j + I) = r_i r_j + I \in (R_{i+j} + I)/I$. Hence $(R_i + I)/I \cdot (R_j + I)/I \subset (R_{i+j} + I)/I$. \hfill \qed

We know that the polynomial rings $K[x_1, \ldots, x_n]$, where $K$ is a field, are Noetherian graded rings. Since most rings on this dissertation will be quotients of these rings by graded ideals (and hence Noetherian graded rings), from now on all rings are Noetherian graded rings.

Definition 0.4.5. A graded $R$-module is an $R$-module $M$ with a decomposition $M = \bigoplus_{i=0}^{\infty} M_i$ such that $R_i M_j \subset M_{i+j}$ for every $i, j \in \mathbb{N}$.

In particular, if $R$ is a graded $R$-module and $I$ is a graded ideal of $R$, then $R/I$ is also a graded $R$-module. If $f \in M_i \setminus \{0\}$, we say that $f$ is homogeneous of degree $i$. Any element $f \in M$ can be written uniquely as $f = \sum_{i=0}^{\infty} f_i$, with $f_i \in M_i$ and only finitely many $f_i \neq 0$. Such $f_i$ are called the homogeneous components of $f$. 
**Proposition 0.4.6.** A finitely generated graded $R$-module $M$ can be generated by a finite system of homogeneous elements.

**Proof.** Pick any finite set of generators of $M$. Then the homogeneous components of such generators are a finite set of generators of $M$. \hfill \box

**Definition 0.4.7.** Let $M$ be a graded $R$-module. An $R$-submodule $U$ of $M$ is called a graded $R$-submodule of $M$ if $U$ is a graded $R$-module with $U_i = U \cap M_i$ for every $i \in \mathbb{N}$.

**Proposition 0.4.8.** If $U$ is a graded submodule of $M$, then $M/U$ is a graded $R$-module with $(M/U)_i = (M_i + U)/U$ for every $i \in \mathbb{N}$.

**Proof.** Since $M = \bigoplus_{i=0}^\infty M_i$, then $M = \sum_{i=0}^\infty (M_i + U)$, hence $M/U = \sum_{i=0}^\infty (M_i + U)/U$. Now we show that this sum is in fact a direct sum.

Let $m_i \in M_i$ such that $m_i + U \in \sum_{j \neq i} (m_j + U)$. Then $m_i + u = \sum_{j \neq i} m_j$, with $u \in U$ and $m_j \in M_j$, thus $u = \sum_{j \neq i} m_j - m_i$, and since $u \in U$ and $U$ is graded, $m_i \in U$, therefore $m_i + U = 0$. Hence $M/U = \bigoplus_{i=0}^\infty (M_i + U)/U$.

By last, if $r_i \in R_i$ and $m_j \in M_j$, then $r_i m_j \in M_{i+j}$, therefore $r_i (m_j + U) = r_i m_j + U \in (M_{i+j} + U)/U$. Hence $R_i \cdot (M_j + U)/U \subset (M_{i+j} + U)/U$. \hfill \box

**Definition 0.4.9.** Let $M$ and $N$ be graded $R$-modules. An $R$-homomorphism $\varphi : M \to N$ is called homogeneous if $\varphi(M_i) \subset N_i$ for every $i \in \mathbb{N}$.

**Proposition 0.4.10.** The kernel and the image of an homogeneous $R$-homomorphism $\varphi : M \to N$ are graded submodules of $M$ and $N$, respectively.

**Proof.** Let $m = \sum_{i=0}^\infty m_i \in \ker \varphi$. Then $0 = \varphi(m) = \sum_{i=0}^\infty \varphi(m_i)$, and since $\varphi(m_i) \in N_i$ for every $i \in \mathbb{N}$, it follows that $\varphi(m_i) = 0$ for every $i \in \mathbb{N}$, that is, $m_i \in \ker \varphi$ for every $i \in \mathbb{N}$. Hence $\ker \varphi = \bigoplus_{i=0}^\infty (\ker \varphi \cap M_i)$, that is, ker $\varphi$ is graded.

Let $m = \sum_{i=0}^\infty m_i \in M$. Since $M = \bigoplus_{i=0}^\infty M_i$, then $\varphi(M) = \sum_{i=0}^\infty \varphi(M_i)$. Since $N = \sum_{i=0}^\infty N_i$ is a direct sum and $\varphi(M_i) \subset N_i$ holds for every $i \in \mathbb{N}$, it follows that $\varphi(M) = \sum_{i=0}^\infty \varphi(M_i)$ is also a direct sum, that is, $\varphi(M)$ is graded. \hfill \box

**Definition 0.4.11.** Let $K$ be a field. A ring $R$ is called a graded $K$-algebra if $R$ is a graded ring such that $R_0 = K$.

As a consequence of this definition, if $R$ is a $K$-algebra, then each $R_i$ is a $K$-vector space.

**Proposition 0.4.12.** The only maximal graded ideal of a graded $K$-algebra $R$ is $m = \bigoplus_{i=1}^\infty R_i$.

**Proof.** Suppose $I \not\subset m$ is a graded ideal of $R$. Then there exists $f = \sum_{i=0}^\infty f_i \in I$ with $f_0 \neq 0$. Since $I$ is a graded ideal, all homogeneous components of $f$ belong to $I$ and in particular $f_0 \in I$. But $f_0 \in K \setminus \{0\}$, thus $1 = f_0^{-1} f_0 \in I$ and so $I = R$. \hfill \box
Corollary 0.4.13. If $R$ is a graded $K$-algebra, then every proper graded ideal of $R$ is contained in $m$.

Proposition 0.4.14. If $R$ is graded $K$-algebra and $I$ is a proper graded ideal of $R$, then $R/I$ is also a graded $K$-algebra.

Proof. Consider the $K$-linear map $\phi : K \to (R/I)_0$ given by $\phi(k) = k + I$ for every $k \in K$. This map is surjective since $(R/I)_0 = (K + I)/I$. Since any proper graded ideal of $R$ is contained in $m$, then in particular $I \subset m$ and so $K \cap I = \{0\}$ implies that $K \cap I = \{0\}$. Hence $\phi$ is also injective and so $\phi$ is a natural $K$-isomorphism between $K$ and $(R/I)_0$. Hence $R/I$ is a graded $K$-algebra.

We know that the polynomial rings $K[x_1, \cdots, x_n]$, where $K$ is a field, are Noetherian graded $K$-algebras. Since most rings on this dissertation will be quotients of these rings by graded ideals (and hence Noetherian graded $K$-algebras), for the rest of the dissertation all rings are Noetherian graded $K$-algebras.

Let $M$ be a graded $R$-module. In general, the multiplication $\varphi : M \to M$ by an homogeneous element $r \in R$ is not an homogeneous $R$-homomorphism, for $\varphi(M_i) \subset M_{i+\deg r}$ for every $i \in \mathbb{N}$. This problem is solved by shifting the homogeneous components of $M$.

Definition 0.4.15. Given a graded $R$-module $M$ and $j \in \mathbb{N}$, then $M(-j)$ is defined to be the graded $R$-module whose graded components are $M(-j)_i = M_{i-j}$ if $i \geq j$ and $M(-j)_i = 0$ otherwise.

Now the multiplication $\varphi : M(-\deg r) \to M$ by an homogeneous element $r \in R$ is a graded $R$-homomorphism.

Lemma 0.4.16 (Graded Nakayama). Let $M$ be a graded $R$-module such that $M = mM$. Then $M = 0$.

Proof. Suppose $M \neq 0$. Let $\alpha$ be the smallest integer such that $M_\alpha \neq 0$. Then $M_\alpha \subset M = mM \subset \bigoplus_{i=\alpha+1}^{\infty} M_i$. Since $M_\alpha \subset \bigoplus_{i=\alpha+1}^{\infty} M_i$, it follows that $M_\alpha = 0$, a contradiction. □

Corollary 0.4.17. If $U$ is a graded $R$-submodule of $M$ such that $M = U + mM$, then $M = U$.

Proof. From $M = U + mM$ it follows that $M/U = m(M/U)$, hence $M/U = 0$, as desired. □

Corollary 0.4.18. If $m_1, \cdots, m_r$ are homogeneous elements whose residue classes modulo $mM$ form a $K$-basis for $M/mM$, then $m_1, \cdots, m_r$ generate $M$.

Proof. Let $U$ be the $R$-submodule of $M$ generated by $m_1, \cdots, m_r$. Then $U$ is a graded submodule such that $M = U + mM$, hence $M = U$. □

Corollary 0.4.19. All homogeneous minimal systems of generators of $M$ have the same cardinality, namely $\dim_K(M/mM)$. 

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0.5 Cohen-Macaulay graded rings

In this section, $R$ will denote a Noetherian graded algebra over a field $K$ with maximal homogeneous ideal $m$. Moreover, all modules considered are finitely generated graded $R$-modules.

**Definition 0.5.1.** Let $M$ be a finitely generated graded $R$-module. The depth of $M$ is $\text{depth}(M) = \text{grade}(m, M)$.

**Proposition 0.5.2.** For every graded ideal $I$, there exists an $M$-regular sequence of length $\text{grade}(I, M)$ consisting of homogeneous elements of $I$.

**Proof.** See [4, 1.5].

Hence, to determine $\text{depth}(M)$, it is enough to consider regular sequences of homogeneous elements of $M$.

**Proposition 0.5.3.** If $r \in m$ is $M$-regular, then $\text{depth}(M/rM) = \text{depth}(M) - 1$.

**Proof.** This is a direct corollary of proposition 0.3.16.

**Proposition 0.5.4.** Let $I$ be a graded ideal of $R$. Then the depth of $R/I$ is the same either as a graded ring or as a graded $R$-module.

**Proof.** Just notice that $r_1, \ldots, r_n$ is $R/I$-regular if and only if $r_1 + I, \ldots, r_n + I$ is $R/I$-regular.

**Proposition 0.5.5.** Let $M$ be a graded $R$-module. Then $\text{depth}(M) \leq \text{dim}(R/P)$ for every $P \in \text{Ass}(M)$.

**Proof.** Induct on $n = \text{depth}(M)$. The case $n = 0$ is trivial.

Suppose $n > 0$. Let $P \in \text{Ass}(M)$. Let $r \in m$ be a $M$-regular homogeneous element. By proposition 0.3.16, one has $\text{depth}(M/rM) = n - 1$. Let $\Sigma$ be the set of cyclic $R$-submodules of $M$ annihilated by $P$. Since $P \in \text{Ass}(M)$, then $\Sigma$ contains a non-zero cyclic $R$-submodule of $M$, and since $R$ is Noetherian, then $\Sigma$ has a maximal element $(y)$, with $y \in M \setminus \{0\}$. Suppose $y \in rM$. Then $y = rm$ for some $m \in M$, and since $r \in m$ and $M$ is a graded $R$-module, then $(y) \subseteq (m)$. On the other hand, $r(Pm) = 0$, and since $r$ is $M$-regular, $Pm = 0$, hence $(m) \in \Sigma$, contradicting the maximality of $(y)$. Hence $y \notin rM$, and since $Py = 0$, it follows that $P \subseteq Z(M/rM)$, therefore $P$ is contained in some $Q \in \text{Ass}(M/rM)$. By induction, $\text{dim}(R/Q) \geq \text{depth}(M/rM) = n - 1$. Since $r(M/rM) = 0$ and $r \notin P$, then $P \notin \text{Ass}(M/rM)$, thus $P \subseteq Q$, therefore $\text{dim}(R/P) > \text{dim}(R/Q) \geq n - 1$, hence $\text{dim}(R/P) \geq n$, as desired.

**Corollary 0.5.6.** Let $R$ be a graded $K$-algebra. Then $\text{depth}(R) \leq \text{dim}(R)$.

**Definition 0.5.7.** Let $K$ be a field and let $R$ be a graded $K$-algebra. Then $R$ is a Cohen-Macaulay graded ring if $\text{depth}(R) = \text{dim}(R)$.

In an abuse of notation, one says that a graded ideal $I$ is a Cohen-Macaulay ideal if $R/I$ is a Cohen-Macaulay ring.
Proposition 0.5.8. If $I$ is a Cohen-Macaulay ideal, then $\text{depth}(R/I) = \dim(R/I) = \dim(R/P)$ for every $P \in \text{Ass}(R/I)$.

Proof. If $P \in \text{Ass}(R/I)$, by proposition 0.5.5 one has $\text{depth}(R/I) \leq \dim(R/P) \leq \dim(R/I)$, and since $\text{depth}(R/I) = \dim(R/I)$, the assertion follows. □

In particular, if $I$ is a Cohen-Macaulay ideal, then $\text{Min}(I) = \text{Ass}(R/I)$.

Proposition 0.5.9. If $R$ is a Cohen-Macaulay ring and $r \in R$ is regular, then $R/(r)$ is a Cohen-Macaulay ring as well.

Proof. Since $R$ is a Cohen-Macaulay ring, then $\text{depth}(R) = \dim(R)$. On the other hand, since $r$ is regular, by proposition 0.3.16 one has $\text{depth}(R/(r)) = \dim(R) - 1$ and by corollary 0.3.18 one has $\dim(R) \geq \dim(R/(r)) + 1$. Hence $\dim(R/(r)) \leq \dim(R) - 1 = \text{depth}(R) - 1 = \text{depth}(R/(r))$, as desired. □

Corollary 0.5.10. Any complete intersection in a Cohen-Macaulay ring is a Cohen-Macaulay ideal.

Proposition 0.5.11. The polynomial ring $R = K[x_1, \cdots, x_n]$ satisfies $\dim(R) = n$.


Since $x_1, \cdots, x_n \in m$ is a regular sequence of length $n$, it follows that $K[x_1, \cdots, x_n]$ is a Cohen-Macaulay ring.

Proposition 0.5.12. If $I$ is an ideal of $R = K[x_1, \cdots, x_n]$, then $\text{ht}(I) + \dim(R/I) = n$.


Corollary 0.5.13. If $I \subset K[x_1, \cdots, x_n]$ is a Cohen-Macaulay ideal, then $I$ is unmixed.

Proof. Let $P \in \text{Ass}(R/I)$. Since $I$ is a Cohen-Macaulay ideal, by proposition 2.2.34 one has $\dim(R/I) = \dim(R/P)$, and from the equalities $\text{ht}(I) + \dim(R/I) = n$ and $\text{ht}(P) + \dim(R/P) = n$ it follows that $\text{ht}(I) = \text{ht}(P)$. □

Notation 0.5.14. Let $K[x]$ and $K[y]$ be two polynomial rings over a field $K$. If $I_1$ and $I_2$ are graded ideals of $K[x]$ and $K[y]$, respectively, then $I_1 + I_2$ denotes the graded ideal of $K[x,y]$ generated by $I_1$ and $I_2$.

Proposition 0.5.15. If $I_1$ and $I_2$ are graded ideals of $K[x]$ and $K[y]$, respectively, then $\text{depth}(K[x]/I_1) + \text{depth}(K[y]/I_2) = \text{depth}(K[x,y]/(I_1 + I_2))$ and $\dim(K[x]/I_1) + \dim(K[y]/I_2) = \dim(K[x,y]/(I_1 + I_2))$.


Corollary 0.5.16. The ring $K[x,y]/(I_1 + I_2)$ is Cohen-Macaulay if and only if the rings $K[x]/I_1$ and $K[y]/I_2$ are Cohen-Macaulay.

Corollary 0.5.17. If $I_1$ and $I_2$ are graded ideals of $K[x_1, \cdots, x_n]$ and $K[y_1, \cdots, y_m]$, respectively, then $\text{ht}(I_1 + I_2) = \text{ht}(I_1) + \text{ht}(I_2)$. 

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Proof. By proposition 0.5.12 the following equalities hold:
\[
\text{ht}(I_1) + \dim(K[x_1, \cdots, x_n]/I_1) = n, \quad \text{ht}(I_2) + \dim(K[y_1, \cdots, y_m]/I_2) = m,
\]
\[
\text{ht}(I_1 + I_2) + \dim(K[x_1, \cdots, x_n, y_1, \cdots, y_m]/(I_1 + I_2)) = n + m.
\]

By proposition 0.5.15 the following equality holds:
\[
\dim(K[x_1, \cdots, x_n, y_1, \cdots, y_m]/(I_1 + I_2)) = \dim(K[x_1, \cdots, x_n]/I_1) + \dim(K[y_1, \cdots, y_m]/I_2).
\]

Combining these four equalities yields \(\text{ht}(I_1 + I_2) = \text{ht}(I_1) + \text{ht}(I_2)\), as desired. \(\square\)

### 0.6 Free resolutions and Castelnuovo-Mumford regularity

Recall that \(R\) is a Noetherian graded algebra over a field \(K\).

Let \(M\) be a finitely generated graded \(R\)-module with homogeneous generators \(m_1, \cdots, m_r \in M\) and \(\deg m_i = a_i\) for \(i = 1, \cdots, r\). Let \(F_0\) be the free \(R\)-module with generators \(e_1, \cdots, e_r\). Then there exists an \(R\)-epimorphism \(\varepsilon : F_0 = \bigoplus_{i=1}^r R e_i \to M\) with \(e_i \mapsto m_i\). Assigning to \(e_i\) the degree \(a_i\) for \(i = 1, \cdots, r\), the map \(F_0 \to M\) becomes graded and \(F_0\) becomes isomorphic to \(\bigoplus_{i=1}^r R(-a_i)\). Let \(U = \ker \varepsilon\). Since \(U\) is a submodule of \(F_0\) and \(R\) is a Noetherian ring, then \(U\) is a finitely generated graded \(R\)-module. Now we find again a graded \(R\)-epimorphism \(F_1 \to U\). Composing it with the inclusion map \(U \to F_0\) we obtain the exact sequence \(F_1 \to F_0 \to M \to 0\) of graded \(R\)-modules. Proceeding in this way we obtain a long exact sequence
\[
F : \cdots \to F_2 \to F_1 \to F_0 \to M \to 0
\]

of graded \(R\)-modules and graded \(R\)-homomorphisms. Such an exact sequence is called a graded free resolution of \(M\).

**Notation 0.6.1.** The \(R\)-homomorphism from \(F_i\) to \(F_{i-1}\) will be called \(\varphi_i\).

**Lemma 0.6.2.** Let \(m_1, \cdots, m_r\) be a homogeneous set of generators of \(M\). Let \(F_0 = \bigoplus_{i=1}^r R e_i \to M\) be the epimorphism with \(e_i \mapsto m_i\) for \(i = 1, \cdots, r\). Then \(m_1, \cdots, m_r\) is a minimal system of generators of \(M\) if and only if \(\ker \varepsilon \subset mF_0\).

**Proof.** See [5] A.2. \(\square\)

**Definition 0.6.3.** A graded free resolution \(F\) is called minimal if the image of each \(F_i \to F_{i-1}\) is contained in \(mF_{i-1}\).

The previous lemma implies that each finitely generated \(R\)-module admits a graded minimal free resolution.
Proposition 0.6.4. Let \( M \) be a finitely generated graded \( R \)-module and let \( \mathbb{F} \) and \( \mathbb{G} \) be two minimal graded free resolutions of \( M \). Then \( \mathbb{F} \) and \( \mathbb{G} \) are isomorphic, that is, there exist graded \( R \)-isomorphisms \( \alpha_i : F_i \to G_i \) such that the diagram

\[
\begin{array}{cccccc}
\cdots & F_1 & \xrightarrow{\alpha_1} & F_0 & \xrightarrow{\alpha_0} & M & \to 0 \\
\downarrow & \downarrow & & \downarrow & \downarrow & \\
\cdots & G_1 & \xrightarrow{\beta_1} & G_0 & \xrightarrow{\beta_0} & M & \to 0
\end{array}
\]

is commutative.

Proof. See [5, A.2]. \( \square \)

Let \( \mathbb{F} \) be a graded minimal free resolution of \( M \) with \( F_i = \bigoplus_j R(-j)^{\beta_{ij}(M)} \). Proposition 0.6.4 tells us that the numbers \( \beta_{ij}(M) \) are uniquely determined by \( M \). They are called the graded Betti numbers of \( M \).

Definition 0.6.5. The projective dimension of a graded module \( M \) is

\[
\text{projdim}(M) = \sup \{ i : \exists j \geq 0 : \beta_{ij}(M) \neq 0 \}.
\]

For the remaining of this section we will only consider finitely generated graded modules over the polynomial ring \( S = K[x_1, \cdots, x_n] \).

Theorem 0.6.6 (Auslander-Buchsbaum). For any \( S \)-module \( M \) one has \( \text{projdim}(M) + \text{depth}(M) = n \).

Proof. See [5, A.4]. \( \square \)

Corollary 0.6.7. For any \( S \)-module \( M \) one has \( \text{projdim}(M) \leq n \) (and in particular the projective dimension is always finite).

Definition 0.6.8. The Castelnuovo-Mumford regularity of \( M \) is \( \text{reg}(M) = \max \{ j : \exists i \geq 0 : \beta_{i,i+j}(M) \neq 0 \} \).

The corollary of Auslander-Buchsbaum theorem tells us that \( \text{reg}(M) < \infty \).

Proposition 0.6.9. If \( I \) is a graded ideal of \( S \), then \( \text{reg}(I) = \text{reg}(S/I) + 1 \).

Proof. If

\[
\cdots \to F_2 \to F_1 \to S \to S/I \to 0
\]

is a minimal graded free resolution for \( S/I \), then

\[
\cdots \to F_2 \to F_1 \to I \to 0
\]

is a minimal graded free resolution for \( I \). Hence \( \beta_{i,j}(I) = \beta_{i+1,j}(S/I) \), and since \( \beta_{0,j}(S/I) = 0 \) for \( j > 0 \), it follows that \( \text{reg}(I) = \max \{ j : \exists i \geq 0 : \beta_{i,i+j}(I) \neq 0 \} = \max \{ j : \exists i \geq 1 : \beta_{i,i+(j-1)}(S/I) \neq 0 \} = \max \{ j : \exists i \geq 0 : \beta_{i,i+(j-1)}(S/I) \neq 0 \} = \max \{ j-1 : \exists i \geq 0 : \beta_{i,i+(j-1)}(S/I) \neq 0 \} + 1 = \text{reg}(S/I) + 1. \) \( \square \)

Proposition 0.6.10. For every \( S \)-module \( M \) and \( k \in \mathbb{N} \) one has \( \text{reg}(M(-k)) = \text{reg}(M) + k \).
Thus the equality follows.

Proof. If
\[ \cdots \rightarrow F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0 \]
is a minimal graded free resolution for \( M \), then
\[ \cdots \rightarrow F_2(-k) \rightarrow F_1(-k) \rightarrow F_0(-k) \rightarrow M(-k) \rightarrow 0 \]
is a minimal graded free resolution for \( M(-k) \). Hence \( \beta_{i,j}(M(-k)) = \beta_{i,j-k}(M) \), therefore \( \text{reg}(M(-k)) = \max\{j : \exists i \geq 0 : \beta_{i,j-k}(M) \neq 0\} = \max\{j \geq 0 : \beta_{i,j-k}(M) \neq 0\} = k = \text{reg}(M) + k \).

\( \square \)

Proposition 0.6.11. If \( 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \) is a short exact sequence of graded \( S \)-modules, then:

1. \( \text{reg}(A) \leq \max\{\text{reg}(B), \text{reg}(C) + 1\} \);
2. \( \text{reg}(B) \leq \max\{\text{reg}(A), \text{reg}(C)\} \);
3. \( \text{reg}(C) \leq \max\{\text{reg}(A) - 1, \text{reg}(B)\} \).


\( \square \)

Corollary 0.6.12. If \( A \) and \( B \) are graded \( S \)-modules, then \( \text{reg}(A \oplus B) \leq \max\{\text{reg}(A), \text{reg}(B)\} \).

Proof. Just consider the short exact sequence \( 0 \rightarrow A \rightarrow A \oplus B \rightarrow B \rightarrow 0 \) and apply inequality (2).

\( \square \)

Proposition 0.6.13. If \( f \in S \) is an \( M \)-regular homogeneous element of degree \( k > 0 \), then \( \text{reg}(M/fM) = \text{reg}(M) + k - 1 \).

Proof. Consider the graded exact sequence \( 0 \rightarrow M(-k) \rightarrow M \rightarrow M/fM \rightarrow 0 \), where the \( S \)-homomorphism \( M(-k) \rightarrow M \) is the multiplication by \( f \). By inequality (1) of proposition 0.6.11,
\[ \text{reg}(M(-k)) \leq \max\{\text{reg}(M), \text{reg}(M/fM) + 1\}, \]
that is, \( \text{reg}(M) + k \leq \max\{\text{reg}(M), \text{reg}(M/fM) + 1\} \), hence \( \text{reg}(M) + k \leq \text{reg}(M/fM) + 1 \), that is, \( \text{reg}(M/fM) \geq \text{reg}(M) + k - 1 \). On the other hand, by inequality (3),
\[ \text{reg}(M/fM) \leq \max\{\text{reg}(M(-k)) - 1, \text{reg}(M)\} = \max\{\text{reg}(M) + k - 1, \text{reg}(M)\} = \text{reg}(M) + k - 1, \]
thus the equality follows.

\( \square \)

Corollary 0.6.14. Let \( f_1, \cdots, f_r \in S \) be an \( M \)-regular sequence of homogeneous elements of degrees \( k_1, \cdots, k_r \), respectively. Then \( \text{reg}(M/(f_1, \cdots, f_r)M) = \text{reg}(M) + k_1 + \cdots + k_r - r \).

Proof. Use induction on \( r \) and the previous proposition.

\( \square \)

Lemma 0.6.15. Let \( M \) and \( N \) be graded modules over \( K[x] \) and \( K[y] \), respectively. Then \( M \otimes_K N \) is a graded module over \( K[x,y] \) such that \( \text{depth}(M \otimes_K N) = \text{depth}(M) + \text{depth}(N) \) and \( \text{reg}(M \otimes_K N) = \text{reg}(M) + \text{reg}(N) \).
0.7 Graph theory

Definition 0.7.1. A graph $G$ is an ordered pair of disjoint finite sets $(V, E)$ such that $E$ is a subset of the set of unordered pairs of $V$.

The set $V$ is the set of vertices and the set $E$ is called the set of edges. An edge $z = \{u, v\}$, with $u, v \in V$, is said to join the vertices $u$ and $v$. We also say that the edge $z$ is incident with $u$ and $v$ or that vertices $u$ and $v$ are adjacent or neighbouring vertices of $G$.

Definition 0.7.2. The order of a graph is its number of vertices.

Notation 0.7.3. The vertex set and the edge set of $G$ are often denoted by $V(G)$ and $E(G)$, respectively.

Definition 0.7.4. The degree $\deg(v)$ of a vertex $v$ is the number of edges incident with $v$.

Definition 0.7.5. A vertex with degree zero is called an isolated vertex.

Example 0.7.6. The vertex $w$ in the graph below is isolated.

Definition 0.7.7. Two graphs $G$ and $H$ are isomorphic if there exists a bijective map $\phi$ from $V(G)$ to $V(H)$ such that $\{u, v\} \in E(G)$ if and only if $\{\phi(u), \phi(v)\} \in E(H)$.

Definition 0.7.8. Let $G$ and $H$ be two graphs. Then $H$ is called a subgraph of $G$ if $V(H) \subset V(G)$ and $E(H) \subset E(G)$.

Definition 0.7.9. A subgraph $H$ is called an induced subgraph if $H$ contains all the edges $\{u, v\} \in E(G)$ with $u, v \in V(H)$.

In this case $H$ is said to be the subgraph induced by $V(H)$.

Notation 0.7.10. An induced subgraph is denoted by $H = G_{V(H)}$.

Example 0.7.11. Let $G$ be the triangle with $V(G) = \{1, 2, 3\}$ and $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$. Then the subgraph $H_1$ with $V(H_1) = \{1, 2\}$ and $E(H_1) = \{\{1, 2\}\}$ is an induced subgraph of $G$. On the other hand, the subgraph $H_2$ with $V(H_2) = \{1, 2, 3\}$ and $E(H_2) = \{\{1, 2\}, \{2, 3\}\}$ is not an induced subgraph of $H_2$ since $1, 3 \in V(H_2)$ and $\{1, 3\} \in E(G)$ but $\{1, 3\} \not\in E(H_2)$. 

Proof. See [9, 2] (the proof of this lemma uses local cohomology, a tool which is beyond this dissertation). 

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Definition 0.7.12. Given two graphs \( G \) and \( H \) such that \( V(G) \cup V(H) \subseteq [n] \), their intersection is the graph \( G \cap H \) such that \( V(G \cap H) = V(G) \cap V(H) \) and \( E(G \cap H) = E(G) \cap E(H) \).

Definition 0.7.13. Given two graphs \( G \) and \( H \) such that \( V(G) \cup V(H) \subseteq [n] \), their union is the graph \( G \cup H \) such that \( V(G \cup H) = V(G) \cup V(H) \) and \( E(G \cup H) = E(G) \cup E(H) \).

Definition 0.7.14. A walk of length \( n \) in \( G \) is a sequence of vertices \( v_0, \ldots, v_n \) such that, for each \( 1 \leq i \leq n \), \( \{v_{i-1}, v_i\} \in E(G) \).

Definition 0.7.15. A path of length \( n \) is a walk \( v_0, \ldots, v_n \) whose vertices are all distinct.

We say that a graph \( G \) with \( V(G) = \{v_0, \ldots, v_n\} \) and \( E(G) = \{\{v_{i-1}, v_i\} : 1 \leq i \leq n\} \) is a path of length \( n \), denoted by \( P_n \).

Definition 0.7.16. Given two vertices \( u \) and \( v \), the distance between \( u \) and \( v \), denoted by \( d(u, v) \) is the minimum of the lengths of the paths from \( u \) to \( v \). If there is no path from \( u \) to \( v \) we say that \( d(u, v) = \infty \).

Definition 0.7.17. We say that \( G \) is connected if for every pair of vertices \( u \) and \( v \) there is a path in \( G \) from \( u \) to \( v \).

Proposition 0.7.18. Every graph \( G \) can be decomposed as \( G = \bigcup_{i=1}^{r} G_i \), where \( G_1, \ldots, G_r \) are the maximal connected subgraphs of \( G \), also called the connected components of \( G \).

Definition 0.7.19. If all the vertices of \( G \) are isolated, then \( G \) is called a discrete graph.

Example 0.7.20. The graph below is a discrete graph with four vertices.

![Discrete Graph](image)

Definition 0.7.21. A cycle of length \( n \) is a walk \( v_0, \ldots, v_n = v_0 \) in which \( n \geq 3 \) such that the vertices \( v_0, \ldots, v_{n-1} \) are distinct.
We say that a graph $G$ with $V(G) = \{v_0, \cdots, v_{n-1}\}$ and $E(G) = \{\{v_{i-1}, v_i\} : 1 \leq i < n\}$ (with $v_n = v_0$) is a cycle of length $n$, denoted by $C_n$.

**Example 0.7.22.** The graph below is a cycle of length 5.

![Cycle of length 5](image)

**Definition 0.7.23.** A cycle is even (respectively, odd) if its length is even (respectively, odd), that is, if it has an even (respectively, odd) number of vertices.

**Definition 0.7.24.** A chord of a cycle $C$ in the graph $G$ is an edge of $G$ joining two non-adjacent vertices of $C$.

**Example 0.7.25.** In the graph below, the cycle of length 4 has a chord.

![Cycle with chord](image)

**Definition 0.7.26.** A graph $G$ is called chordal if every cycle of $G$ of length greater than 3 has a chord in $G$.

**Proposition 0.7.27.** Any induced subgraph of a chordal graph is chordal as well.

Suppose $G$ is a chordal graph and consider any cycle $C$ in $G$. Seeing $C$ as a convex polygon, $G$ being chordal implies that we can divide $C$ in triangles in a way such that each triangle is a cycle in $G$. In other words, given an edge $\{u, v\} \in E(C)$, we get that there exists $w \in V(C)$ such that $\{u, w\}, \{v, w\} \in E(G)$.

**Definition 0.7.28.** The complete graph $K_n$ is the graph such that every pair of its $n$ vertices is adjacent.

**Example 0.7.29.** The graph below is the complete graph $K_5$.

![Complete graph](image)
Definition 0.7.30. A clique of a graph $G$ is a set of vertices that induces a complete subgraph.

We will also call a complete subgraph of $G$ a clique.

Definition 0.7.31. A graph consisting of three different edges that share a common vertex is called a claw.

Example 0.7.32. The graph below is a claw.

Definition 0.7.33. A graph $G$ is bipartite if its vertex set $V(G)$ can be bipartitioned into two disjoint subsets $A$ and $B$ such that every edge of $G$ has one vertex in $A$ and one vertex in $B$.

The pair $(A, B)$ is called a bipartition of $G$.

Example 0.7.34. The graph below is a bipartite graph.

Proposition 0.7.35. If $G$ is connected, such a bipartition is uniquely determined.

Proposition 0.7.36. A graph $G$ is bipartite if and only if all the cycles of $G$ are even.

Proof. See [5, 9.1].

Definition 0.7.37. A forest is a graph without cycles.

Corollary 0.7.38. Every forest is a bipartite graph.

Definition 0.7.39. A tree is a connected forest.
Proposition 0.7.40. A chordal graph has a cycle of length 3 unless it is a forest.

Proof. Suppose $G$ is a chordal graph which is not a forest and let $C$ be a cycle in $G$ of smallest length. If $C$ is a cycle $v_0, \cdots, v_k = v_0$ with $k > 3$, then it has a chord, say $\{v_i, v_j\}$ with $j > i + 1$. But then $v_i, v_{i+1}, \cdots, v_j, v_i$ is a cycle of length $j - i + 1 < k$, a contradiction. \qed

Definition 0.7.41. Let $A$ be a set of vertices of a graph $G$. The neighbour set of $A$, denoted by $N_G(A)$ or simply by $N(A)$ if $G$ is understood, is the set of vertices of $G$ that are adjacent with at least one vertex of $A$.

One may wonder if any of the inequalities $A \subset N(A)$ and $N(A) \subset A$ holds. The first inequality holds if and only if the induced subgraph $G_A$ is connected and the second inequality holds if and only if $G_A$ is the union of non-singular connected components of $G$. In particular, $N(A) = A$ holds if and only if $G_A$ is a non-singular connected component of $G$.

Proposition 0.7.42. Let $A$ and $B$ be subsets of $V(G)$. Then $N(A \cup B) = N(A) \cup N(B)$.

Definition 0.7.43. A subset $C \subset V(G)$ is a vertex cover of $G$ if every edge of $G$ is incident with at least one vertex in $C$.

Example 0.7.44. If $G$ is a bipartite graph with bipartition $(V, V')$, then both $V$ and $V'$ are vertex covers of $G$.

Definition 0.7.45. A graph $G$ is called unmixed or well-covered if any two minimal vertex covers of $G$ have the same cardinality.

Proposition 0.7.46. Let $G$ be an unmixed bipartite graph without isolated vertices and with bipartition $(V, V')$. Let $U$ be a subset of $V$. Then $(V \setminus U) \cup N(U)$ is a vertex cover of $G$.

Proof. Let $v$ be an arbitrary vertex on $G$.

If $v \in V'$, then $v \in N(V) = N(U) \cup N(V \setminus U)$ since $G$ has no isolated vertices.

If $v \in V$, then $v \in (V \setminus U) \cup U \subset (V \setminus U) \cup N(N(U))$ since $G$ has no isolated vertices. \qed

It is convenient to regard the empty set as the only minimal vertex cover of a discrete graph.

Definition 0.7.47. A subset of vertices of a graph is called independent or stable if no two of them are adjacent.

Example 0.7.48. If $G$ is a bipartite graph with bipartition $(V, V')$, then both $V$ and $V'$ are independent subsets of vertices of $G$.

Definition 0.7.49. The vertex covering number of $G$, denoted by $\alpha_0(G)$, is the number of vertices in any vertex cover of $G$ of smallest size.

Definition 0.7.50. The vertex independence number of $G$, denoted by $\beta_0(G)$, is the number of vertices in any independent set of vertices of $G$ of largest size.
Definition 0.7.51. Given a graph $G$ on $[n]$, its complement $\overline{G}$ is the graph on $[n]$ such that $\{i, j\} \in E(\overline{G})$ if and only if $\{i, j\} \notin E(G)$.

Example 0.7.52. The two graphs below are complements of each other.

![Example Graphs](image)

Proposition 0.7.53. A subset of vertices of a graph is independent if and only if its complement is a vertex cover.

Proof. Let $C \subset [n]$ be a subset of vertices of $G$. Then $C$ is independent if and only if $\{i, j\} \notin E(G)$ for every $i, j \in C$. In other words, $C$ is independent if and only if each edge in $G$ contains a vertex in $[n] \setminus C$, that is, if and only if $[n] \setminus C$ is a vertex cover of $G$. \qed

Corollary 0.7.54. For any graph $G$ on $[n]$, one has $\alpha_0(G) + \beta_0(G) = n$.

Proof. Since a subset of $[n]$ is independent if and only if its complement is a vertex cover, it follows that a subset of $[n]$ is a maximal independent set if and only if its complement is a maximal vertex cover. Hence the result follows. \qed

Definition 0.7.55. A set of edges in a graph $G$ is called independent if no two of them have a vertex in common.

Example 0.7.56. In the graph below, the three vertical edges form an independent set of edges.

![Example Graph with Independent Edges](image)

Definition 0.7.57. A set of independent edges that covers all vertices of a graph $G$ is called a perfect matching.

Thus $G$ has a perfect matching if and only if $G$ has an even number of vertices and there is an independent set of edges containing all the vertices.

Example 0.7.58. The independent set of edges of the graph above is a perfect matching.
**Theorem 0.7.59** (Marriage theorem). Let $G$ be a bipartite graph on the vertex set $V \cup V'$ with $|V| = |V'|$. Suppose that $|N(U)| \geq |U|$ for every $U \subset V$. Then there exist labellings $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ such that $\{x_i, y_i\} \in E(G)$ for every $i \in [n]$.

**Proof.** See [5, 9.1].

**Definition 0.7.60.** An induced matching of a graph is an induced subgraph consisting of pairwise disjoint edges.

**Example 0.7.61.** Any pair of opposite edges of the hexagon below is an induced matching.

![Image of a hexagon with opposite edges highlighted]

**Definition 0.7.62.** Given a graph $G$, its induced matching number, denoted by $\text{indmatch}(G)$, is the number of edges in an induced matching of $G$ of largest size.
Chapter 1

Monomial ideals and Gröbner bases

Monomial ideals, simplicial complexes and Stanley-Reisner ideals are fundamental pre-requisites in order to study combinatorical Commutative Algebra and so the inclusion of this chapter, instead of just stating the most important results and giving references for their proofs (as we did in chapter 0), is very helpful for a first reader.

In section 1.1, monomial ideals are introduced and their properties are studied in detail.

In section 1.2, simplicial complexes are defined and, to each simplicial complex, we associate its Stanley-Reisner ideal. In fact, simplicial complexes are uniquely determined by their Stanley-Reisner ideals.

In section 1.3, it is shown that a Gröbner basis of an ideal $I$ is also a set of generators of $I$ and the Buchberger’s criterion gives necessary and sufficient conditions for a set of generators of an ideal to be a Gröbner basis. We end by stating some relations between the algebraic properties of an arbitrary ideal of polynomials (such as its depth, dimension and regularity) and the same properties of its initial ideal. Such relations will be used in chapters 2 and 3 when studying of binomial edge ideals of closed graphs.

1.1 Monomial ideals

Monomials form a natural $K$-basis for the polynomial ring $S = K[x_1, \cdots, x_n]$. A monomial ideal $I$ also has a $K$-basis of monomials. As a consequence, a polynomial $f \in S$ belongs to $I$ if and only if all monomials in $f$ with a non-zero coefficient also belong to $I$. This is one of the reasons why algebraic operations with monomial ideals are easy to perform and one may take advantage of this fact when studying general ideals in $S$ by considering its initial ideal with respect to some monomial order.

**Definition 1.1.1.** A monomial in $S$ is a product $x^a = x_1^{a_1} \cdots x_n^{a_n}$, with $a = (a_1, \cdots, a_n) \in \mathbb{N}^n$.

**Notation 1.1.2.** If $I$ is an ideal of $S$, then the set of monomials of $I$ is $\text{Mon}(I)$.
Definition 1.1.3. An ideal $I \subset S$ is called a monomial ideal if it is generated by monomials.

It is well known that $S$ is a Noetherian ring, that is, every ideal of $S$ is finitely generated. As it would be expected, every monomial ideal of $S$ has a finite system of monomial generators.

Proposition 1.1.4. Any monomial ideal has a finite system of monomial generators.

Proof. Given a non-zero monomial ideal $I$, let $\Sigma$ be the set of ideals with a finite system of monomial generators in $I$. Since $S$ is Noetherian and $\Sigma \neq \emptyset$, then $\Sigma$ has a maximal ideal $J$. Suppose $J \neq I$. Since $I$ is a monomial ideal, then there exists $u \in \operatorname{Mon}(I) \setminus J$ and thus $J + (u)$ is an ideal with a finite system of monomial generators in $I$ which strictly contains $J$, a contradiction. Hence $I = J$ has a finite system of monomial generators.

The set $\operatorname{Mon}(S)$ is a $K$-basis of $S$, that is, every polynomial $f \in S$ is a unique finite $K$-linear combination of monomials

$$f = \sum_{u \in \operatorname{Mon}(S)} a_u u, \text{ with } a_u \in K. \quad (1.1)$$

Definition 1.1.5. The support of $f$ is $\operatorname{supp}(f) = \{u \in \operatorname{Mon}(S) : a_u \neq 0\}$.

Theorem 1.1.6. If $I$ is a monomial ideal, then the set $\mathcal{N}$ of monomials belonging to $I$ is a $K$-basis of $I$.

Proof. It is clear that the elements of $\mathcal{N}$ are linearly independent, as $\mathcal{N} \subset \operatorname{Mon}(S)$.

To show that $\mathcal{N}$ generates the $K$-vector space $I$, it is enough to show that $\operatorname{supp}(f) \subset \mathcal{N}$ for any $f \in I$. In fact, if $f \in I$, then there exist monomials $u_1, \ldots, u_m \in I$ and polynomials $f_1, \ldots, f_m \in S$ such that $f = \sum_{i=1}^m f_i u_i$, hence $\operatorname{supp}(f) \subset \bigcup_{i=1}^m \operatorname{supp}(f_i u_i)$. Since each monomial in $\operatorname{supp}(f_i u_i)$ is of the form $w u_i$ with $w \in \operatorname{supp}(f_i)$, it follows that $\operatorname{supp}(f_i u_i) \subset \mathcal{N}$ for every $1 \leq i \leq m$, hence $\operatorname{supp}(f) \subset \mathcal{N}$, as desired.

Recall definition [0.4.3] Monomial ideals can be characterized similarly.

Corollary 1.1.7. The ideal $I \subset S$ is a monomial ideal if and only if $\operatorname{supp}(f) \subset I$ for every $f \in I$.

Proof. Suppose $I$ is a monomial ideal and let $f \in I$. According to theorem [1.1.6] there exist monomials $u_1, \ldots, u_m \in I$ and constants $a_1, \ldots, a_m \in K$ such that $f = a_1 u_1 + \cdots + a_m u_m$, hence $\operatorname{supp}(f) \subset \{u_1, \ldots, u_m\} \subset I$.

Suppose $\operatorname{supp}(f) \subset I$ for every $f \in I$ and let $f_1, \ldots, f_m \in I$ be a set of generators of $I$. Since $\operatorname{supp}(f_i) \subset I$ for every $i$, it follows that $\bigcup_{i=1}^m \operatorname{supp}(f_i)$ is a set of monomial generators of $I$ and thus $I$ is a monomial ideal.

Definition 1.1.8. For monomials $x^a = x_1^{a_1} \cdots x_n^{a_n}$ and $x^b = x_1^{b_1} \cdots x_n^{b_n}$ of $S$, we say that $x^b$ divides $x^a$, and we write $x^b | x^a$, if $b_i \leq a_i$ for each $i$.

Proposition 1.1.9. Let $\{u_1, \ldots, u_m\}$ be a monomial system of generators of $I$. Then the monomial $v$ belongs to $I$ if and only if $u_i | v$ for some $i \in [m]$. 

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Proof. Suppose \( v \in I \). Then there exist polynomials \( f_1, \cdots, f_m \in S \) such that \( v = f_1u_1 + \cdots + f_mu_m \), therefore \( v \in \text{supp}(f_iu_i) \) for some \( i \in [m] \), that is, \( v = wu_i \) for some \( w \in \text{supp}(f_i) \), hence \( u_i \mid v \).

**Proposition 1.1.10.** A monomial set of generators \( G \) of an ideal \( I \) is minimal if and only if there is no pair of distinct monomials \( u, v \in G \) such that \( u \mid v \).

**Proof.** We will show that \( G \) is not minimal if and only if there exists two distinct monomials \( u, v \in G \) such that \( u \mid v \).

Suppose \( G \) is not minimal. Then there exists a monomial set of generators \( G' \) of \( I \) such that \( G' \subset G \). Let \( v \in G' \setminus G \). By proposition 1.1.9 there exists \( u \in G' \) such that \( u \mid v \).

Suppose there exists two distinct monomials \( u, v \in G \) such that \( u \mid v \). Then \( G \setminus \{v\} \) is a monomial set of generators of \( I \) strictly contained in \( G \), hence \( G \) is not minimal.

**Proposition 1.1.11.** Each monomial ideal has a unique minimal monomial set of generators.

**Proof.** Let \( G_1 = \{u_1, \cdots, u_r\} \) and \( G_2 = \{v_1, \cdots, v_s\} \) be two minimal sets of generators of the monomial ideal \( I \). Let \( i \in [r] \). Since \( u_i \in \text{Mon}(I) \), then \( v_j \mid u_i \) for some \( j \in [s] \). Similarly, \( u_k \mid v_j \) for some \( k \in [r] \), therefore \( u_k \mid u_i \), and since \( G_1 \) is a minimal set of generators, \( u_i = u_k \), therefore \( u_i = v_j \in G_2 \). This shows that \( G_1 \subset G_2 \). By symmetry we also have \( G_2 \subset G_1 \).

**Notation 1.1.12.** The unique minimal set of monomial generators of the monomial ideal \( I \) is denoted by \( G(I) \).

**Definition 1.1.13.** Let \( \mathcal{M} \) be a non-empty subset of \( \text{Mon}(S) \). A monomial \( x^a \in \mathcal{M} \) is said to be a minimal element of \( \mathcal{M} \) with respect to divisibility if whenever \( x^b \mid x^a \) with \( x^b \in \mathcal{M} \), then \( x^b = x^a \).

**Notation 1.1.14.** The set of minimal elements of \( \mathcal{M} \) is denoted by \( \mathcal{M}^{\text{min}} \).

**Theorem 1.1.15** (Dickson’s lemma). Let \( \mathcal{M} \) be a non-empty subset of \( \text{Mon}(S) \). Then \( \mathcal{M}^{\text{min}} \) is a finite non-empty set.

**Proof.** Let \( I \subset S \) be the ideal with generator set \( \mathcal{M} \). Since \( G(I) \) is the only minimal set of monomial generators of \( I \), then \( G(I) \subset \mathcal{M} \). We will show that \( G(I) = \mathcal{M}^{\text{min}} \), from where the result follows.

Let \( u \in G(I) \). Suppose there exists \( v \in \mathcal{M} \) such that \( v \mid u \). Then \( v \in \text{Mon}(I) \) and by proposition 1.1.9 there exists \( w \in G(I) \) such that \( w \mid v \). Since \( w \mid v \) and \( v \mid u \), then \( w \mid u \), and since \( u, w \in G(I) \), it follows that \( w = u \) and so \( v = u \). Hence \( u \in \mathcal{M}^{\text{min}} \).

Let \( u \in \mathcal{M}^{\text{min}} \). Then \( u \in I \), therefore there exists \( v \in G(I) \) such that \( v \mid u \). But then \( v \in \mathcal{M} \), and since \( v \mid u \), it follows that \( u = v \in G(I) \).

It is obvious that sums and products of monomial ideals are again monomial ideals. More precisely, if \( I \) and \( J \) are monomial ideals, then so are \( I + J \) and \( IJ \), with \( G(I + J) \subset G(I) \cup G(J) \) and \( G(IJ) \subset G(I)G(J) \).
Notation 1.1.16. Given two monomials \( u \) and \( v \), we denote by \( \gcd(u, v) \) their greatest common divisor and by \( \lcm(u, v) \) their least common multiple.

Proposition 1.1.17. Let \( I \) and \( J \) be monomial ideals. Then \( I \cap J \) is a monomial ideal and

\[
\{ \lcm(u, v) : u \in G(I), v \in G(J) \}
\]

is a set of generators of \( I \cap J \).

Proof. Let \( f \in I \cap J \). By corollary 1.1.14, \( \text{supp}(f) \subset I \cap J \). Since \( f \in I \cap J \) is arbitrary, applying corollary 1.1.14 again we see that \( I \cap J \) is a monomial ideal.

Let \( w \in \text{Mon}(I \cap J) \). Then there exist \( u \in G(I) \) and \( v \in G(J) \) such that \( u \mid w \) and \( v \mid w \), therefore \( \lcm(u, v) \mid w \). It is clear that \( \{ \lcm(u, v) : u \in G(I), v \in G(J) \} \subset I \cap J \), hence \( \{ \lcm(u, v) : u \in G(I), v \in G(J) \} \) is a set of generators of \( I \cap J \). \( \square \)

Proposition 1.1.18. Let \( I \) and \( J \) be monomial ideals. Then \( I : J \) is a monomial ideal such that

\[
I : J = \bigcap_{v \in G(J)} I : (v).
\]

Moreover, \( \{ u / \gcd(u, v) : u \in G(I) \} \) is a set of generators of \( I : (v) \).

Proof. Let \( f \in I : J \). Then \( fv \in I \) for every \( v \in G(J) \). In view of corollary 1.1.14, we have \( \text{supp}(f)v = \text{supp}(fv) \subset I \). This implies that \( \text{supp}(f) \subset I : J \). Since \( f \in I : J \) is arbitrary, corollary 1.1.14 yields that \( I : J \) is a monomial ideal.

The given interpretation of \( I : J \) as an intersection is obvious, and it is clear that \( \{ u / \gcd(u, v) : u \in G(I) \} \subset I : (v) \). So now let \( w \in \text{Mon}(I : (v)) \). Then \( vw \in I \), therefore there exists \( u \in G(I) \) such that \( u \mid vw \), hence \( u / \gcd(u, v) \mid w \), as desired. \( \square \)

Proposition 1.1.19. The radical of a monomial ideal \( I \) is again a monomial ideal.

Proof. Let \( f \in \sqrt{I} \). Then \( f^k \in I \) for some \( k \geq 1 \) and by corollary 1.1.14, one has \( \text{supp}(f^k) \subset I \) since \( I \) is a monomial ideal. Let \( \text{supp}(f) = \{ x^{a_1}, \ldots, x^{a_r} \} \). Then some \( a_i \), say \( a_1 \), does not belong to the convex hull of the set \( \{ a_1, \ldots, a_r \} \setminus \{ a_i \} \).

Assume \( (x^{a_1})^k = (x^{a_1})^{k_1}(x^{a_2})^{k_2} \cdots (x^{a_r})^{k_r} \), with \( k = k_1 + \cdots + k_r \) and \( k_1 < k \). Then \( a_1 = \sum_{i=2}^{r} \frac{k_i}{k-k_1} a_i \), with \( \sum_{i=2}^{r} \frac{k_i}{k-k_1} = 1 \), so \( a_1 \) belongs to the convex hull of \( \{ a_2, \ldots, a_r \} \), a contradiction. It follows that the monomial \( (x^{a_1})^k \) cannot cancel against other terms in \( f^k \) and hence \( (x^{a_1})^k \) belongs to \( \text{supp}(f^k) \), which is a subset of \( I \). Therefore \( x^{a_1} \in \sqrt{I} \) and \( f - cx^{a_1} \in \sqrt{I} \), where \( c \) is the coefficient of \( f \) in the monomial \( x^{a_1} \). By induction on the cardinality of \( \text{supp}(f) \) we conclude that \( \text{supp}(f) \subset \sqrt{I} \). Thus corollary 1.1.14 implies that \( \sqrt{I} \) is a monomial ideal. \( \square \)

Definition 1.1.20. A monomial \( x_1^{a_1} \cdots x_n^{a_n} \in S \) is called square-free if \( a_1, \ldots, a_n \in \{0,1\} \).

Notation 1.1.21. For \( u = x_1^{a_1} \cdots x_n^{a_n} \) we set \( \sqrt{u} = \prod_{i: a_i \neq 0} x_i \).
One has $u = \sqrt{u}$ if and only if $u$ is square-free.

**Proposition 1.1.22.** Let $I$ be a monomial ideal. Then $\{\sqrt{u}, u \in G(I)\}$ is a set of generators of $\sqrt{I}$.

**Proof.** Obviously $\{\sqrt{u}, u \in G(I)\} \subset \sqrt{I}$. Since $\sqrt{I}$ is a monomial ideal, it is enough to show that each monomial $v \in \sqrt{I}$ is a multiple of some $\sqrt{u}$ with $u \in G(I)$. In fact, if $v \in \sqrt{I}$, then $v^k \in I$ for some integer $k \geq 1$ and therefore $u \mid v^k$ for some $u \in G(I)$. This yields the desired conclusion. □

**Definition 1.1.23.** A monomial ideal $I$ is called a square-free monomial ideal if it is generated by square-free monomials.

**Corollary 1.1.24.** A monomial ideal is a radical ideal if and only if it is a square-free monomial ideal.

**Theorem 1.1.25.** Let $I \subset K[x_1, \cdots, x_n]$ be a monomial ideal. Then $I = \bigcap_{i=1}^m Q_i$, where each $Q_i$ is generated by pure powers of the variables. In other words, each $Q_i$ is of the form $(x_{i_1}^{a_1}, \cdots, x_{i_k}^{a_k})$. Moreover, an irredundant presentation of this form is unique.

**Proof.** Let $G(I) = \{u_1, \cdots, u_r\}$, and suppose some $u_i$, say $u_1$, is not a pure power of a variable. Then we can write $u_1 = vw$ where $v, w \in \text{Mon}(S)$ are such that $\gcd(v, w) = 1$ and $v \neq 1 \neq w$. We claim that $I = I_1 \cap I_2$, where $I_1 = \langle v, u_2, \cdots, u_r \rangle$ and $I_2 = \langle w, u_2, \cdots, u_r \rangle$. Obviously, $I$ is contained in the intersection. Conversely, let $u \in \text{Mon}(I_1 \cap I_2)$. If $u$ is the multiple of some $u_i$, with $2 \leq i \leq r$, then $u \in I$. If not, then $u$ is a multiple of both $v$ and $w$, and since $\gcd(v, w) = 1$, $u$ is a multiple of $u_1$. In any case, $u \in I$.

If either $G(I_1)$ or $G(I_2)$ contains an element which is not a pure power, we proceed as before and, after a finite number of steps, we obtain a presentation of $I$ as an intersection of monomial ideals generated by pure powers. By omitting those ideals which contain the intersection of the others we end up with an irredundant presentation.

Let $Q_1 \cap \cdots \cap Q_r = Q'_1 \cap \cdots \cap Q'_s$ be two such irredundant presentations of $I$. We will show that for each $i \in [r]$ there exists $j \in [s]$ such that $Q'_j \subset Q_i$. By symmetry we then also have that for each $k \in [s]$ there exists $l \in [r]$ such that $Q_l \subset Q'_k$. This will then imply that $r = s$ and $\{Q_1, \cdots, Q_r\} = \{Q'_1, \cdots, Q'_s\}$.

In fact, let $i \in [r]$. We may assume that $Q_i = \langle x_{i_1}^{a_1}, \cdots, x_{i_k}^{a_k} \rangle$. Suppose that $Q'_j \not\subset Q_i$ for all $j \in [s]$. Then for each $j$ there exists $x_{j_i}^{b_j} \in Q'_j \setminus Q_i$. It follows that either $l_j \not\in [k]$ or $b_j < a_{i_j}$. Let $u = \text{lcm}(x_{i_1}^{b_1}, \cdots, x_{i_s}^{b_s})$. We have $u \in \bigcap_{j=1}^s Q'_j = I \subset Q_i$. Thus there exists $i \in [k]$ such that $x_{i_i}^{a_i}$ divides $u$. But this is obviously impossible. □

**Example 1.1.26.** Let $I = (x_1^2x_2, x_1^2x_3, x_2^2, x_2x_3^2) \subset K[x_1, x_2, x_3]$. Then:

$I = (x_1^2x_2, x_1^2x_3, x_2^2, x_2x_3^2) = (x_1^2, x_1x_3^2, x_2^2, x_2x_3^2) \cap (x_2, x_1^2x_3, x_2, x_2x_3^2) =$

$= (x_1^2, x_2^2, x_2x_3^2) \cap (x_2, x_2x_3^2) = (x_1^2, x_2^2, x_2) \cap (x_2, x_2x_3^2) \cap (x_2, x_1^2) \cap (x_2, x_2x_3^2) =$

$= (x_1^2, x_2^2, x_3^2) \cap (x_2, x_1^2) \cap (x_2, x_2x_3^2)$. 

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Definition 1.1.27. A monomial ideal is called irreducible if it cannot be written as proper intersection of two other monomial ideals. It is called reducible if it is not irreducible.

Corollary 1.1.28. A monomial ideal is irreducible if and only if it is generated by pure powers of the variables.

Proof. Let $Q = \langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle$ and suppose $Q = I \cap J$, where $I$ and $J$ are monomial ideals properly containing $Q$. By theorem [1.1.25] we have $I = \bigcap_{i=1}^{r} Q_i$ and $J = \bigcap_{j=1}^{s} Q_j'$, where the ideals $Q_i$ and $Q_j'$ are generated by pure powers. Therefore we get the presentation

$$Q = \bigcap_{i=1}^{r} Q_i \cap \bigcap_{j=1}^{s} Q_j'.$$

By omitting suitable ideals in the intersection on the right hand side, we obtain an irredundant presentation of $Q$. The uniqueness of the irredundant presentation implies that $Q = Q_i$ for some $i \in \{r\}$ or $Q = Q_j'$ for some $j \in \{s\}$, a contradiction.

Conversely, if $G(Q)$ contains a monomial $u = vw$ with $\gcd(v, w) = 1$ and $v \neq 1 \neq w$, then, as in the proof of theorem [1.1.25], $Q$ can be written as the intersection of monomial ideals properly containing $Q$.

If $I$ is a square-free monomial ideal, the above procedure yields that the irreducible monomial ideals appearing in the intersection of $I$ are all of the form $(x_i_1, \ldots, x_i_k)$. These monomial ideals are precisely the monomial prime ideals.

Corollary 1.1.29. A square-free monomial ideal is the intersection of monomial prime ideals.

Combining this corollary with lemma [0.2.13] we obtain:

Corollary 1.1.30. Let $I \subset S$ be a square-free monomial ideal. Then $I = \bigcap_{P \in \Min(I)} P$, and each $P \in \Min(I)$ is a monomial prime ideal.

Proposition 1.1.31. The irreducible ideal $(x_1^{a_1}, \ldots, x_k^{a_k})$ is $(x_1, \ldots, x_k)$-primary.

Proof. Let $Q = \langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle$ and $P = \langle x_1, \ldots, x_k \rangle$. By proposition [0.2.25] $Q$ is $P$-primary if and only if $\Ass(S/Q) = \{ P \}$. Notice that $\Min(Q) = \{ P \}$. Since $P \in \Min(Q)$, by proposition [0.2.21] it follows that $P \in \Ass(S/Q)$. Suppose there exists another $P' \in \Ass(S/Q)$.

Let $g \in S \setminus Q$ such that $P' = \Ann(g + Q)$ and consider its decomposition as a finite $K$-linear combination of monomials. Removing from such $K$-linear combination the monomials which are multiples of some monomial in the set $(x_1^{a_1}, \ldots, x_k^{a_k})$, we get a polynomial $g' \in S \setminus Q$ whose support does not intersect $\Mon(Q)$ and such that $g + Q = g' + Q$. Hence we can suppose without loss of generality that $\supp(g) \cap \Mon(Q) = \emptyset$.

Since $Q \subset P'$ and $\Min(Q) = \{ P \}$, then $P \subset P'$.

Pick $f \in P' \setminus P$ and consider its decomposition as a finite $K$-linear combination of monomials. Removing from such $K$-linear combination the monomials which are multiples of some monomial in the set $(x_1, \ldots, x_k)$, we get a polynomial in $P' \setminus P$ whose support does not intersect $\Mon(P)$. Hence we can suppose without loss of generality that $\supp(f) \cap \Mon(P) = \emptyset$.
Since \( f \neq 0 \) and \( g \neq 0 \), then \( fg \neq 0 \). Let \( w \in \text{supp}(fg) \). Then there exist \( u \in \text{supp}(f) \) and \( v \in \text{supp}(g) \) such that \( w = uv \). Since none of the monomials \( x_{i_1} \cdots x_{i_k} \) divide \( u \) and none of the monomials \( x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \) divide \( v \), it follows that none of the monomials \( x_{i_1}^{a_1} \cdots x_{i_k}^{a_k} \) divide \( w \). Since \( w \in \text{supp}(fg) \) is arbitrary, then \( \text{supp}(fg) \cap \text{Mon}(Q) = \emptyset \), hence \( fg \not\in Q \). But since \( f \in P' \) and \( P' = \text{Ann}(g + Q) \), by definition it follows that \( fg \in Q \), a contradiction.

Corollary 1.1.32. The irredundant decomposition of a monomial ideal as the intersection of monomial ideals generated by pure powers is in fact a primary irredundant decomposition.

Corollary 1.1.33. The associated prime ideals of a monomial ideal are monomial prime ideals.

Even though a primary irredundant decomposition of a monomial ideal \( I \) may not be unique, the primary decomposition, obtained from an irredundant intersection of irreducible ideals as described above, is unique. We call it the standard primary decomposition of \( I \).

Example 1.1.34. As we saw in example 1.1.26, if \( I = (x_1^2 x_2, x_1^2 x_3, x_2^2, x_2 x_3^2) \), then

\[
I = (x_1^2, x_2^2, x_3^2) \cap (x_2, x_1^2) \cap (x_2, x_3^2)
\]

is the standard primary decomposition of \( I \) and in particular \( \text{Ass}(R/I) = \{(x_1, x_2), (x_2, x_3), (x_1, x_2, x_3)\} \).

Notice that, in this particular case, \( \text{Min}(I) \neq \text{Ass}(R/I) \).

We end this section by stating a result which will be important in the study of monomial edge ideals:

Proposition 1.1.35. If \( P \) is a monomial prime ideal generated by \( k \) variables, then

\[
\text{ht}(P) = k \quad \text{and} \quad \text{dim}(S/P) = n - k.
\]

Proof. Let \( P = (x_{i_1}, \cdots, x_{i_k}) \). Since \( P \) is generated by \( k \) elements of \( S \), theorem 0.3.2 implies that \( \text{ht}(P) \leq k \). On the other hand, \( (0) \subseteq (x_{i_1}) \subseteq (x_{i_1}, x_{i_2}) \subseteq \cdots \subseteq (x_{i_1}, \cdots, x_{i_{k-1}}) \subseteq P \) is an ascending chain of prime ideals of length \( k \) and so \( \text{ht}(P) = k \). By proposition 0.5.12, \( \text{ht}(P) + \text{dim}(S/P) = n \) and so \( \text{dim}(S/P) = n - k \). \( \square \)

1.2 Simplicial complexes and Stanley-Reisner ideals

Sometimes, when studying graphs, is it is a good idea to think about them as simplicial complexes. In fact, for each graph \( G \), we can associate a simplicial complex \( \Delta(G) \), called its clique complex, such that there is an obvious correspondence between the faces of \( G \) and the cliques of \( \Delta(G) \). Since each graph is uniquely determined by its clique complex, then graphs are just a special class of simplicial complexes.

Definition 1.2.1. A simplicial complex \( \Delta \) on the vertex set \([n]\) is a collection of subsets of \([n]\) that contains all one-element subsets and such that if \( F \in \Delta \) and \( F' \subset F \), then \( F' \in \Delta \).
Definition 1.2.2. A face of $\Delta$ is just an element of $\Delta$.

Definition 1.2.3. The dimension of a face $F$ is $|F| - 1$.

Definition 1.2.4. A vertex is a face of dimension 0.

Definition 1.2.5. An edge is a face of dimension 1.

Definition 1.2.6. A facet is a maximal face with respect to inclusion.

Notation 1.2.7. The set of facets of $\Delta$ is denoted by $\mathcal{F}(\Delta)$.

It is clear that $\mathcal{F}(\Delta)$ determines $\Delta$ and so we write $\Delta = (F_1, \ldots, F_m)$ when $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$. More generally, given a set $\{G_1, \ldots, G_s\}$ of faces of $\Delta$, we denote by $(G_1, \ldots, G_s)$ the subcomplex of $\Delta$ consisting of those faces of $\Delta$ which are contained in some $G_i$.

Definition 1.2.8. A non-face of $\Delta$ is a subset of $[n]$ such that $F \notin \Delta$.

Notation 1.2.9. The set of minimal non-faces of $\Delta$ is denoted by $\mathcal{N}(\Delta)$.

Definition 1.2.10. A simplicial complex $\Delta$ is pure if all its facets have the same dimension. More precisely, if such dimension is $d$, we say $\Delta$ is a pure $d$-dimensional simplicial complex.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over $K$ and let $\Delta$ be a simplicial complex on $[n]$.

Notation 1.2.11. For $F \subset [n]$, we denote $x_F = \prod_{i \in F} x_i$.

Definition 1.2.12. The Stanley-Reisner ideal of $\Delta$ is the ideal $I_\Delta \subset S$ generated by the square-free monomials $x_F$ with $F \notin \Delta$.

Proposition 1.2.13. If $\Delta$ is a simplicial complex, then $I_\Delta$ is a monomial ideal with $G(I_\Delta) = \{x_F : F \in \mathcal{N}(\Delta)\}$.

Notation 1.2.14. For each subset $F \subset [n]$ we denote $\overline{F} = [n] \setminus F$.

Notation 1.2.15. For each subset $F \subset [n]$, $P_F$ is the prime ideal of $S$ generated by the variables $x_i$ with $i \in F$.

Lemma 1.2.16. The standard primary decomposition of $I_\Delta$ is $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}}$.

Proof. Let $F \in \mathcal{F}(\Delta)$. If $G$ is a non-face of $\Delta$, then $G \notin F$. Pick $i \in G \setminus F$. Then $x_i \in P_{\overline{F}}$ and $x_i \mid x_G$, hence $x_G \in P_{\overline{F}}$. Since $G$ is an arbitrary non-face of $\Delta$, $I_\Delta \subset P_{\overline{F}}$.

Let $u \in \text{Mon}(S)$ such that $u \in \bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}}$. Then for each $F \in \mathcal{F}(\Delta)$ there exists $i_F \in F$ such that $x_{i_F} \mid u$ and so $\text{lcm}(x_{i_F} : F \in \mathcal{F}(\Delta)) \mid u$. On the other hand, $\{i_F : F \in \mathcal{F}(\Delta)\}$ cannot be contained in any facet of $\Delta$ and so it is a non-face of $\Delta$, which implies $u \in I_\Delta$. Since $\bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}}$ is a monomial ideal (for it is a finite intersection of monomial ideals), it follows that $\bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}} \subset I_\Delta$.

Since the Stanley-Reisner ideal of a simplicial complex is uniquely determined by its facets, the primary decomposition $I_\Delta = \bigcap_{F \in \mathcal{F}(\Delta)} P_{\overline{F}}$ turns out to be irredundant.
Definition 1.2.17. A simplicial complex $\Delta$ is Cohen-Macaulay over a field $K$ if the ideal $I_\Delta$ is Cohen-Macaulay.

Proposition 1.2.18. If $\Delta$ is a Cohen-Macaulay simplicial complex, then $\Delta$ is a pure simplicial complex.

Proof. By lemma 1.2.16, \[ \text{Ass}(S/I_\Delta) = \{P_F : F \in \mathcal{F}(\Delta)\}. \]
If $\Delta$ is a Cohen-Macaulay simplicial complex on $[n]$, that is, $I_\Delta$ is Cohen-Macaulay, then by proposition 2.2.34, $\dim(S/P_F) = \dim(S/I_\Delta)$ for every $F \in \mathcal{F}(\Delta)$. But $P_F$ is a monomial prime ideal generated by $n - |F|$ variables and so, by proposition 1.1.35, $\dim(S/P_F) = |F|$, hence $|F| = \dim(S/I_\Delta)$. This means that all facets of $\Delta$ have dimension $\dim(S/I_\Delta) - 1$ and so $\Delta$ is a pure simplicial complex. \qed

Definition 1.2.19. Let $F \in \Delta$. The link of $F$ in $\Delta$ is the subcomplex $\text{link}_\Delta(F) = \{G \in \Delta : F \cup G \in \Delta, F \cap G = \emptyset\}$.

Example 1.2.20. Let $\Delta$ be the simplicial complex below. If $F$ is the edge on the left, then $\text{link}_\Delta(F)$ is the clique whose only facet is the triangle on the right.

![Diagram of a simplicial complex with a clique and a triangle as the link of an edge.]

Proposition 1.2.21. Let $F' \in \Delta$ and let $F \subset F'$. Then $F' \setminus F \in \text{link}_\Delta(F)$.

Proof. Let $G = F' \setminus F$. Since $G \subset F'$, then $G \in \Delta$. We also have $G \cup F = F' \in \Delta$ and $G \cap F = \emptyset$, and by definition, $G \in \text{link}_\Delta(F)$. \qed

On the other hand, by definition, it immediately follows that $G \in \text{link}_\Delta(F)$ implies $F \cup G \in \Delta$. This provides us a natural bijection between the faces in $\text{link}_\Delta(F)$ and the faces in $\Delta$ containing $F$. Moreover, such bijection provides us a natural bijection between the facets in $\text{link}_\Delta(F)$ and the facets in $\Delta$ containing $F$.

Proposition 1.2.22. Let $\Delta$ be a Cohen-Macaulay simplicial complex and let $F \in \Delta$. Then $\text{link}_\Delta(F)$ is also Cohen-Macaulay.

Proof. See [5] 8.1 (the proof of this proposition uses local cohomology). \qed

Definition 1.2.23. A simplicial complex $\Delta$ is connected if each pair of facets $F, G$ can be connected by a sequence of facets $F = F_0, F_1, \cdots, F_q = G$, that is, such sequence satisfies $F_{i-1} \cap F_i \neq \emptyset$ for each $i \in [q]$.

Example 1.2.24. The simplicial complex below is connected.
Proposition 1.2.25. Every Cohen-Macaulay simplicial complex is connected.

Proof. See [5, 8.1] (the proof of this proposition uses local cohomology).

Definition 1.2.26. A pure $d$-dimensional simplicial complex $\Delta$ is strongly connected (or connected in codimension one) if each pair of facets $F, G$ can be connected by a sequence of facets $F = F_0, F_1, \ldots, F_q = G$ such that $\dim(F_{i-1} \cap F_i) = d - 1$ for each $i \in [q]$.

If $d > 0$, then every $d$-dimensional strongly connected simplicial complex is clearly connected. However, 0-dimensional non-singular simplicial complexes are not connected even though they are strongly connected (recall that, by definition, $\emptyset$ is a face such that $\dim \emptyset = -1$).

Proposition 1.2.27. Every Cohen-Macaulay simplicial complex is strongly connected.

Proof. Let $\Delta$ be a pure $d$-dimensional simplicial complex. If $d = 0$, then the assertion is trivial (recall that $\dim \emptyset = -1$). Therefore we assume $d > 0$. Let $F, G \in \mathcal{F}(\Delta)$. Since $\Delta$ is connected, there exists a sequence of facets $F = F_0, F_1, \ldots, F_q = G$ such that $F_{i-1} \cap F_i \neq \emptyset$ for each $i \in [q]$. For $i \in [q]$, pick $y_i \in F_{i-1} \cap F_i$. Since facets in $\text{link}_\Delta(\{y_i\})$ are naturally given by facets in $\Delta$ containing $y_i$ and $\Delta$ is a pure $d$-dimensional Cohen-Macaulay simplicial complex, then $\text{link}_\Delta(\{y_i\})$ is pure $(d-1)$-dimensional Cohen-Macaulay simplicial complex. By working with induction on the dimension of $\Delta$, we may assume that $\text{link}_\Delta(\{y_i\})$ is strongly connected. Moreover, there exists a sequence of facets $F_{i-1} = H_0, \ldots, H_r = F_i$ of $\Delta$, where all $H_j$ contain $y_i$ and $\dim(H_j \cap H_{j+1}) = d - 1$. Composing all these sequences of facets which we have between $F_{i-1}$ and $F_i$ for $i \in [q]$ yields the desired sequence between $F$ and $G$.

Definition 1.2.28. A facet $F$ of $\Delta$ is called a leaf if either $F$ is the only facet, or else there exists a facet $G \neq F$, called a branch of $F$, such that, for each facet $H$ of $\Delta$ with $H \neq F$ one has $H \cap F \subset G \cap F$.

If $\Delta$ has at most two facets, then such facets are leaves.

Example 1.2.29. The simplicial complex below has three facets: $\{1, 2\}$, $\{2, 3, 4\}$ and $\{3, 4, 5\}$. The facets $\{1, 2\}$ and $\{3, 4, 5\}$ are clearly leaves while the facet $\{2, 3, 4\}$ is not a leaf.
Proposition 1.2.30. Each leaf $F$ has at least one free vertex, that is, a vertex which belongs to $F$ but to no other facet.

Proof. If $F$ is the only facet of $\Delta$, the result is obvious. Otherwise, let $G$ be a branch of $F$ and suppose $F$ has no free vertices. Let $i \in F$. Then there exists another facet $H$ such that $i \in H$ and so $i \in H \cap F \subset G \cap F$. Since $i \in F$ is arbitrary, it follows that $F \subset G \cap F$, which is absurd. Hence $F$ has at least one free vertex. \qed

Definition 1.2.31. The simplicial complex $\Delta$ is called a quasi-forest if its facets $F_1, \cdots, F_r$ can be ordered in a way such that for all $i > 1$ the facet $F_i$ is a leaf of the simplicial complex with facets $F_1, \cdots, F_{i-1}, F_i$.

Such an order of the facets is called a leaf order. As one would expect, a connected quasi-forest is called a quasi-tree.

Definition 1.2.32. The clique complex of a graph $G$ on $[n]$ is the simplicial complex $\Delta(G)$ on $[n]$ whose faces are the cliques of $G$.

Since any graph is uniquely determined by its clique complex, it follows that each graph can be viewed as a simplicial complex $\Delta$ on $[n]$ with the property that if $F \subset [n]$ is such that all two-element subsets of $F$ are faces of $\Delta$, then $F \in \Delta$.

Definition 1.2.33. A graph $G$ has a perfect elimination order if its vertices can be labelled $1, \cdots, n$ such that for all $j \in [n]$, the set $C_j = \{i : i \leq j, \{i, j\} \in E(G)\} \cup \{j\}$ is a clique of $G$.

Example 1.2.34. If $G$ is a path, then we get a perfect elimination order by labelling its vertices in a way such that every two consecutive vertices are adjacent.

Theorem 1.2.35. The following conditions are equivalent:

1. $G$ is chordal.
2. $G$ has a perfect elimination order.
3. $\Delta(G)$ is a quasi-forest.

Proof. In 1961, in [13], Dirac showed that (1) and (2) are equivalent. In [5][9.2] it is shown that (2) and (3) are equivalent. Hence the conclusion follows. \qed

1.3 Gröbner bases

Gröbner basis theory has become a fundamental field in algebra which provides a wide range of theoretical and computational methods in many areas of mathematics and other sciences. Bruno Buchberger defined the notion of Gröbner basis in 1965. An intensive research in this theory, related algorithms and applications developed, and many books on this topic have appeared since then.

Let $S = K[x_1, \cdots, x_n]$ be a polynomial ring in $n$ variables over a field $K$. 

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Definition 1.3.1. A monomial order on $S$ is a total order $<$ on $\text{Mon}(S)$ such that:

- $1 < u$ for all $u \in \text{Mon}(S) \setminus \{1\}$;
- if $u, v \in \text{Mon}(S)$ and $u < v$, then $uw < vw$ for all $w \in \text{Mon}(S)$.

Example 1.3.2. Consider the monomial order $<$ defined as follows: if $x^a, x^b \in S$, with $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, then $x^a < x^b$ if and only if the leftmost non-zero component of $a - b$ is negative. This monomial order is called the pure lexicographic order on $S$ induced by the ordering $x_1 > \cdots > x_n$.

Example 1.3.3. Consider the monomial order $<$ defined as follows: if $x^a, x^b \in S$, with $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, then $x^a < x^b$ if and only if $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$ or $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the leftmost non-zero component of $a - b$ is negative. This monomial order is called the lexicographic order on $S$ induced by the ordering $x_1 > \cdots > x_n$.

Example 1.3.4. Consider the monomial order $<$ defined as follows: if $x^a, x^b \in S$, with $a = (a_1, \cdots, a_n)$ and $b = (b_1, \cdots, b_n)$, then $x^a < x^b$ if and only if $\sum_{i=1}^{n} a_i < \sum_{i=1}^{n} b_i$ or $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ and the rightmost non-zero component of $a - b$ is positive. This monomial order is called the reverse lexicographic order on $S$ induced by the ordering $x_1 > \cdots > x_n$.

Definition 1.3.5. The initial monomial of $f \in S \setminus \{0\}$ with respect to $<$ is the maximal element of $\text{supp}(f)$ with respect to $<$.

Notation 1.3.6. We write $\text{in}_<(f)$ for the initial monomial of $f$ with respect to $<$.

Definition 1.3.7. The leading coefficient of $f$ is the coefficient of $\text{in}_<(f)$.

Lemma 1.3.8. Let $u, v$ be monomials of $S$ and $f, g$ non-zero polynomials in $S$. Then:

1. if $u$ divides $v$, then $u \leq v$;
2. $\text{in}_<(uf) = u \text{in}_<(f)$;
3. $\text{in}_<(fg) = \text{in}_<(f) \text{in}_<(g)$;
4. $\text{in}_<(f + g) \leq \max\{\text{in}_<(f), \text{in}_<(g)\}$ with equality if $\text{in}_<(f) \neq \text{in}_<(g)$.

Proof. 1. Let $w \in \text{Mon}(S)$ such that $v = uw$. Since $1 \leq w$, then $u \cdot 1 \leq u \cdot w$, that is, $u \leq v$.

2. If $w \in \text{supp}(f)$, then $w \leq \text{in}_<(f)$, thus $uw \leq u \text{in}_<(f)$. Since $\text{supp}(uf) = u \text{supp}(f)$, the conclusion follows.

3. Let $w \in \text{supp}(f)$ and $w' \in \text{supp}(g)$. Then $w \leq \text{in}_<(f)$ and $w' \leq \text{in}_<(g)$, therefore $ww' \leq w \text{in}_<(g) \leq \text{in}_<(f) \text{in}_<(g)$. Since $\text{supp}(fg) \subset \text{supp}(f) \cup \text{supp}(g)$, the conclusion follows.

4. Recall that $\text{supp}(f + g) \subset \text{supp}(f) \cup \text{supp}(g)$.

The lemma is thus shown. \qed
**Proposition 1.3.9.** Let $\mathcal{M}$ be a non-empty set of monomials in $S$ and let $<$ be a monomial order in $S$. Then $\mathcal{M}$ has a unique minimal element with respect to $<$.

**Proof.** By Dickson’s lemma, $\mathcal{M}^{\text{min}}$ is a finite non-empty set. Let $u$ be the smallest monomial in $\mathcal{M}^{\text{min}}$ with respect to $<$. Let $v \in \mathcal{M}$. Then there exists $w \in \mathcal{M}^{\text{min}}$ such that $w \mid v$, hence $w \leq v$. On the other hand, $u \leq w$ and so $u \leq v$. Since $v \in \mathcal{M}$ is arbitrary, then $u$ is the unique minimal element of $\mathcal{M}$.

**Definition 1.3.10.** Let $I$ be a non-zero ideal of $S$. The initial ideal of $I$ with respect to $<$ is the monomial ideal of $S$ which is generated by the set $\{\text{in}_<(f) : f \in I \setminus \{0\}\}$.

**Notation 1.3.11.** We write $\text{in}_<(I)$ for the initial ideal of $I$.

Since $\text{in}_<(I)$ is a monomial ideal such that $\text{Mon}(\text{in}_<(I)) = \{\text{in}_<(f) : f \in I \setminus \{0\}\}$, then by proposition 1.1.4 there exists a finite number of non-zero polynomials $g_1, \cdots, g_s \in I$ such that $\text{in}_<(I) = (\text{in}_<(g_1), \cdots, \text{in}_<(g_s))$.

**Definition 1.3.12.** Let $I$ be a non-zero ideal of $S$. A finite set of non-zero polynomials $\{g_1, \cdots, g_s\} \subset I$ is said to be a Gröbner basis of $I$ with respect to $<$ if the initial ideal $\text{in}_<(I)$ is generated by the monomials $\text{in}_<(g_1), \cdots, \text{in}_<(g_s)$.

**Theorem 1.3.13.** Let $I$ be a non-zero ideal of $S$ and $\{g_1, \cdots, g_s\}$ a Gröbner basis of $I$ with respect to a monomial order $<$ on $S$. Then $I = (g_1, \cdots, g_s)$. In other words, every Gröbner basis of $I$ is a generator set of $I$.

**Proof.** Suppose $I \neq (g_1, \cdots, g_s)$ and let $\mathcal{M} = \{\text{in}_<(g) : g \in I \neq (g_1, \cdots, g_s)\}$. Pick $f \in I \setminus (g_1, \cdots, g_s)$ such that $\text{in}_<(f)$ is the minimal element of $\mathcal{M}$ with respect to $<$. Since $\{g_1, \cdots, g_s\}$ is a Gröbner basis of $I$, by proposition 1.1.9 some $\text{in}_<(g_i)$ must divide $\text{in}_<(f)$. Let $w \in \text{Mon}(S)$ such that $\text{in}_<(f) = w \text{in}_<(g_i)$. Let $c_0, c_i \in K \setminus \{0\}$ be the coefficients of $\text{in}_<(f)$ and $\text{in}_<(g_i)$ in $f$ and $g_i$, respectively. Then $f - c_0c_i^{-1}wg_i \in I \setminus (g_1, \cdots, g_s)$ is such that $\text{in}_<(f - c_0c_i^{-1}wg_i) < \text{in}_<(f)$, a contradiction. It is natural to ask if the converse of theorem 1.3.13 is true or false. That is to say, if $I = (f_1, \cdots, f_s)$ is an ideal of $S$, then does there exist a monomial order $<$ on $S$ such that $\{f_1, \cdots, f_s\}$ is a Gröbner basis of $I$ with respect to $<$?

**Example 1.3.14.** Let $S = K[x_1, \cdots, x_{10}]$ and $I$ the ideal generated by $f_1 = x_1x_8 - x_2x_6$, $f_2 = x_2x_9 - x_3x_7$, $f_3 = x_3x_{10} - x_4x_8$, $f_4 = x_4x_6 - x_5x_9$, and $f_5 = x_5x_7 - x_1x_{10}$. There exists no monomial order $<$ on $S$ such that $\{f_1, \cdots, f_5\}$ is a Gröbner basis of $I$ with respect to $<$. Suppose, on the contrary, that there exists a monomial order $<$ on $S$ such that $G = \{f_1, \cdots, f_5\}$ is a Gröbner basis of $I$ with respect to $<$. First, notice that each of the five polynomials

\[
x_1x_8x_9 - x_3x_6x_7, \quad x_2x_9x_{10} - x_4x_7x_8, \quad x_2x_6x_{10} - x_5x_7x_8, \quad x_3x_6x_{10} - x_5x_8x_9, \quad x_1x_9x_{10} - x_4x_6x_7
\]

41
belongs to $I$. Let, say, $x_1x_8x_9 > x_3x_6x_7$. Since $x_1x_8x_9 \in \text{in}_<(I)$, there exists $g \in \mathcal{G}$ such that $\text{in}_<(g) \mid x_1x_8x_9$.

Such $g$ must be $f_1$, hence $x_1x_8 > x_2x_6$. Thus $x_2x_6 \not\in \text{in}_<(I)$. Hence there exists no $g \in \mathcal{G}$ such that $\text{in}_<(g) \mid x_2x_6x_{10}$, hence $x_2x_6x_{10} < x_5x_7x_8$. Thus $x_5x_7 > x_1x_{10}$. Continuing these arguments yields

$$x_1x_8x_9 > x_3x_6x_7, \ x_2x_9x_{10} > x_4x_7x_8, \ x_2x_6x_{10} < x_5x_7x_8, \ x_3x_6x_{10} > x_5x_8x_9, \ x_1x_9x_{10} < x_4x_6x_7$$

and

$$x_1x_8 > x_2x_6, \ x_2x_9 > x_3x_7, \ x_3x_{10} > x_4x_8, \ x_4x_6 > x_5x_9, \ x_5x_7 > x_1x_{10}.$$  

Hence $(x_1x_8)(x_2x_9)(x_3x_{10})(x_4x_6)(x_5x_7) > (x_2x_6)(x_3x_7)(x_4x_8)(x_5x_9)(x_1x_{10})$. However, both sides of the inequality coincide with $x_1 \cdots x_{10}$, a contradiction.

It is well known that, given polynomials $f$ and $g \neq 0$ in one variable $x$, there exist unique polynomials $q$ and $r$ such that $f = qg + r$, where either $r = 0$ or $\deg r < \deg g$. The division algorithm generalizes this result.

**Theorem 1.3.15** (The division algorithm). Fix a monomial order $<$ on $S$. Let $g_1, \cdots, g_s$ be non-zero polynomials. Then, given a polynomial $f \in S \setminus \{0\}$, there exist polynomials $f_1, \cdots, f_s$ and $f'$ in $S$ with

$$f = f_1g_1 + \cdots + f_sg_s + f', \quad (1.2)$$

such that the following conditions are satisfied:

- if $f' \neq 0$ and $u \in \text{supp}(f')$, then none of the initial monomials $\text{in}_<(g_1), \cdots, \text{in}_<(g_s)$ divides $u$, that is, no monomial $u \in \text{supp}(f')$ belongs to $(\text{in}_<(g_1), \cdots, \text{in}_<(g_s))$;

- if $f_i \neq 0$, then $\text{in}_<(f) \geq \text{in}_<(f_ig_i)$.

The right-hand side of (1.2) is said to be a standard expression for $f$ with respect to $g_1, \cdots, g_s$, and the polynomial $f'$ is said to be a remainder of $f$ with respect to $g_1, \cdots, g_s$. One also says that $f$ reduces to $f'$ with respect to $g_1, \cdots, g_s$.

**Proof.** If $\text{supp}(f) \cap (\text{in}_<(g_1), \cdots, \text{in}_<(g_s)) = \emptyset$, then $f$ has a standard expression with $f_1 = \cdots = f_s = 0$ and $f' = f$. For $f \in S \setminus \{0\}$ such that $\text{supp}(f) \cap (\text{in}_<(g_1), \cdots, \text{in}_<(g_s)) \neq \emptyset$, let $u_f$ be the maximal monomial in $\text{supp}(f) \cap (\text{in}_<(g_1), \cdots, \text{in}_<(g_s))$ with respect to $<$. Let $\Sigma$ be the set of non-zero polynomials in $S$ without a standard expression and suppose $\Sigma \neq \emptyset$. Let $\mathcal{M} = \{u_g : g \in \Sigma\}$. Pick $f \in \Sigma$ such that $u_f$ is the minimal element of $\mathcal{M}$ with respect to $<$. Some $\text{in}_<(g_i)$ must divide $u_f$. Let $w \in \text{Mon}(S)$ such that $u_f = w\text{in}_<(g_i)$. Let $c_0, c_i \in K \setminus \{0\}$ be the coefficients of $u_f$ and $\text{in}_<(g_i)$ in $f$ and $g_i$, respectively. Suppose $f = c_0c_i^{-1}wg_i$. Then $\text{in}_<(f) \geq u_f = \text{in}_<(wg_i)$, hence $f$ has a standard expression with $f_j = 0$ for $j \neq i$, $f_i = c_0c_i^{-1}w$ and $f' = 0$. Suppose $f \neq c_0c_i^{-1}wg_i$. Then $f - c_0c_i^{-1}wg_i$ is such that $u_f \not\in \text{supp}(f - c_0c_i^{-1}wg_i)$. If $f - c_0c_i^{-1}wg_i$ has a standard expression, say $f - c_0c_i^{-1}wg_i = f_1g_1 + \cdots + f_sg_s + f'$, then $f = \sum_{j \neq i}(f_jg_j + (f_i + c_0c_i^{-1}w)g_i + f')$ is a standard expression for $f$, a contradiction. In fact, $\text{in}_<(f_ig_i)(f_i + c_0c_i^{-1}w)g_i) \leq \max\{\text{in}_<(f_ig_i), \text{in}_<(wg_i)\} = \max\{\text{in}_<(f_ig_i), u_f\} \leq \text{in}_<(f)$.
Hence \( f - c_0c_i^{-1}wg_i \) does not have a standard expression, that is, \( f - c_0c_i^{-1}wg_i \in \Sigma \) and in particular \( \text{supp}(f - c_0c_i^{-1}wg_i) \cap (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) \neq \emptyset \). Since \( u_f \notin \text{supp}(f - c_0c_i^{-1}wg_i) \) and \( \text{in}_<(w_g) = u_f \), then every element of \( \text{supp}(f - c_0c_i^{-1}wg_i) \cap (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) \) is smaller than \( u_f \) with respect to \( < \), hence \( u_f - c_0c_i^{-1}wg_i < u_f \), which contradicts the minimality of \( u_f \). \( \square \)

**Example 1.3.16.** Consider the lexicographic order on \( S = K[x, y, z] \) induced by \( x > y > z \). Let \( g_1 = x^2 - z \), \( g_2 = xy - 1 \) and \( f = x^3 - x^2 y - x^2 - 1 \). Each of \( f = (x - y - 1)g_1 + (xz - yz - z - 1) \) and \( f = (x - 1)g_1 - xg_2 + (xz - x - z - 1) \) is a standard expression for \( f \) with respect to \( g_1 \) and \( g_2 \), and each of \( xz - yz - z - 1 \) and \( xz - x - z - 1 \) is a remainder of \( f \).

This example says that in the division algorithm a remainder of \( f \) is, in general, not unique. However, the following lemma holds:

**Lemma 1.3.17.** If \( \{g_1, \ldots, g_s\} \) is a Gröbner basis of \( I = \langle g_1, \ldots, g_s \rangle \), then for any non-zero polynomial \( f \) in \( S \), there is a unique remainder of \( f \) with respect to \( g_1, \ldots, g_s \).

**Proof.** Suppose there exist two distinct remainders \( f' \) and \( f'' \) of \( f \) with respect to \( g_1, \ldots, g_s \). Then \( f' - f'' \in I \) and \( \text{supp}(f' - f'') \subseteq \text{supp}(f') \cup \text{supp}(f'') \). On the other hand, \( \text{supp}(f') \cap (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) = \emptyset \), hence \( \text{supp}(f' - f'') \cap (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) = \emptyset \), which contradicts the fact that \( \{g_1, \ldots, g_s\} \) is a Gröbner basis of \( I \). \( \square \)

**Lemma 1.3.18.** If \( \{g_1, \ldots, g_s\} \) is a Gröbner basis of \( I = \langle g_1, \ldots, g_s \rangle \), then a non-zero polynomial \( f \) of \( S \) belongs to \( I \) if and only if the unique remainder of \( f \) with respect to \( g_1, \ldots, g_s \) is 0.

**Proof.** If there exists a standard expression for \( f \) with remainder 0, then it is clear that \( f \in I \).

Suppose \( f \in I \) is a non-zero polynomial with standard expression \( f = f_1g_1 + \cdots + f_sg_s + f' \) such that \( f' \neq 0 \). Then \( \text{supp}(f') \cap (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) = \emptyset \) and in particular \( \text{in}_<(f') \notin (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) \). On the other hand, \( f' \in I \), which contradicts the fact that \( \{g_1, \ldots, g_s\} \) is a Gröbner basis of \( I \). \( \square \)

**Proposition 1.3.19.** Let \( I \) be a non-zero ideal of \( S \) and \( < \) a monomial order on \( S \). Then the set of monomials which do not belong to \( \text{in}_<(I) \) form a \( K \)-basis of \( S/I \).

**Proof.** Let \( \{g_1, \ldots, g_s\} \) be a Gröbner basis for \( I \). Let \( f \in S \setminus I \) and let \( f' \) be a remainder of \( f \) with respect to \( g_1, \ldots, g_s \). Then \( f + I = f' + I \) and \( \text{supp}(f') \cap \text{in}_<(I) = \emptyset \), hence \( f + I \) is a linear combination of monomials which do not belong to \( \text{in}_<(I) \). Hence the set of monomials which do not belong to \( \text{in}_<(I) \) generates \( S/I \) as a \( K \)-vector space. Suppose there exists a non-zero linear combination \( f = \sum a_uu \in I \) of monomials which do not belong to \( \text{in}_<(I) \). Then \( \text{in}_<(f) \) is a monomial in such linear combination belonging to \( \text{in}_<(I) \), a contradiction. Hence the set of monomials which do not belong to \( \text{in}_<(I) \) is \( K \)-linearly independent as a subset of \( S/I \). \( \square \)

**Proposition 1.3.20.** Let \( I \subset J \) be non-zero ideals of \( S \) and let \( < \) and \( <' \) be monomial orders on \( S \). Then:
• $\text{in}(I) \subset \text{in}(J)$ and $\text{in}(I) \neq \text{in}(J)$;

• $\text{in}(I) \subset \text{in}(I)$ implies $\text{in}(I) = \text{in}(I)$.

**Proof.** It is obvious that $\text{in}(I) \subset \text{in}(J)$. Let $f \in J \setminus I$. Let $f'$ be the remainder of $f$ with respect to a Gröbner basis for $I$. Then $f' \neq 0$ and $\text{supp}(f') \cap \text{in}(I) = \emptyset$, therefore $\text{in}(f') \in \text{in}(J) \setminus \text{in}(I)$. Hence $\text{in}(I) \neq \text{in}(J)$.

For the second part, recall that both sets $\text{Mon}(S) \setminus \text{Mon}(\text{in}(I))$ and $\text{Mon}(S) \setminus \text{Mon}(\text{in}(I))$ are $K$-bases for $S/I$, and since $\text{Mon}(S) \setminus \text{Mon}(\text{in}(I)) \subset \text{Mon}(S) \setminus \text{Mon}(\text{in}(I))$, such bases must coincide, that is, $\text{in}(I) = \text{in}(I)$. \hfill \Box

**Definition 1.3.21.** A Gröbner basis $\mathcal{G} = \{g_1, \cdots, g_s\}$ is called reduced if the following conditions are satisfied:

• The coefficient of $\text{in}(g_i)$ in $g_i$ is 1 for all $1 \leq i \leq s$;

• If $i \neq j$, then none of the monomials of $\text{supp}(g_i)$ is divisible by $\text{in}(g_i)$.

From the second condition of definition 1.3.21 it follows that if $\mathcal{G} = \{g_1, \cdots, g_s\}$ is a reduced Gröbner basis of an ideal $I$, then $G(\text{in}(I)) = \{\text{in}(g_1), \cdots, \text{in}(g_s)\}$.

**Theorem 1.3.22.** A reduced Gröbner basis exists and is uniquely determined.

**Proof.** Let $I$ be a non-zero ideal of $S$ and let $G(\text{in}(I)) = \{u_1, \cdots, u_s\}$. For each $1 \leq i \leq s$ we choose a polynomial $g_i \in I$ with $\text{in}(g_i) = u_i$.

Let $g_i = f_2 g_2 + \cdots + f_s g_s + h_1$ be a standard expression of $g_i$ with respect to $g_2, \cdots, g_s$. Since $\text{in}(g_i) \geq \text{in}(f_i g_i)$ for $2 \leq i \leq s$, it follows that $\text{in}(g_i)$ coincides with some of the monomials $\text{in}(f_2)$, $\text{in}(f_3)$, $\ldots$, $\text{in}(f_s)$, $\text{in}(g_2)$, $\text{in}(h_1)$.

Since $u_1 = \text{in}(g_1)$ is not divisible by any of the monomials $\text{in}(g_2), \cdots, \text{in}(g_s)$, one has $\text{in}(h_1) = \text{in}(g_1)$. Hence $\{h_1, g_2, \cdots, g_s\}$ is a Gröbner basis of $I$. Since $h_1$ is a remainder of $g_1$ with respect to $g_2, \cdots, g_s$, then each monomial in $\text{supp}(h_1)$ is not divisible by any of the monomials $\text{in}(g_2), \cdots, \text{in}(g_s)$.

Similarly, if $h_2$ is a remainder of $g_2$ with respect to $h_1, g_3, \cdots, g_s$, then $\text{in}(h_2) = \text{in}(g_2)$ and each monomial in $\text{supp}(h_2)$ is not divisible by any of the monomials $\text{in}(h_1), \text{in}(g_3), \cdots, \text{in}(g_s)$. Moreover, $\{h_1, h_2, g_3, \cdots, g_s\}$ is a Gröbner basis of $I$. Since $\text{in}(h_2) = \text{in}(g_2)$, each monomial in $\text{supp}(h_1)$ is not divisible by any of the monomials $\text{in}(h_2), \text{in}(g_3), \cdots, \text{in}(g_s)$. Continuing these procedures yields polynomials $h_3, \cdots, h_s$ such that $\{h_1, h_2, h_3, \cdots, h_s\}$ is a Gröbner basis satisfying the second condition of definition 1.3.21.

Dividing each $h_i$ by the coefficient of $\text{in}(h_i)$, we obtain a reduced Gröbner basis of $I$.

Let $\{g_1, \cdots, g_s\}$ and $\{h_1, \cdots, h_t\}$ be two reduced Gröbner bases of $I$. Then

$$G(\text{in}(I)) = \{\text{in}(g_1), \cdots, \text{in}(g_s)\} = \{\text{in}(h_1), \cdots, \text{in}(h_t)\},$$
thus we may assume that \( s = t \) and \( \text{in}_<(g_i) = \text{in}_<(h_i) \) for \( i \in [s] \). If \( g_i \neq h_i \), then \( g_i - h_i \in I \setminus \{0\} \) and \( \text{in}_<(g_i - h_i) < \text{in}_<(g_i) \) and in particular \( \text{in}_<(g_i) \) cannot divide \( \text{in}_<(g_i - h_i) \). Since \( \text{in}_<(g_i - h_i) \in \text{supp}(g_i) \cup \text{supp}(h_i) \), it follows that \( \text{in}_<(g_i - h_i) \) is not divisible by any \( \text{in}_<(g_j) = \text{in}_<(h_j) \) with \( j \neq i \). Hence \( \text{in}_<(g_i - h_i) \notin \text{in}_<(I) \), which contradicts \( g_i - h_i \in I \).

**Notation 1.3.23.** We write \( G_{\text{red}}(I; <) \) for the reduced Gröbner basis of \( I \) with respect to \( < \).

**Corollary 1.3.24.** Let \( I \) and \( J \) be non-zero ideals of \( S \). Then \( I = J \) if and only if \( G_{\text{red}}(I; <) = G_{\text{red}}(J; <) \).

**Proof.** Just recall that the Gröbner basis of a given ideal is a set of generators for that ideal.

Let \( f \) and \( g \) be non-zero polynomials of \( S \). Let \( c_f \) denote the coefficient of \( \text{in}_<(f) \) in \( f \) and let \( c_g \) denote the coefficient of \( \text{in}_<(g) \) in \( g \). The polynomial
\[
S(f, g) = \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_f \text{in}_<(f)} \cdot f - \frac{\text{lcm}(\text{in}_<(f), \text{in}_<(g))}{c_g \text{in}_<(g)} \cdot g
\]
is called the \( S \)-polynomial of \( f \) and \( g \).

**Lemma 1.3.25.** Let \( f \) and \( g \) be non-zero polynomials and suppose that \( \gcd(\text{in}_<(f), \text{in}_<(g)) = 1 \). Then \( S(f, g) \) reduces to \( 0 \) with respect to \( f, g \).

**Proof.** To simplify notation we will assume that each of the coefficients of \( \text{in}_<(f) \) in \( f \) and of \( \text{in}_<(g) \) in \( g \) is equal to \( 1 \). Let \( f = \text{in}_<(f) + f_1 \) and \( g = \text{in}_<(g) + g_1 \). Then \( S(f, g) = \text{in}_<(g) f - \text{in}_<(f) g = (g - g_1) f - (f - f_1) g = f_1 g - g_1 f \). Suppose \( \text{in}_<(f_1 g) = \text{in}_<(g_1 f) \). Then \( \text{in}_<(f_1) \text{in}_<(g) = \text{in}_<(g_1) \text{in}_<(f) \).

Since \( \gcd(\text{in}_<(f), \text{in}_<(g)) = 1 \), then \( \text{in}_<(f) \mid \text{in}_<(f_1) \) and so \( \text{in}_<(f) \leq \text{in}_<(f_1) \), which is absurd. Hence \( \text{in}_<(f_1 g) \neq \text{in}_<(g_1 f) \) and so \( \text{in}_<(S(f, g)) = \max\{\text{in}_<(f_1 g), \text{in}_<(g_1 f)\} \). Hence \( S(f, g) = f_1 g - g_1 f \) is a standard expression for \( S(f, g) \) with respect to \( f, g \) and so \( S(f, g) \) reduces to \( 0 \) with respect to \( f, g \).

**Lemma 1.3.26.** Let \( w \in \text{Mon}(S) \) and \( f_1, \ldots, f_s \in S \) be polynomials whose initial ideal is \( w \). Let \( g = \sum_{i=1}^s b_i f_i \) be a non-zero \( K \)-linear combination of \( f_1, \ldots, f_s \) (that is, \( b_1, \ldots, b_s \in K \)) and suppose that \( \text{in}_<(g) < w \). Then \( g \) is a linear combination of the \( S \)-polynomials \( S(f_j, f_k) \) with \( j, k \in [s] \).

**Proof.** Let \( c_i \) denote the coefficient of \( w \) in \( f_i \). Let \( g_i = f_i/c_i \). Then \( S(f_j, f_k) = g_j - g_k \) for \( j, k \in [s] \), hence \( \text{in}_<(S(f_j, f_k)) < w \).

Now
\[
g = \sum_{i=1}^s b_i f_i = \sum_{i=1}^s b_i c_i g_i = b_1 c_1 (g_1 - g_2) + (b_1 c_1 + b_2 c_2) (g_2 - g_3) + \cdots + (b_1 c_1 + \cdots + b_{s-1} c_{s-1}) (g_{s-1} - g_s) + (b_1 c_1 + \cdots + b_s c_s) g_s = \sum_{i=2}^s (b_1 c_1 + \cdots + b_{i-1} c_{i-1}) S(f_{i-1}, f_i) + (b_1 c_1 + \cdots + b_s c_s) g_s.
\]

Since \( \text{in}_<(g) < w \), then \( b_1 c_1 + \cdots + b_s c_s \) must be zero, hence
\[
g = \sum_{i=2}^s (b_1 c_1 + \cdots + b_{i-1} c_{i-1}) S(f_{i-1}, f_i),
\]
as desired.
Theorem 1.3.27 (Buchberger’s criterion). Let $I$ be a non-zero ideal and let $\mathcal{G} = \{g_1, \cdots, g_s\}$ be a system of generators of $I$. Then $\mathcal{G}$ is a Gröbner basis of $I$ if and only if, for all $i \neq j$, $S(g_i, g_j)$ reduces to $0$ with respect to $g_1, \cdots, g_s$.

Proof. For all $i \neq j$, $S(g_i, g_j) \in I$. Thus, if $\mathcal{G}$ is a Gröbner basis of $I$, by lemma 1.3.18 it follows that $S(g_i, g_j)$ reduces to $0$ with respect to $g_1, \cdots, g_s$.

Suppose that, for all $i \neq j$, $S(g_i, g_j)$ reduces to $0$ with respect to $g_1, \cdots, g_s$. If $f \in I \setminus \{0\}$, then we write $\mathcal{H}_f$ for the set of sequences $h = (h_1, \cdots, h_s)$ with each $h_i \in S$ such that $f = \sum_{i=1}^{s} h_i g_i$. We associate each sequence $h \in \mathcal{H}_f$ with the monomial $\delta_h = \max\{\text{in}_<(h_i g_i) : h_i g_i \neq 0\}$. Among such monomials $\delta_h$ with $h \in \mathcal{H}_f$, we are interested in the monomial $\delta_f = \min\{\delta_h : h \in \mathcal{H}_f\}$.

Since $\text{in}_<(f) \leq \delta_h$ for every $h \in \mathcal{H}_f$, then $\text{in}_<(f) \leq \delta_f$. It is not hard to show that $\mathcal{G}$ is a Gröbner basis of $I$ if $\text{in}_<(f) = \delta_f$ for every $f \in I \setminus \{0\}$. In fact, if $\text{in}_<(f) = \delta_f$ and $\delta_f = \delta_h$ with $h = (h_1, \cdots, h_s) \in \mathcal{H}_f$, then $\text{in}_<(f) = \text{in}_<(h_i g_i)$ for some $1 \leq i \leq s$, hence $\text{in}_<(f) \in \{\text{in}_<(g_1), \cdots, \text{in}_<(g_s)\}$.

So our goal is to show that $\text{in}_<(f) = \delta_f$ for every $f \in I \setminus \{0\}$. Suppose there exists $f \in I \setminus \{0\}$ such that $\text{in}_<(f) < \delta_f$ and choose a sequence $h = (h_1, \cdots, h_s) \in \mathcal{H}_f$ with $\delta_f = \delta_h$. Then

$$f = \sum_{\text{in}_<(h_i g_i) = \delta_f} h_i g_i + \sum_{\text{in}_<(h_i g_i) < \delta_f} h_i g_i = \sum_{\text{in}_<(h_i g_i) = \delta_f} c_i \text{in}_<(h_i g_i) + \sum_{\text{in}_<(h_i g_i) = \delta_f} (h_i - c_i \text{in}_<(h_i)) g_i + \sum_{\text{in}_<(h_i g_i) < \delta_f} h_i g_i,$$

where $c_i$ is the coefficient of $\text{in}_<(h_i g_i)$ in $h_i$. Since $\text{in}_<(f) < \delta_f$ and

$$\text{in}_<\left(\sum_{\text{in}_<(h_i g_i) = \delta_f} (h_i - c_i \text{in}_<(h_i)) g_i + \sum_{\text{in}_<(h_i g_i) < \delta_f} h_i g_i\right) < \delta_f,$$

it follows that

$$\text{in}_<\left(\sum_{\text{in}_<(h_i g_i) = \delta_f} c_i \text{in}_<(h_i) g_i\right) < \delta_f.$$

By lemma 1.3.26

$$\sum_{\text{in}_<(h_i g_i) = \delta_f} c_i \text{in}_<(h_i) g_i$$

is a linear combination of those $S$-polynomials $S(\text{in}_<(h_j g_j), \text{in}_<(h_k g_k))$ with $\text{in}_<(h_j g_j) = \text{in}_<(h_k g_k) = \delta_f$. One can easily check that

$$S(\text{in}_<(h_j g_j), \text{in}_<(h_k) g_k) = (\delta_f/\text{lcm}(\text{in}_<(g_j), \text{in}_<(g_k))) S(g_j, g_k).$$

Let $u_{jk} = \delta_f/\text{lcm}(\text{in}_<(g_j), \text{in}_<(g_k))$. It then follows that there exists an expression of the form

$$\sum_{\text{in}_<(h_i g_i) = \delta_f} c_i \text{in}_<(h_i) g_i = \sum_{j,k} c_{jk} u_{jk} S(g_j, g_k),$$

with $c_{jk} \in K$ and $\text{in}_<(u_{jk} S(g_j, g_k)) < \delta_f$ since $\text{in}_<(S(g_j, g_k)) < \text{lcm}(\text{in}_<(g_j), \text{in}_<(g_k))$.
Since each \( S(g_j, g_k) \) reduces to 0 with respect to \( g_1, \ldots, g_s \), then there exists an expression of the form

\[
S(g_j, g_k) = \sum_{i=1}^{s} p_i^{jk} g_i,
\]

with \( p_i^{jk} \in S \) and \( \text{in}_<(p_i^{jk} g_i) \leq \text{in}_<(S(g_j, g_k)) \).

Combining these two equalities yields

\[
\sum_{\text{in}_<(h_i) = \delta_f} c_i \text{in}_<(h_i) g_i = \sum_{i=1}^{s} \left( \sum_{j,k} c_{jk} u_{jk} p_i^{jk} \right) g_i.
\]

If we define \( h_i' = \sum_{j,k} c_{jk} u_{jk} p_i^{jk} \) for \( 1 \leq i \leq s \), then \( \text{in}_<(h_i' g_i) < \delta_f \) since \( \text{in}_<(p_i^{jk} g_i) \leq \text{in}_<(S(g_j, g_k)) < \text{lcm}(\text{in}_<(g_j), \text{in}_<(g_k)) \) for every \( j, k \). Consequently, the polynomial \( f \) finally can be expressed as \( f = \sum_{i=1}^{s} h_i'' g_i \), with \( \text{in}_<(h_i'' g_i) < \delta_f \). The existence of such an expression contradicts the definition of \( \delta_f \).

Combining lemma 1.3.25 with Buchberger’s criterion we get:

**Corollary 1.3.28.** If \( g_1, \ldots, g_s \) are non-zero polynomials of \( S \) such that \( \gcd(\text{in}_<(g_1), \text{in}_<(g_j)) = 1 \) for every \( i \neq j \), then \( \{g_1, \ldots, g_s\} \) is a Gröbner basis of \( I = (g_1, \ldots, g_s) \).

**Proposition 1.3.29.** Let \( < \) be a monomial order on \( K[x_1, x_2] \) and let \( <_1 \) and \( <_2 \) be the monomial orders on \( K[x_1] \) and \( K[x_2] \), respectively, induced by \( < \). For \( i \in \{1, 2\} \), let \( I_i \) be an ideal of \( K[x_i] \) with Gröbner basis \( G_i \) with respect to \( <_i \). Then \( G_1 \cup G_2 \) is a Gröbner basis of \( I_1 + I_2 \subset K[x_1, x_2] \).

**Proof.** Since \( G_1 \) and \( G_2 \) generate \( I_1 \) and \( I_2 \), respectively, then \( G_1 \cup G_2 \) generates \( I_1 + I_2 \). Since \( G_1 \) is a Gröbner basis of \( I_1 \), Buchberger’s criterion implies that, for every \( f, g \in G_1 \), \( S(f, g) \) reduces to 0 with respect to \( G_1 \). Similarly, for every \( f, g \in G_2 \), \( S(f, g) \) reduces to 0 with respect to \( G_2 \). To show that \( G_1 \cup G_2 \) is a Gröbner basis of \( I_1 + I_2 \), it remains to check that if \( f \in G_1 \) and \( g \in G_2 \), then \( S(f, g) \) reduces to 0 with respect to \( G_1 \cup G_2 \). But in this case \( \text{in}_<(f) \) and \( \text{in}_<(g) \) have no common factors and by lemma 1.3.25 the result follows.

The Buchberger criterion supplies an algorithm to compute a Gröbner basis of an ideal \( I \) starting from a system of generators of \( I \).

Let \( \{g_1, \ldots, g_s\} \) be a system of generators of a non-zero ideal \( I \) of \( S \). Compute the \( S \)-polynomials \( S(g_i, g_j) \). If all \( S(g_i, g_j) \) reduce to 0 with respect to \( g_1, \ldots, g_s \), then, by the Buchberger criterion, \( \{g_1, \ldots, g_s\} \) is a Gröbner basis. Otherwise one of the \( S(g_i, g_j) \) has a nonzero remainder \( g_{s+1} \). and so none of the monomials \( \text{in}_<(g_1), \ldots, \text{in}_<(g_s) \) divides \( \text{in}_<(g_{s+1}) \). In other words, the inclusion \( (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) \subset (\text{in}_<(g_1), \ldots, \text{in}_<(g_s), \text{in}_<(g_{s+1})) \) is strict.

Notice that \( g_{s+1} \in I \). Now we replace \( \{g_1, \ldots, g_s\} \) by \( \{g_1, \ldots, g_s, g_{s+1}\} \) and compute all the \( S \)-polynomials for this new system of generators. If all \( S \)-polynomials reduce to 0 with respect to \( g_1, \ldots, g_s, g_{s+1} \), then \( \{g_1, \ldots, g_s, g_{s+1}\} \) is a Gröbner basis. Otherwise there is a non-zero remainder \( g_{s+2} \) and we obtain the new system of generators \( \{g_1, \ldots, g_{s+1}, g_{s+2}\} \), and the inclusion

\[
(\text{in}_<(g_1), \ldots, \text{in}_<(g_s), \text{in}_<(g_{s+1})) \subset (\text{in}_<(g_1), \ldots, \text{in}_<(g_{s+1}), \text{in}_<(g_{s+2}))
\]
Example 1.3.30. Let $S = K[x_1, \cdots, x_7]$ and let $\prec$ be the lexicographic order on $S$ induced by $x_1 > \cdots > x_7$. Let $f = x_1 x_4 - x_2 x_3$ and $g = x_4 x_7 - x_5 x_6$ with their initial monomials $\text{in}_\prec(f) = x_1 x_4$ and $\text{in}_\prec(g) = x_4 x_7$. Let $I = (f, g)$. Then $\{f, g\}$ is not a Gröbner basis of $I$ with respect to $\prec$. Now, as a remainder of $S(f, g) = x_7 f - x_1 g = x_1 x_5 x_6 - x_2 x_3 x_7$ with respect to $f$ and $g$, we choose $S(f, g)$ itself. Let $h = S(f, g)$ with $\text{in}_\prec(h) = x_1 x_5 x_6$. Then $\gcd(\text{in}_\prec(g), \text{in}_\prec(h)) = 1$. On the other hand, $S(f, h) = x_2 x_3 g$ reduces to 0 with respect to $f, g, h$. It follows from the Buchberger criterion that $\{f, g, h\}$ is a Gröbner basis of $I$ with respect to $\prec$.

Example 1.3.31. Let $S = K[x_1, \cdots, x_7]$ and let $\prec$ be the reverse lexicographic order on $S$ induced by $x_1 > \cdots > x_7$. Let $f$ and $g$ as in the previous example. In this case, $\text{in}_\prec(f) = x_2 x_3$ and $\text{in}_\prec(g) = x_5 x_6$, therefore $\gcd(\text{in}_\prec(f), \text{in}_\prec(g)) = 1$, hence $\{f, g\}$ is a Gröbner basis of $I = (f, g)$. As a consequence of this, we see that a Gröbner basis with respect to a given monomial order is not necessarily a Gröbner basis for every other monomial order.

Definition 1.3.32. Let $t \in [n]$. An elimination order on $S$ for $x_1, \cdots, x_t$ is a monomial order $\prec$ on $S$ which satisfies the following condition: for any two monomials $u, v \in S$ such that $x_j \mid u$ for some $j \in [t]$ and $x_j \nmid v$ for all $j \in [t]$ one has $u > v$.

In other words, $\prec$ is an elimination order for the first $t$ variables if any monomial which is divisible by some variable $x_j$ with $j \in [t]$ is strictly greater than any monomial in the last $n - t$ variables. Obviously, this is equivalent to saying that if a polynomial $f \in S$ has the property that its initial monomial with respect to $\prec$ belongs to the subring $K[x_{t+1}, \cdots, x_n]$, then $f \in K[x_{t+1}, \cdots, x_n]$.

Example 1.3.33. For every $t \in [n-1]$, the pure lexicographic order on $S$ induced by the ordering $x_1 > \cdots > x_n$ is an elimination order for $x_1, \cdots, x_t$.

Example 1.3.34. The lexicographic order induced by the ordering $x_1 > \cdots > x_n$ is not an induced order for $x_1, \cdots, x_t$. In fact, $f = x_t + x_{t+1}^2$ is such that $\text{in}_\prec(f) = x_{t+1}^2 \in K[x_{t+1}, \cdots, x_n]$ but $f \notin K[x_{t+1}, \cdots, x_n]$.

Example 1.3.35. Given two monomial orders $\prec_1$ and $\prec_2$ on $K[x_1, \cdots, x_t]$ and $K[x_{t+1}, \cdots, x_n]$, respectively, we can construct a monomial order $\prec$ induced by these two orders. In fact, define $\prec$ as following: let $u, v \in \text{Mon}(S)$ and let $u = u_1 u_2$ and $v = v_1 v_2$, with $u_1, v_1 \in K[x_1, \cdots, x_t]$ and $u_2, v_2 \in K[x_{t+1}, \cdots, x_n]$. Then $u \prec v$ if and only if $u_1 < v_2$ or $u_1 = u_2$ and $v_1 \prec v_2$. 

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Theorem 1.3.36 (Elimination theorem). Let \( I \subseteq S \) be an ideal and let \( t \in [n] \). If \( \mathcal{G} \) is a Gröbner basis of \( I \) with respect to some elimination order \( < \) for \( x_1, \ldots, x_t \), then \( \mathcal{G} \cap K[x_{t+1}, \ldots, x_n] \) is a Gröbner basis of \( I \cap K[x_{t+1}, \ldots, x_n] \) with respect to the induced order on the subring \( K[x_{t+1}, \ldots, x_n] \).

Proof. Let \( \mathcal{G} = \{g_1, \ldots, g_s\} \) and assume that \( \mathcal{G} \cap K[x_{t+1}, \ldots, x_n] = \{g_1, \ldots, g_r\} \). By the choice of the monomial order, we have \( \text{in}_<(g_j) \notin K[x_{t+1}, \ldots, x_n] \) for all \( j > r \). We show that the set \( \{\text{in}_<(g_1), \ldots, \text{in}_<(g_r)\} \) generates \( \text{in}_<(I \cap K[x_{t+1}, \ldots, x_n]) \). Let \( f \in I \cap K[x_{t+1}, \ldots, x_n] \) be a non-zero polynomial. Since \( \text{in}_<(f) \in \text{in}_<(I) \) and \( \text{in}_<(f) \) is a monomial in the last \( n-t \) variables, we must have \( \text{in}_<(g_j) \mid \text{in}_<(f) \) for some \( j \in [r] \), hence \( \text{in}_<(f) \in (\text{in}_<(g_1), \ldots, \text{in}_<(g_r)) \). Therefore we have \( \text{in}_<(I \cap K[x_{t+1}, \ldots, x_n]) \subseteq (\text{in}_<(g_1), \ldots, \text{in}_<(g_r)) \). The other inclusion is obvious. \( \square \)

Corollary 1.3.37. Let \( I \subseteq S \) be an ideal and let \( t \in [n] \). Then \( \text{in}_<(I \cap K[x_{t+1}, \ldots, x_n]) = \text{in}_<(I) \cap K[x_{t+1}, \ldots, x_n] \) for any elimination order \( < \) on \( S \) for \( x_1, \ldots, x_t \).

The following results will be useful when studying binomial edge ideals:

Theorem 1.3.38. If \( I \subseteq S \) is a graded ideal and \( < \) is a monomial order on \( S \), then:

- \( \dim(S/I) = \dim(S/\text{in}_<(I)) \);
- \( \text{projdim}(S/I) \leq \text{projdim}(S/\text{in}_<(I)) \);
- \( \text{reg}(S/I) \leq \text{reg}(S/\text{in}_<(I)) \);
- \( \text{depth}(S/I) \geq \text{depth}(S/\text{in}_<(I)) \).

Proof. See [5, 3.3]. \( \square \)

Corollary 1.3.39. Let \( I \) be a graded ideal such that \( \text{in}_<(I) \) is a Cohen-Macaulay ideal. Then \( I \) is also Cohen-Macaulay.

Proof. By theorem 1.3.38, \( \text{depth}(S/\text{in}_<(I)) \leq \text{depth}(S/I) \) and \( \dim(S/I) = \dim(S/\text{in}_<(I)) \). But since \( \text{in}_<(I) \) is Cohen-Macaulay, then \( \text{depth}(S/\text{in}_<(I)) = \dim(S/\text{in}_<(I)) \). Hence \( \dim(S/I) \leq \text{depth}(S/I) \) and so \( I \) is Cohen-Macaulay. \( \square \)

Proposition 1.3.40. If \( I \subseteq S \) is a graded ideal and \( \text{in}_<(I) \) is a square-free monomial ideal, then \( I \) is a radical ideal.

Proof. See [5, 3.3]. \( \square \)

Proposition 1.3.41. Let \( I \) be an ideal with graded Gröbner basis \( I = (g_1, \ldots, g_s) \) such that \( \text{in}_<(I) = (\text{in}_<(g_1), \ldots, \text{in}_<(g_s)) \) is a complete intersection. Then \( I = (g_1, \ldots, g_s) \) is also a complete intersection.

Proof. See [5, 3.3]. \( \square \)
Chapter 2

Binomial edge ideals and closed graphs

The algebraic properties of determinantal ideals have been explored by considering their initial ideal, which for a suitable monomial order is a square-free monomial ideal and hence is accessible to powerful techniques. In [8] such properties are studied and in particular it is shown that determinantal ideals are Cohen-Macaulay prime ideals.

Determinantal ideals of 2-minors of a $2 \times n$ matrix of indeterminates have a natural generalization: binomial edge ideals. As shown in [11], binomial edge ideals are radical ideals and so they are the intersection of their minimal primes. In section 2.2, we determine such minimal primes in terms of the algebraic properties of the associated graphs.

Similar to what happens with monomial edge ideals, a general classification of Cohen-Macaulay binomial edge ideals seems to be hopeless. Thus we have to restrict our attention to special classes of graphs. In section 2.3 we consider the class of chordal graphs such that any two distinct maximal cliques intersect in at most one vertex (this class includes in particular all forests) and in section 2.4 we consider the class of closed graphs. More precisely, a graph is closed if and only if the standard generators of its binomial edge ideal form a Gröbner basis. If $G$ is a closed graph, then $\text{in}_{<}(J_G)$, where $<$ is the lexicographic order, is the monomial edge ideal of a bipartite graph $H$. This motivates the study of monomial edge ideals in section 2.1 and, in particular, the study of Cohen-Macaulay bipartite graphs.

2.1 Monomial edge ideals and bipartite graphs

Monomial edge ideals are the simplest class of ideals that can be built of graphs. The same way a Cohen-Macaulay simplicial complex is defined as a simplicial complex whose Stanley-Reisner ideal is Cohen-Macaulay, a Cohen-Macaulay graph will be defined as a graph whose monomial edge ideal is Cohen-Macaulay.

In this section we classify Cohen-Macaulay bipartite graphs, and such classification will provide us a
classification of Cohen-Macaulay binomial edge ideals of closed graphs in section 2.4.

Let $G$ be a graph on $[n]$ and let, as usual, $S = K[x_1, \ldots, x_n]$ be the polynomial ring in $n$ variables over a field $K$.

**Definition 2.1.1.** The monomial edge ideal of $G$ is the ideal $I(G)$ of $S$ generated by all square-free monomials $x_ix_j$ such that $\{i, j\} \in E(G)$. If $G$ is a discrete graph, we set $I(G) = (0)$.

Hence monomial edge ideals are Stanley-Reisner ideals generated by degree two monomials.

**Proposition 2.1.2.** A subset $C \subset [n]$ is a vertex cover of $G$ if and only if the monomial prime ideal $P_C$ contains $I(G)$.

**Proof.** Suppose $C$ is a vertex cover of $G$. Let $\{i, j\} \in E(G)$. Since $C$ is a vertex cover of $G$, then $i \in C$ or $j \in C$ and so $x_i \in P_C$ or $x_j \in P_C$, hence $x_ix_j \in P_C$. Since $\{i, j\} \in E(G)$ is arbitrary, $I(G) \subset P_C$.

Suppose $I(G) \subset P_C$. Let $\{i, j\} \in E(G)$. Then $x_ix_j \in I(G)$ and so $x_ix_j \in P_C$. Since $P_C$ is a prime ideal, then $x_i \in P_C$ or $x_j \in P_C$, and so $i \in C$ or $j \in C$. Since $\{i, j\} \in E(G)$ is arbitrary, $C$ is vertex cover of $G$.

Since $I(G)$ is a square-free monomial ideal, then by corollary 1.1.30 $I(G)$ is the intersection of its minimal prime ideals, which are monomial ideals. Moreover, thanks to the above proposition it is possible to state combinatorial properties of a graph $G$ as algebraic properties of $I(G)$.

**Corollary 2.1.3.** A subset $C \subset [n]$ is a minimal vertex cover of $G$ if and only if $P_C$ is a minimal prime ideal of $I(G)$.

Thanks to this corollary, we can determine the minimal vertex covers of $G$ using Macaulay2: we just have to use Macaulay2 to determine the minimal prime ideals of $I(G)$ (and it makes no difference which base field we consider).

**Corollary 2.1.4.** For every graph $G$, one has $\alpha_0(G) = \height(I(G))$.

**Proof.** By proposition 1.1.35 one has $\height(P_C) = |C|$ for every subset $C \subset [n]$.

Hence corollary 2.1.3 implies that $\height(I(G)) = \min\{\height(P_C) : C \text{ is a minimal vertex cover of } G\} = \min\{|C| : C \text{ is a minimal vertex cover of } G\} = \alpha_0(G)$.

**Corollary 2.1.5.** For every graph $G$, one has $\beta_0(G) = \dim(S/I(G))$.

**Proof.** By corollary 0.7.54 one has $\alpha_0(G) + \beta_0(G) = n$ and by proposition 0.5.12 one has $\height(I(G)) + \dim(S/I(G)) = n$. Combining these two equalities with corollary 2.1.4 it follows that $\beta_0(G) = \dim(S/I(G))$.

**Corollary 2.1.6.** A graph $G$ is unmixed if and only if the ideal $I(G)$ is unmixed.

**Proof.** Since $I(G)$ is the intersection of its minimal primes, then $\text{Ass}(S/I(G)) = \text{Min}(I(G)) = \{P_C : C \text{ is a minimal vertex cover of } G\}$. Combining corollary 2.1.4 and proposition 1.1.35 the result follows.
Example 2.1.7. Let \( G \) be the graph as below.

Using Macaulay2 we find out that the minimal prime ideals of \( I(G) \) are the following:

\[
\{x_1, x_2, x_3, x_5\}, \{x_1, x_2, x_4, x_5\}, \{x_1, x_2, x_3, x_6\}, \{x_1, x_2, x_4, x_6\}, \{x_1, x_3, x_5, x_6\}, \{x_2, x_3, x_4, x_5\}.
\]

Hence \( G \) has the following minimal vertex covers:

\[
\{1, 2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 2, 3, 6\}, \{1, 2, 4, 6\}, \{1, 3, 4, 5, 6\}, \{2, 3, 4, 5, 6\}.
\]

Hence it is not unmixed.

Definition 2.1.8. A graph \( G \) is called weakly chordal if all cycles of length greater than 4 in \( G \) and in \( \overline{G} \) have a chord.

If \( G \) is a weakly chordal graph, then its induced matching number is given by the regularity of the ideal \( I(G) \).

Theorem 2.1.9. If \( G \) is a weakly chordal graph on the vertex set \([n]\), then

\[
\text{reg}(K[x_1, \ldots, x_n]/I(G)) = \text{indmatch}(G).
\]

Proof. See \([14]\).

Definition 2.1.10. A graph \( G \) is called Cohen-Macaulay over a field \( K \) if \( I(G) \) is a Cohen-Macaulay ideal.

Proposition 2.1.11. If \( G \) is a Cohen-Macaulay graph, then \( G \) is unmixed.

Proof. If \( G \) is a Cohen-Macaulay graph, then \( I(G) \) is a Cohen-Macaulay ideal, therefore it is unmixed by corollary \(0.5.13\) therefore \( G \) is unmixed.

Example 2.1.12. The graph \( G \) as in the previous example is not unmixed, therefore it is not Cohen-Macaulay.

Proposition 2.1.13. A graph is Cohen-Macaulay if and only if its connected components are all Cohen-Macaulay.

Proof. Just use corollary \(0.5.16\) and induction.

Now suppose \( G \) is a graph on \([n]\) and that \( i \in [n] \) is an isolated vertex of \( G \). Let \( G' \) be the induced subgraph of \( G \) on \([n] \setminus \{i\}\). As a corollary of this proposition, \( G \) is Cohen-Macaulay over \( K \) if and only if
$G'$ is Cohen-Macaulay over $K$. Hence, when studying which graphs are Cohen-Macaulay or not, we can always assume that $G$ has no isolated vertices. Our main goal in this section is to classify Cohen-Macaulay bipartite graphs.

Let $P = \{p_1, \ldots, p_n\}$ be a finite partially ordered set (poset for short) with a partial order $\leq$ and let $V_n = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. The bipartite graph on $V_n$ whose edges are the two-element subset $\{x_i, y_j\}$ such that $p_i \leq p_j$ is denoted $G(P)$.

**Definition 2.1.14.** We say that a bipartite graph $G$ with bipartition $(V, V')$ comes from a poset if $|V| = |V'|$ and if there is a finite poset on $[n]$, where $n = |V|$, such that after relabelling the vertices of $G$ one has $G(P) = G$.

**Proposition 2.1.15.** A bipartite graph coming from a poset is Cohen-Macaulay.

**Proof.** See [5, 9.1] (the proof of this proposition uses local cohomology).

**Proposition 2.1.16.** Let $G$ be a bipartite graph with bipartition $(V, V')$. Then $G$ is a Cohen-Macaulay graph if and only if $|V| = |V'|$ and the vertices $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ can be labelled such that:

- For every $i \in [n]$, $\{x_i, y_i\} \in E(G)$.
- If $i > j$, then $\{x_i, y_j\} \notin E(G)$.
- If $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$, then $\{x_i, y_k\} \in E(G)$.

**Proof.** Suppose $|V| = |V'|$ and $V = \{x_1, \ldots, x_n\}$ and $V' = \{y_1, \ldots, y_n\}$ can be labelled such that those three conditions hold. Let $P = \{p_1, \ldots, p_n\}$ be the poset with $p_i \leq p_j$ if and only if $\{x_i, y_j\} \in E(G)$. We need to check that $P$ is indeed a poset.

- If $i \in [n]$, then $\{x_i, y_i\} \in E(G)$ and so $p_i \leq p_i$.
- If $p_i \leq p_j$ and $p_j \leq p_i$, then $\{x_i, y_j\}, \{x_j, y_i\} \in E(G)$. If $i \neq j$, then either $i > j$ or $j > i$ and so $\{x_i, y_j\} \notin E(G)$ or $\{x_j, y_i\} \notin E(G)$, a contradiction. Hence $i = j$.
- If $p_i \leq p_j$ and $p_j \leq p_k$, that is, $\{x_i, y_j\} \in E(G)$ and $\{x_j, y_k\} \in E(G)$, then $\{x_i, y_k\} \in E(G)$, that is $p_i \leq p_k$.

Hence $P$ is a poset such that $G = G(P)$, therefore $G$ is a Cohen-Macaulay graph.

Suppose $G$ is a Cohen-Macaulay graph. By corollary 2.1.11, $G$ is unmixed. Since $G$ has no isolated vertices, both $V$ and $V'$ are minimal vertex covers of $G$, hence $|V| = |V'|$. Let $U$ be a subset of $V$. By proposition 0.7.46, $(V \setminus U) \cup N(U)$ is a vertex cover of $G$, and since $G$ is unmixed and $V$ is a minimal vertex cover of $G$, $|V| \leq |(V \setminus U) \cup N(U)| \leq |V \setminus U| + |N(U)| = |V| - |U| + |N(U)|$, hence $|N(U)| \geq |U|$. Now the marriage theorem (theorem 0.7.59) tells us that there exist labellings $W = \{x_1, \ldots, x_n\}$ and $W' = \{y_1, \ldots, y_n\}$ such that $\{x_i, y_i\} \in E(G)$ for every $i \in [n]$. 

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Let $\Delta = \Delta(G)$. Since $\{x_i, y_i\} \in E(G)$ for $i \in [n]$, then $\{x_i, y_i\} \notin \Delta$ for $i \in [n]$. In particular, this implies that $I_\Delta = I(G)$.

Since $G$ is a Cohen-Macaulay graph and $I_\Delta = I(G)$, it follows that $\Delta$ is a Cohen-Macaulay simplicial complex and proposition 1.2.27 implies $\Delta$ is strongly connected. Since $G$ has no isolated vertices, both $V$ and $V'$ are facets of $\Delta$.

On the other hand, since $\Delta$ is strongly connected, then there exist facets $V = F_0, \cdots, F_k = V'$ such that $F_i$ and $F_{i-1}$ intersect in codimension one for $i \in [k]$. This means that each set $F_i \setminus F_{i-1}$ has only one element, and since $(V, V')$ is a bipartition of $G$ such that $|V| = |V'| = n$, it follows that $k = n$.

Now $F_1 \setminus F_0$ has only one element, say $y_1$, and since $\{x_1, y_1\} \notin \Delta$, it follows that $F_1 = \{x_1, x_2, \cdots, x_n\}$. Similarly, $F_2 \setminus F_1$ has only one element, say $y_2$, and since $\{x_2, y_2\} \notin \Delta$, it follows that $F_2 = \{y_1, y_2, x_3, \cdots, x_n\}$.

Hence by induction we may assume that $F_i = \{y_1, \cdots, y_i, x_{i+1}, \cdots, x_n\}$ for $i \in [n-1]$. In particular, if $i > j$, then $\{x_i, y_j\} \subset F_j$ and so $\{x_i, y_j\} \in \Delta$, that is, $\{x_i, y_j\} \notin E(G)$.

By last, let $\{x_i, y_j\}, \{x_j, y_k\} \in E(G)$. Then $i \leq j \leq k$. If either $i = j$ or $j = k$, there is nothing to prove, so assume $i < j < k$. Suppose $\{x_i, y_k\} \notin E(G)$. Then $\{x_i, y_k\} \in \Delta$, thus there exists $F \in F(\Delta)$ such that $\{x_i, y_k\} \subset F$, and since $\Delta$ is pure and $V$ is a facet of $\Delta$, $|F| = n$. Since $\{x_i, y_j\} \in E(G)$, that is, $\{x_i, y_j\} \notin \Delta$, and $x_i \in F$, it follows that $y_j \notin F$. Similarly $y_k \in F$ implies $x_j \notin F$. On the other hand, since $\{x_i, y_i\} \in E(G)$ for every $l \in [n]$, then $\{x_i, y_l\} \notin \Delta$ for every $l \in [n]$ and so the facet $F$ never contains both $x_i$ and $y_l$, that is, $F = \{x_{i_1}, \cdots, x_{i_k}, y_{j_1}, \cdots, y_{j_{n-k}}\}$ with $\{i_1, \cdots, i_k, j_1, \cdots, j_{n-k}\} = [n]$. So in particular either $x_j \in F$ or $y_j \in F$, a contradiction.

Corollary 2.1.17. The Cohen-Macaulayness of a bipartite graph does not depend on the base field $K$.

Proposition 2.1.16 will essentially serve to classify Cohen-Macaulay binomial edge ideals of closed graphs in section 2.4. Nevertheless, we will end this section by studying the Cohen-Macaulayness of some bipartite graphs, such as paths and even cycles.

Corollary 2.1.18. If $G$ is a path of length $l$, then $G$ is Cohen-Macaulay if and only if $l \in \{1, 3\}$.

Proof. Note that a path is a bipartite graph.

It is clear that $G$ satisfies the conditions of proposition 2.1.16 if $l \in \{1, 3\}$.

If $l$ is even, then $G$ is a bipartite graph with an odd number of vertices, therefore it cannot be Cohen-Macaulay (in fact, it is not even unmixed).

Suppose now that $l = 2n - 1$ for $n > 2$ and $G$ is Cohen-Macaulay. Then there exists a labelling $V(G) = \{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$ satisfying the conditions of proposition 2.1.16. Since $y_1$ and $x_n$ are vertices of degree 1, then they must be the endpoints of $G$. Now $y_2$ can only be adjacent to $x_1$ and $x_2$, and since $y_2$ has degree 2, it follows that $\{x_1, y_2\}, \{x_2, y_2\} \in E(G)$. On the other hand, $y_3$ can only be adjacent to $x_1$, $x_2$ and $x_3$. Since $\{x_1, y_1\}, \{x_1, y_2\} \in E(G)$ and $G$ is a path, then $x_1$ cannot be adjacent to other vertices of $G$, hence $\{x_1, y_3\} \notin E(G)$, and since $y_3$ has degree 2, it follows that $\{x_2, y_3\}, \{x_3, y_3\} \in E(G)$. But $\{x_1, y_2\}, \{x_2, y_3\} \in E(G)$ implies $\{x_1, y_3\} \in E(G)$, a contradiction. □
In particular, paths of odd length greater than 3 are examples of unmixed graphs which are not Cohen-Macaulay.

Proposition 2.1.16 can be used in a simpler way for even cycles.

Corollary 2.1.19. If $G$ is an even cycle, then $G$ is not Cohen-Macaulay.

Proof. Suppose $G$ is Cohen-Macaulay with $|V(G)| = 2n$. Since $G$ is a bipartite graph, then there exists a labelling $V(G) = \{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\}$ as in the proposition 2.1.16. But in particular $y_1$ and $x_n$ are vertices of degree 1, which is an absurd since $G$ is a cycle.

In particular, if $G$ is a cycle of length 4, then $G$ is an unmixed graph which is not Cohen-Macaulay.

Proposition 2.1.16 cannot be used when $G$ is an odd cycle for in this case $G$ is not a bipartite graph.

2.2 Binomial edge ideals and their minimal primes

The class of binomial edge ideals is a natural generalization of the ideal of 2-minors of a $2 \times n$ matrix of indeterminates. Indeed, the ideal of 2-minors of a $2 \times n$ matrix of indeterminates may be interpreted as the binomial edge ideal of a complete graph on $[n]$. In the case of a path our binomial edge ideal may be interpreted as an ideal of adjacent minors. In this section, we classify the Cohen-Macaulayness of some binomial edge ideals and, for every graph $G$, we determine the minimal primes of $J_G$.

Definition 2.2.1. A binomial is a polynomial of the form $u - v$, where $u$ and $v$ are monomials.

Let $G$ be a graph on the vertex set $[n]$, let $K$ be a field and $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be the polynomial ring in $2n$ variables.

Notation 2.2.2. If $i, j \in [n]$ are such that $i < j$, we denote $f_{ij} = x_i y_j - x_j y_i$.

Definition 2.2.3. The binomial edge ideal $J_G \subset S$ is the ideal generated by the binomials $f_{ij} = x_i y_j - x_j y_i$ such that $i < j$ and $\{i, j\} \in E(G)$. If $G$ is a discrete graph, we set $J_G = (0)$.

Proposition 2.2.4. If $G$ and $G'$ are graphs on $[n]$ such that $J_G \subset J_{G'}$, then $G$ is a subgraph of $G'$.

Proof. Let $\{i, j\} \in E(G)$ with $i < j$. Then $f_{ij} \in J_G$, and so $f_{ij} \in J_{G'}$. Since $\deg(f_{ij}) = 2$ and the generators of $J_{G'}$ have degree 2, it follows that $f_{ij}$ is a $K$-linear combination of the binomials $f_{kl}$ with $k < l$ and $\{k, l\} \in E(G')$, say $f_{ij} = \sum_{(k, l) \in E(G')} a_{kl} f_{kl}$. Since the binomials $f_{kl}$ with $k < l$ are a $K$-linearly independent subset of $S$, it follows that $\{i, j\} \in E(G')$. Since $\{i, j\} \in E(G)$ is arbitrary, $E(G) \subset E(G')$. □
Corollary 2.2.5. If $G$ and $G'$ are graphs on $[n]$ such that $J_G = J_G'$, then $G$ and $G'$ are the same graph, that is, $E(G) = E(G')$.

Hence any graph is completely characterized by its binomial edge ideal.

Let $G$ be a graph on $[n]$. From now on $<$ will be the lexicographic order on $S = K[x_1, \ldots, x_n, y_1, \ldots, y_n]$ induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$. It is pertinent to ask for which graphs $G$ the standard generators of $J_G$ form a Gröbner basis with respect to $<$. These graphs are precisely the closed graphs.

Definition 2.2.6. A graph $G$ is closed if for all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ and $\{i, k\} \in E(G)$ if $j = l$.

Theorem 2.2.7. A graph $G$ is closed if and only if the generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis.

Proof. Suppose the generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis. Suppose $\{i, j\}$ and $\{i, k\}$ are edges with $i < j < k$. Since $f_{ik}, f_{ij} \in J_G$, then $S(f_{ik}, f_{ij}) \in J_G$, but $S(f_{ik}, f_{ij}) = y_if_{jk}$, hence $\text{in}_< S(f_{ik}, f_{ij}) = x_jy_ik$, therefore $x_j$ must be divisible by the initial monomial of a quadratic generator of $J_G$, and since $i < j < k$, such initial monomial can only be $x_jy_ik$, hence $f_{jk} \in J_G$, that is, $\{j, k\} \in E(G)$. The case where $\{i, j\}$ and $\{k, j\}$ are edges with $i < k < j$ is entirely analogous.

Suppose $G$ is closed. We apply Buchberger’s criterion and show that all $S$-pairs $S(f_{ij}, f_{kl})$, with $\{i, j\}, \{k, l\} \in E(G)$, $i < j$ and $k < l$, reduce to 0. If $i \neq k$ and $j \neq l$, then $\text{in}_< (f_{ij}) = x_jy_j$ and $\text{in}_< (f_{kl}) = x_ky_l$ have no common factor, hence lemma 1.3.25 implies $S(f_{ij}, f_{kl}) = 0$. On the other hand, if $i = k$, then $\{j, l\} \in E(G)$ and $S(f_{ij}, f_{kl}) = y_i(x_1y_j - x_jy_i)$ and so $S(f_{ij}, f_{kl})$ reduces to 0. Similarly, if $j = l$, then $\{i, k\} \in E(G)$ and $S(f_{ij}, f_{kl}) = x_j(x_1y_k - x_ky_1)$ and so $S(f_{ij}, f_{kl})$ reduces to 0. In both cases the $S$-pair reduces to 0.

Being closed does not only depend on the isomorphism type of the graph, but also on the labelling of its vertices.

Example 2.2.8. The graphs $G$ and $G'$ with 3 vertices and such that $E(G) = \{\{1, 2\}, \{2, 3\}\}$ and $E(G') = \{\{1, 2\}, \{1, 3\}\}$ are isomorphic, but $G$ is closed while $G'$ is not.
However, algebraic properties of $J_G$, where $G$ is closed, do not depend on the label considered but only on the isomorphism type of $G$, so most results shown for closed graphs will also hold for graphs that are isomorphic to a closed graph. The properties of closed graphs will be studied in section 2.4. We only introduced closed graphs in this section since we want to use the fact that complete graphs are closed graphs, which is equivalent to saying that if $G$ is a complete graph on $[n]$, then the binomials $f_{ij}$ with $1 \leq i < j \leq n$ form a Gröbner basis for $J_G$.

Let $G$ be a graph which is not necessarily closed. In [11], a reduced Gröbner basis for $J_G$ is determined.

**Definition 2.2.9.** Let $i$ and $j$ be two vertices in $G$ with $i < j$. A path $i = i_0, i_1, \cdots, i_r = j$ from $i$ to $j$ is called admissible if, for any proper subset $\{j_1, \cdots, j_s\}$ (which may or may not be empty) of $\{i_1, \cdots, i_{r-1}\}$, the sequence $i, j_1, \cdots, j_s, j$ is not a path.

In other words, a path from $i$ to $j$ is admissible if we cannot obtain a shorter path from $i$ to $j$ by removing vertices from the original path.

**Example 2.2.10.** Let $G$ be a cycle of length 4 with $E(G) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$. Then $1, 2, 3$ is an admissible path and $1, 2, 3, 4$ is not an admissible path. In fact, by removing the vertices 2 and 3 from the path $1, 2, 3, 4$ we get the path $1, 4$.

**Definition 2.2.11.** Given an admissible path $\pi : i = i_0, i_1, \cdots, i_r = j$ we associate the monomial

$$u_\pi = \left( \prod_{i_k > j} x_{i_k} \right) \left( \prod_{i_l < j} y_{i_l} \right), \text{ where } k, l \in \{0, \cdots, r\}.$$ 

**Theorem 2.2.12.** Let $G$ be a graph on $[n]$. The set of binomials

$$G = \bigcup_{i<j} \{u_\pi f_{ij} : \pi \text{ is an admissible path from } i \text{ to } j\}$$

is a reduced Gröbner basis for $J_G$.

**Proof.** See [11, 2].

**Proposition 2.2.13.** The binomial edge ideal $J_G$ is a radical ideal.

**Proof.** By proposition [1.3.40] it is enough to show that $\text{in}_<(J_G)$ is a square-free monomial ideal.

Since $G$ is a reduced Gröbner basis for $J_G$, then

$$G(\text{in}_<(J_G)) = \bigcup_{i<j} \{u_\pi x_i y_j : \pi \text{ is an admissible path from } i \text{ to } j\}.$$
The monomials $u_{x_i y_j}$ are all square-free, hence $\text{in}_{\mathcal{C}}(J_G)$ is a square-free monomial ideal, as desired.

Since $J_G$ is a radical ideal, then $J_G$ is the intersection of its minimal prime ideals. We want to determine such prime ideals. One starts by considering the case when $G$ is a complete graph. In this case, $S/J_G$ turns out to be a determinantal ring. The study of determinantal rings is beyond this dissertation, so only the necessary results will be stated.

**Definition 2.2.14.** Let $X$ be an $m \times n$ matrix of indeterminates over $K$ and let $K[X]$ be the polynomial ring whose variables are the entries of $X$. Let $t$ be an integer such that $1 \leq t \leq \max\{m, n\}$. Then $I_t(X)$ is the ideal generated by the $t$-minors of the matrix $X$ and the determinantal ring $R_t$ is the quotient $K[X]/I_t(X)$.

**Proposition 2.2.15.** The determinantal ring $R_t$ has dimension $\dim(R_t) = (t-1)(m+n-t+1)$.

**Proof.** See [8] 1.C.

**Proposition 2.2.16.** The determinantal ring $R_t$ is a Cohen-Macaulay domain, that is, $I_t(X)$ is a Cohen-Macaulay prime ideal.

**Proof.** See [8] 2.B.

**Corollary 2.2.17.** If $G$ is a complete graph, then $J_G$ is a Cohen-Macaulay prime ideal such that $\dim(S/J_G) = n+1$.

**Proof.** Just notice that $J_G = I_2(X)$, where $X = \begin{bmatrix} x_1 & \cdots & x_n \\ y_1 & \cdots & y_n \end{bmatrix}$.

**Corollary 2.2.18.** If $G$ is a complete graph, then $\text{ht}(J_G) = n-1$.

**Proof.** Just recall that, by proposition 0.5.12 one has $\text{ht}(J_G) + \dim(S/J_G) = 2n$.

**Proposition 2.2.19.** Let $f_1, \ldots, f_q \in K[x_1, \ldots, x_n]$ and let $\varphi : K[t_1, \ldots, t_q] \to K[x_1, \ldots, x_n]$ be the ring homomorphism induced by $\varphi(t_i) = f_i$ for $i \in [q]$. Then $\ker \varphi = (f_1 - t_1, \ldots, f_q - t_q) \cap K[t_1, \ldots, t_q]$.

**Proof.** See [7][8.2].

Let $\varphi : S \to K[t, z_1, \ldots, z_n]$ be the ring homomorphism induced by $\varphi(x_i) = z_i$ and $\varphi(y_i) = t z_i$. Then $\ker \varphi = I \cap S$ where $I = (z_1 - x_1, \ldots, z_n - x_n, t z_1 - y_1, \ldots, t z_n - y_n)$. Note that $t z_i - y_i = t(z_i - x_i) + (t x_i - y_i)$ for $i \in [q]$ and so $I = (z_1 - x_1, \ldots, z_n - x_n, t x_1 - y_1, \ldots, t x_n - y_n)$. Consider any elimination order $<$ on $K[t, z_1, \ldots, z_n, x_1, \ldots, x_n, y_1, \ldots, y_n]$ for $t, z_1, \ldots, z_n$ whose restriction to $S$ is the lexicographic order on $S$ induced by $x_1 > \cdots > x_n > y_1 > \cdots > y_n$ (in example 1.3.35 we saw how such an order can be obtained).

**Proposition 2.2.20.** The set $\mathcal{G} = \{z_1 - x_1, \ldots, z_n - x_n, t x_1 - y_1, \ldots, t x_n - y_n\} \cup \{f_{ij} : 1 \leq i < j \leq n\}$ is a Gröbner basis of $I$ with respect to the elimination order considered above.
Proof. We will use Buchberger’s criterion. Recall corollary 1.3.28: If \( g, h \) are distinct elements in \( \mathcal{G} \), then the only cases when \( \text{in}_< (g) \) and \( \text{in}_< (h) \) have common factors (and so \( S(g, h) \) may not reduce to 0 with respect to \( g, h \)) are the following cases:

- If \( g = tx_i - y_i \) and \( h = tx_j - y_j \) for \( i \neq j \). In this case, suppose without loss of generality that \( i < j \). Then \( \text{in}_< (g) = tx_i \) and \( \text{in}_< (h) = tx_j \), hence \( S(g, h) = x_jg - x_ih = f_{ij} \) and so \( S(g, h) \) reduces to 0 with respect to \( \mathcal{G} \).

- If, without loss of generality, \( g = tx_i - y_i \) and \( h = f_{ij} \) for \( i < j \). In this case \( S(g, h) = y_i(tx_j - y_j) \) and so \( S(g, h) \) reduces to 0 with respect to \( \{ f_{ij} : 1 \leq i < j \leq n \} \).

Hence \( \mathcal{G} \) is a Gröbner basis of \( I \).

From the elimination theorem (theorem 1.3.36) and these two propositions it follows that \( \{ f_{ij} : 1 \leq i < j \leq n \} \) is a Gröbner basis of \( \ker \varphi \) and in particular \( \ker \varphi = J_G \), where \( G \) is the complete graph on \( [n] \).

Now suppose \( G_1, \ldots, G_c \) are complete graphs on disjoint sets of vertices. Let \( i \in [c] \).

Let \( \varphi_i : S_i \to K[t_i, \{ z_{ij} \}_{j \in V(G_i)}] \), where \( S_i = K[\{ x_{ij} \}_{j \in V(G_i)}] \), be the ring homomorphism induced by \( \varphi_i(x_j) = z_j \) and \( \varphi_i(y_j) = t_i z_j \). Then \( \ker \varphi_i = I_i \cap S_i \) where \( I_i = (z_j - x_j, t_i z_j - y_j : j \in V(G_i)) \). Let \( \prec \) be an elimination order on \( K[t_i, \{ z_{ij} \}_{j \in V(G_i)}, \{ x_{ij} \}_{j \in V(G_i)}] \) for \( \{ t_i \} \cup \{ z_j : j \in V(G_i) \} \) whose restriction to \( S_i \) is the lexicographic order on \( S_i \) such that, if \( j, k, l, m \in V(G_i) \) are such that \( j < k \) and \( l < m \), then \( x_j > x_k > y_l > y_m \). By proposition 2.2.20, the set \( \mathcal{G}_i = \{ z_j - x_j, t_i z_j - y_j : j \in V(G_i) \} \cup \{ f_{jk} : j < k \wedge j, k \in V(G_i) \} \) is a Gröbner basis of \( I_i \).

Now let
\[
\varphi : S \to K[t_1, \cdots, t_c, \{ z_{ij} \}_{j \in V(G_1)}], \quad S = K[\bigcup_{i=1}^c \{ x_{ij} \}_{j \in V(G_i)}],
\]
be the ring homomorphism induced by \( \varphi(x_j) = z_j \) and \( \varphi(y_j) = t_i z_j \) if \( j \in V(G_i) \). Then \( \ker \varphi = I \cap S \), where \( I = I_1 + \cdots + I_c \). Let \( \prec \) be an elimination order on
\[
K[t_1, \cdots, t_c, \{ z_{ij} \}_{j \in V(G_1)}], \quad \bigcup_{i=1}^c \{ x_{ij} \}_{j \in V(G_i)} \text{ for } \{ t_1, \cdots, t_c \} \cup \bigcup_{i=1}^c \{ z_j : j \in V(G_i) \}
\]
such that, for every \( i \in [c] \), the restriction of \( \prec \) to \( K[t_i, \{ z_{ij} \}_{j \in V(G_i)}, \{ x_{ij} \}_{j \in V(G_i)}] \) is \( \prec_i \). For \( i \in [c] \), \( \mathcal{G}_i \) is a Gröbner basis of \( I_i \), and using induction and proposition 1.3.29 it follows that \( \bigcup_{i=1}^c \mathcal{G}_i \) is a Gröbner basis of \( I \). Since \( \prec \) is an elimination order on
\[
K[t_1, \cdots, t_c, \{ z_{ij} \}_{j \in V(G_1)}], \quad \bigcup_{i=1}^c \{ x_{ij} \}_{j \in V(G_i)} \text{ for } \{ t_1, \cdots, t_c \} \cup \bigcup_{i=1}^c \{ z_j : j \in V(G_i) \}
\]
then, by theorem 1.3.36, \( S \cap \bigcup_{i=1}^c \mathcal{G}_i \) is a Gröbner basis for \( \ker \varphi \). But \( \mathcal{G}_i \subset K[t_i, \{ z_{ij} \}_{j \in V(G_i)}, \{ x_{ij} \}_{j \in V(G_i)}] \) for every \( i \in [c] \), therefore
\[
S \cap \bigcup_{i=1}^c \mathcal{G}_i = \bigcup_{i=1}^c (\mathcal{G}_i \cap S_i) = \bigcup_{i=1}^c \{ f_{jk} : j < k \wedge j, k \in V(G_i) \}.
\]
Since $\bigcup_{t=1}^{r} \{J_{jk} : j < k \wedge j, k \in V(G_t)\}$ is a Gröbner basis of $\ker \varphi$, it follows in particular that $\ker \varphi = J_{G_1} + \cdots + J_{G_c}$ (recall that $G_1, \ldots, G_c$ are complete graphs on disjoint sets of vertices).

**Corollary 2.2.21.** If $G_1, \ldots, G_c$ are complete graphs on disjoint sets of vertices, then $J_{G_1} + \cdots + J_{G_c}$ is a prime ideal.

**Proof.** Recall that $J_{G_1} + \cdots + J_{G_c} = \ker \varphi$, where $\varphi$ is a ring homomorphism between two polynomial rings, which we denote by $S$ and $S'$, respectively, and so $S/\ker \varphi$ and $\varphi(S)$ are isomorphic rings. But $\varphi(S)$ is a subring of $S'$, which is a domain, and so $\varphi(S)$ is also a domain. But $S/\ker \varphi$ is a domain if and only if $\ker \varphi$ is a prime ideal, hence the conclusion follows.

**Notation 2.2.22.** Given a graph $G$, the complete graph on the vertex set $V(G)$ is denoted by $\bar{G}$.

**Notation 2.2.23.** Given a subset $A \subset \llbracket n \rrbracket$, the complete graph on the vertex set $A$ is denoted by $\bar{A}$.

Let $G$ be a graph on $\llbracket n \rrbracket$. For each subset $R \subset \llbracket n \rrbracket$ we define a prime ideal $P_R(G)$. Let $G_1, \ldots, G_{c(R)}$ be the connected components of $G_{\llbracket n \rrbracket \setminus R}$. We define $P_R(G)$ as being the ideal generated by the ideals $J_{G_1}, \ldots, J_{G_{c(R)}}$ and by the variables $x_i, y_i$ with $i \in R$. This also works for $R = \emptyset$ and in this case $P_R(G)$ is the binomial edge ideal of the graph $\bar{G}_1 \cup \cdots \cup \bar{G}_{c(R)}$.

**Proposition 2.2.24.** The ideal $P_R(G)$ is a prime ideal.

**Proof.** Recall that $P_R(G) = J_{\bar{G}_1} + \cdots + J_{\bar{G}_{c(R)}} + (x_i, y_i : i \in R)$ and so, under the ring isomorphism $S/(x_i, y_i : i \in R)$, the ideal $P_R(G)/(x_i, y_i : i \in R)$ corresponds to the ideal $J_{\bar{G}_1} + \cdots + J_{\bar{G}_{c(R)}} \subset K[\{x_i, y_i : i \notin R\}]$. By corollary 2.2.21 $J_{\bar{G}_1} + \cdots + J_{\bar{G}_{c(R)}}$ is a prime ideal of $K[\{x_i, y_i : i \notin R\}]$ and so $P_R(G)/(x_i, y_i : i \in R)$ is a prime ideal of $S/(x_i, y_i : i \in R)$. This implies that $P_R(G)$ is a prime ideal of $S$, as desired.

**Lemma 2.2.25.** Let $G$ be a graph on $\llbracket n \rrbracket$ and let $R \subset \llbracket n \rrbracket$. Then $\dim(P_R(G)) = |R| + (n - c(R))$.

**Proof.** Let $G_1, \ldots, G_{c(R)}$ be the connected components of the induced subgraph $G_{\llbracket n \rrbracket \setminus R}$. As a consequence of 0.5.17 we have $\dim(P_R(G)) = \dim((x_i : i \in R)) + \dim((y_i : i \in R)) + \sum_{j=1}^{c(R)} \dim(J_{G_j}) = 2|R| + \sum_{j=1}^{c(R)} (|V(G_j)| - 1) = 2|R| + \sum_{j=1}^{c(R)} |V(G_j)| - c(R) = 2|R| + (n - |R|) - c(R) = |R| + (n - c(R))$.

**Corollary 2.2.26.** Let $G$ be a graph on $\llbracket n \rrbracket$ and let $R \subset \llbracket n \rrbracket$. Then $\dim(S/P_R(G)) = c(R) + (n - |R|)$.

**Proof.** Just recall that, by proposition 0.5.12 one has $\dim(P_R(G)) + \dim(S/P_R(G)) = 2n$.

**Theorem 2.2.27.** Let $G$ be a graph on the vertex set $\llbracket n \rrbracket$. Then $J_G = \bigcap_{R \subset \llbracket n \rrbracket} P_R(G)$.

**Proof.** It is obvious that each of the prime ideals $P_R(G)$ contains $J_G$. We will show by induction on $\llbracket n \rrbracket$ that each minimal prime ideal containing $J_G$ is of the form $P_R(G)$ for some $R \subset \llbracket n \rrbracket$. Since $J_G$ is a radical ideal, the assertion of the theorem will follow.
Suppose this result is true for connected graphs. Let $G_1, \ldots, G_r$ be the connected components of $G$. Let $P \in \text{Min}(J_G)$. For $i \in [r]$, let $S_i$ be the polynomial ring in the variables $x_j, y_j$ with $j \in V(G_i)$. Since $P$ is a prime ideal of $S_i$, then $P \cap S_i$ is a prime ideal of $S_i$, and since $J_{G_i} \subset J_G \cap S_i$, then $J_{G_i} \subset P \cap S_i$. Let $P_i \in \text{Min}(J_{G_i})$ such that $P_i \subset P \cap S_i$. Since $P_i \in \text{Min}(J_{G_i})$, then there exists $R_i \subset V(G_i)$ such that $P_i = P_{R_i}(G_i)$. Let $R = \bigcup_{i=1}^r R_i$. Then $P_{R}(G) = P_{R_1}(G_1) + \cdots + P_{R_r}(G_r)$ and so $P_{R}(G) \subset P$, and from $P \in \text{Min}(J_G)$ it follows that $P = P_{R}(G)$, as desired.

So let $G$ be connected and let $P \in \text{Min}(J_G)$. Let $T = \{x_i \in P : y_i \notin P\}$ and $T' = \{y_i \in P : x_i \notin P\}$.

Suppose $T = \{x_1, \ldots, x_n\}$. Then $J_G \subset P_R(G) \subseteq (x_1, \ldots, x_n) \subset P$, contradicting the fact that $P \in \text{Min}(J_G)$. Hence $T \neq \{x_1, \ldots, x_n\}$. Similarly, $T' \neq \{y_1, \ldots, y_n\}$.

Suppose that one of the sets $T, T'$, say $T$, is non-empty. Since $T \neq \{x_1, \ldots, x_n\}$ and $G$ is connected, then there exists $\{i, j\} \in E(G)$ such that $x_i \in T$ but $x_j \notin T$. Since $x_i, y_j - x_j, y_j \in J_G \subseteq P$ and $x_i \in P$, it follows that $x_j y_j \in P$. Since $x_i \in T$, then $y_i \notin P$, therefore $x_j y_j \in P$ implies $x_j \in P$, and since $x_j \notin T$, it follows that $y_j \in P$. Since $x_j, y_j \in P$, one can write $P = P + (x_j, y_j)$, where $P$ is an ideal of $K[[x_k, y_k : k \neq j]]$.

Let $G'$ be the restriction of $G$ to the vertex set $[n] \setminus \{j\}$. Then $J_{G'} + (x_j, y_j) = J_G + (x_j, y_j) \subset P$ and in particular $P \in \text{Min}(J_{G'} + (x_j, y_j))$, therefore $P/(x_j, y_j) \in \text{Min}((J_{G'} + (x_j, y_j))/(x_j, y_j))$. But, under the ring isomorphism $S/(x_j, y_j) \simeq K[[x_k, y_k : k \neq j]]$, the ideals $(J_{G'} + (x_j, y_j))/(x_j, y_j)$ and $P/(x_j, y_j)$ correspond to the ideals $J_{G'}$ and $P$ of $K[[x_k, y_k : k \neq j]]$, respectively, hence $P \in \text{Min}(J_{G'})$. By induction hypothesis, $P$ is of the form $P_{R'}(G')$ for some subset $R' \subset [n] \setminus \{j\}$. Since $P' = P_{R'}(G')$, it follows that $P = P + (x_j, y_j) = P_{R'\cup\{j\}}(G)$.

It remains to consider the case where $T = T' = \emptyset$. Let $R = \{i \in [n] : x_i \in P\}$. Then $y_i \in P$ if and only if $i \in R$.

Suppose $R = \emptyset$, that is, $P$ contains no variables. We claim that if $i, j \in [n]$ with $i < j$, then $f_{ij} \in P$. From this it will follow that $J_G \subset P$, and since $J_G = P_R(G)$ (recall that $G$ is connected) is a prime ideal containing $J_G$ and $P \in \text{Min}(J_G)$, we conclude that $P = P_R(G)$.

Let $i = i_0, i_1, \ldots, i_r = j$ be a path from $i$ to $j$. We proceed by induction on $r$ to show that $f_{ij} \in P$. The assertion is trivial for $r = 1$. Suppose now that $r > 1$. Our induction hypothesis says that $f_{i_1j} \in P$. On the other hand, one has $x_{i_1}f_{i_1j} = x_j f_{i_1i} + x_i f_{ij}$. Thus $x_{i_1}f_{i_1j} \in P$. Since $P$ is a prime ideal and $x_{i_1} \notin P$, we see that $f_{ij} \in P$.

Now suppose $R \neq \emptyset$. One can write $P = P + (x_i, y_i : i \notin R)$, where $P$ is an ideal of $K[[x_i, y_i : i \notin R]]$ containing no variables. Let $G'$ be the restriction of $G$ to the vertex set $[n] \setminus R$. Then $J_{G'} + (x_i, y_i : i \in R) = J_G + (x_i, y_i : i \in R) \subset P$ and in particular $P \in \text{Min}(J_{G'} + (x_i, y_i : i \in R))$, therefore $P/(x_i, y_i : i \in R) \in \text{Min}((J_{G'} + (x_i, y_i : i \in R))/(x_i, y_i : i \in R))$. But, under the ring isomorphism $S/(x_i, y_i : i \in R) \simeq K[[x_i, y_i : i \notin R]]$, the ideals $(J_{G'} + (x_i, y_i : i \in R))/(x_i, y_i : i \in R)$ and $P/(x_i, y_i : i \in R)$ correspond to the ideals $J_{G'}$ and $P'$ of $K[[x_i, y_i : i \notin R]]$, respectively, hence $P' \in \text{Min}(J_{G'})$. By induction hypothesis, $P'$ is of the form $P_{R'}(G')$ for some subset $R' \subset [n] \setminus R$. Since $P' = P_{R'}(G')$, it follows that $P = P + (x_i, y_i : i \in R) = P_{R \cup R'}(G)$.
We have seen that all minimal prime ideals of $J_G$ are of the form $P_R(G)$ for some $R \subseteq [n]$. However, not all such prime ideals are minimal prime ideals of $J_G$.

**Example 2.2.28.** If $G$ is a complete graph, then $J_G = P_\emptyset(G)$ and so $P_R(G) \notin \text{Min}(J_G)$ for every non-empty $R \subseteq [n]$. Moreover, in this case it is true that $P_R(G) \subset P_T(G)$ if and only if $R \subset T$.

**Example 2.2.29.** If $G$ is the path with $E(G) = \{\{1, 2\}, \{2, 3\}\}$, then $\text{Min}(J_G) = \{P_\emptyset(G), P_{\{2\}}(G)\}$.

In this section, as well as studying the Cohen-Macaulayness of some binomial edge ideals, we will determine which prime ideals $P_R(G)$ are indeed minimal primes of $J_G$. By now we only show that $P_\emptyset(G) \in \text{Min}(J_G)$ for every graph $G$.

**Corollary 2.2.30.** The ideal $P_\emptyset(G)$ is a minimal prime of $J_G$.

**Proof.** Let $R \subset [n]$ be non-empty. Pick $i \in R$. By definition, $x_i, y_i \in P_R(G)$. But the standard generators of $P_\emptyset(G)$ are all of degree two, therefore $x_i, y_i \notin P_\emptyset(G)$. Hence $P_R(G) \notin P_\emptyset(G)$. Since all minimal primes of $J_G$ are of the form $P_R(G)$, the assertion follows. \(\square\)

**Corollary 2.2.31.** Let $G$ be a graph on $[n]$. Then $\dim(S/J_G) = \max\{(n - |R|) + c(R) : R \subset [n]\}$. In particular, $\dim(S/J_G) \geq n + c$, where $c$ is the number of connected components of $G$.

In general, this inequality is strict.

**Example 2.2.32.** If $G$ is a claw, then using either Macaulay2 or the corollary above we get that $\dim(S/J_G) = 6$ and so $\dim(S/J_G) > n + c$ (where $n = 4$ and $c = 1$).

**Corollary 2.2.33.** Let $G$ be a graph on $[n]$ with $c$ connected components. If the ideal $J_G$ is unmixed, then $\dim(S/J_G) = n + c$.

**Proof.** Since $P_\emptyset(G) \in \text{Min}(J_G)$ and $J_G$ is unmixed, it follows that $\dim(S/J_G) = \dim(S/P_\emptyset(G)) = n + c$. \(\square\)

**Corollary 2.2.34.** If $J_G$ is a Cohen-Macaulay ideal, then $\operatorname{depth}(S/J_G) = \dim(S/J_G) = n + c$.

**Proof.** Recall that, by proposition 0.5.13, Cohen-Macaulay ideals in $S$ are unmixed ideals. \(\square\)

**Proposition 2.2.35.** If $G$ is a path with edges $\{1, 2\}, \cdots, \{n - 1, n\}$, then $J_G = (f_{1,2}, \cdots, f_{n-1,n})$ is a complete intersection.
**Proof.** For \( i \in [n-1] \) one has \( \text{in}_<(f_{i,i+1}) = x_iy_{i+1} \), hence corollary \[3.28\] implies that \( \{f_1, \ldots, f_{n-1,n}\} \) is a Gröbner basis for \( J_G \).

Since \( J_G = (f_1, \ldots, f_{n-1,n}) \) is a graded Gröbner basis such that \( \text{in}_<(J_G) = (x_1y_2, x_2y_3, \ldots, x_{n-1}y_n) \) is a complete intersection, proposition \[1.3.41\] implies \( J_G = (f_1, \ldots, f_{n-1,n}) \) is also a complete intersection, as desired. \( \square \)

**Corollary 2.2.36.** If \( G \) is a path, then \( J_G \) is a Cohen-Macaulay ideal.

**Proof.** Just use corollary \[0.5.10\] \( \square \)

Note in particular that \( G \) being a Cohen-Macaulay graph and \( J_G \) being a Cohen-Macaulay ideal are not equivalent statements.

**Example 2.2.37.** If \( G \) is a path of length 2, then, by corollary \[2.1.16\] \( G \) is not a Cohen-Macaulay graph. However, \( J_G \) is a Cohen-Macaulay ideal.

We now know that \( J_G \) is a Cohen-Macaulay when \( G \) is either a complete graph or a path.

**Example 2.2.38.** Let \( G \) be the path on \([n]\) with \( E(G) = \{\{1, 2\}, \ldots, \{n-1, n\}\} \). Then \( G \) is a connected graph and \( J_G \) is a Cohen-Macaulay ideal, therefore \( \dim(S/J_G) = n+1 \). By corollary \[2.2.26\] the minimal prime ideals of \( J_G \) are the prime ideals \( P_R(G) \) for which \( c(R) = |R| + 1 \). Let \( R \subseteq [n] \). There exist integers \( 1 \leq a_1 \leq b_1 < a_2 \leq b_2 < a_3 \leq b_3 < \cdots < a_r \leq b_r \leq n \) such that \( R = \bigcup_{i=1}^r [a_i, b_i] \). Suppose also that, for each \( i \in [r-1] \), \( b_i \) and \( a_{i+1} \) are not consecutive integers. We see that \( |R| = \sum_{i=1}^r (b_i - a_i + 1) = \sum_{i=1}^r (b_i - a_r) + r \), and that \( c(R) = r - 1 \) if \( a_1 = 1 \) and \( b_r = n \), \( c(R) = r + 1 \) if \( a_1 \neq 1 \) and \( b_r \neq n \), and \( c(R) = r \) otherwise. Thus \( c(R) = |R| + 1 \) if and only if \( a_1 \neq 1, b_r \neq n \) and \( a_i = b_i \) for all \( i \in [r] \). In other words, the minimal prime ideals of \( G \) are those \( P_R(G) \) for which \( R \) is a subset of \([n]\) of the form \( \{a_1, \ldots, a_r\} \), with \( 1 < a_1 < a_2 < \cdots < a_r < n \) and such that no two of these \( r \) integers are consecutive integers.

**Proposition 2.2.39.** The ideal \( J_G \) is a prime ideal if and only if each connected component of \( G \) is a complete graph.

**Proof.** If each of the connected components of \( G \) is a complete graph, then \( J_G = P_\emptyset(G) \) and so \( J_G \) is a prime ideal.

Let \( G_1, \ldots, G_r \) be the connected components of \( G \) and suppose that \( J_G \) is a prime ideal. Since \( P_\emptyset(G) = J_{G_1} + \cdots + J_{G_r} \) is a minimal prime ideal of \( J_G \), it follows that \( J_G = J_{G_1} + \cdots + J_{G_r} = J_{G_1 \cup \cdots \cup G_r} \), therefore, by corollary \[2.2.5\] \( G = G_1 \cup \cdots \cup G_r \) and so each connected component of \( G \) is a complete graph, as desired. \( \square \)

As it happens with monomial edge ideals:

**Proposition 2.2.40.** The binomial edge ideal of a graph is Cohen-Macaulay if and only if each of the binomial edge ideals of its connected components is Cohen-Macaulay.
Proof. As in the proof for monomial edge ideals, one just needs to use corollary 0.5.16 and induction. \(\square\)

Thus, to study the Cohen-Macaulayness of binomial edge ideals (which is the goal of this chapter), it is enough to consider connected graphs.

We have seen that the binomial edge ideal of a complete graph is Cohen-Macaulay. The following result tells us which cycles have a Cohen-Macaulay binomial edge ideal.

**Corollary 2.2.41.** Let \(G\) be a cycle of length \(n\). Then the following conditions are equivalent:

1. \(n = 3\).
2. \(J_G\) is a prime ideal.
3. \(J_G\) is unmixed.
4. \(J_G\) is a Cohen-Macaulay ideal.

*Proof.* The implication (1) \(\Rightarrow\) (2) follows from proposition 2.2.39. Moreover, if \(J_G\) is a prime ideal, then each of the connected components of \(G\) is a complete graph and so, by corollary 2.2.17, the binomial edge ideal of each connected component of \(G\) is Cohen-Macaulay, therefore proposition 2.2.40 implies \(J_G\) is also Cohen-Macaulay. Furthermore, by corollary 0.5.13 any Cohen-Macaulay ideal is unmixed, so all equivalences follow once it is shown that (3) \(\Rightarrow\) (2).

One of the minimal prime ideals of \(J_G\) is \(P_G(G)\) and \(ht(P_G(G)) = n - 1\). Now let \(R \subseteq [n]\) with \(R \neq \emptyset\). We may assume that we have labelled the vertices of the cycle counterclockwise and that \(R = \bigcup_{i=1}^r [a_i, b_i]\) with \(1 = a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_r \leq b_r < n\). Suppose also that, for each \(i \in [r - 1]\), \(b_i\) and \(a_{i+1}\) are not consecutive integers. Then \(c(R) = r\) and \(ht(P_R(G)) = |R| + n - c(R) = \sum_{i=1}^r (b_i - a_i) + r + n - r = n + \sum_{i=1}^r (b_i - a_i) \geq n\). Thus if \(J_G\) is unmixed, then \(\text{Min}(J_G) = \{P_G(G)\}\), and by theorem 2.2.27 it follows that \(J_G = P_G(G)\) is a prime ideal, as desired. \(\square\)

We still have not answered which of the prime ideals \(P_R(G)\), besides from \(P_G(G)\), are the minimal primes of \(J_G\).

**Proposition 2.2.42.** Let \(G\) be a graph on \([n]\) and let \(R\) and \(T\) be subsets of \([n]\). Let \(G_1, \cdots, G_r\) be the connected components of \(G\) \(\setminus R\) and let \(H_1, \cdots, H_t\) be the connected components of \(G\) \(\setminus T\). Then \(P_T(G) \subseteq P_R(G)\) if and only if \(T \subseteq R\) and for all \(i \in [t]\) one has \(V(H_i) \setminus R \subseteq V(G_j)\) for some \(j \in [r]\).

*Proof.* Suppose that \(P_T(G) \subseteq P_R(G)\).

Let \(i \in T\). Then \(x_i, y_i \in P_T(G)\) and so \(x_i, y_i \in P_R(G)\), and since the only variables in \(P_R(G)\) are the variables \(x_j, y_j\) such that \(j \in R\), it follows that \(i \in R\). Since \(i \in T\) is arbitrary, then \(T \subseteq R\).

For \(i \in [t]\), let \(H_i = (H_i)_{n\setminus R}\). Then \(\sum_{i \in T} J_{R_i} \cap \sum_{i \in R} J_{G_i} + \cdots + J_{G_r} + J_{G_r}\), and the standard generators of \(\sum_{i \in T} J_{R_i} \cap \sum_{i \in R} J_{G_i} + \cdots + J_{G_r}\) are polynomials in \(K\{x_i, y_i : i \notin R\}\), it follows that \(J_{R_i} + \cdots + J_{R_t} \subseteq J_{G_1} + \cdots + J_{G_r}\), that is, \(J_{R_1 \cup \cdots \cup R_t} \subseteq J_{G_1 \cup \cdots \cup G_r}\) and so, by proposition 2.2.4
Let $\bar{H}_1' \cup \cdots \cup \bar{H}_r' \subset \bar{G}_1' \cup \cdots \cup \bar{G}_r'$. Since each of the graphs $\bar{H}_1', \cdots, \bar{H}_r'$ is connected, it follows that, for all $i \in [t]$, one has $V(H_i) \setminus R \subset V(G_j)$ for some $j \in [r]$.

Suppose that $T \subset R$ and that for all $i \in [t]$ one has $V(H_i) \setminus R \subset V(G_j)$ for some $j \in [r]$.

Let $i \in T$. Then $i \in R$ and so $x_i, y_i \in P_R(G)$.

Let $a \in [t]$ and let $i, j \in V(H_a)$ with $i < j$. If $\{i, j\} \cap R \neq \emptyset$, then $f_{ij} \in (x_k, y_k : k \in R) \subset P_R(G)$. Otherwise, $i, j \in V(H_a) \setminus R$. Pick $b \in [s]$ such that $V(H_a) \setminus R \subset V(G_b)$. Then $i, j \in V(G_b)$ and so $f_{ij} \in J_{\bar{G}_b} \subset P_R(G)$.

Hence $P_T(G) \subset P_R(G)$.

**Corollary 2.2.43.** Let $G$ be a connected graph on $[n]$ and let $R \subset [n]$. Then $P_R(G) \in \text{Min}(J_G)$ if and only if $S = \emptyset$ or $S \neq \emptyset$ and for each $i \in R$ one has $c(R \setminus \{i\}) < c(R)$.

**Proof.** Let $G_1, \cdots, G_{c(R)}$ be the connected components of $G_{[n]\setminus R}$. For each $i \in R$ set $T_i = R \setminus \{i\}$.

Suppose that $i$ is only adjacent to vertices in $T_i$. Then the connected components of $G_{[n]\setminus T_i}$ are $G_1, \cdots, G_{c(R)} \setminus \{i\}$, thus $c(T_i) = c(R) + 1$.

Suppose that there are exactly $k_i$ connected components of $G_{[n]\setminus R}$, say $G_1, \cdots, G_{k_i}$, with $k_i \geq 1$, containing vertices which are adjacent to $i$. Then the connected components of $G_{[n]\setminus T_i}$ are $G_1', G_{k_i+1}, \cdots, G_{c(R)}$, where $V(G_1') = \bigcup_{i=1}^{k_i} V(G_i) \cup \{i\}$, thus $c(T_i) = c(R) - k_i + 1$.

We get that, for every $i \in R$, $c(T_i) = c(R) - k_i + 1$, where $k_i \geq 0$ is the number of connected components of $G_{[n]\setminus R}$ containing vertices which are adjacent to $i$.

Suppose that $P_R(G) \in \text{Min}(J_G)$. Let $i \in R$. Then $P_{T_i}(G) \not\subset P_R(G)$, and since $T_i \subset R$, then proposition 2.2.42 implies that $G_{[n]\setminus T_i}$ has a connected component $H$ such that $V(H) \setminus R$ is not contained in any of the sets $V(G_1), \cdots, V(G_{c(R)})$. But since the connected components of $G_{[n]\setminus T_i}$ are $G_1', G_{k_i+1}, \cdots, G_{c(R)}$, where $V(G_1') = \bigcup_{i=1}^{k_i} V(G_i) \cup \{i\}$, then this is only possible if $k_i \geq 2$ and $H = G_1'$, hence $c(T_i) < c(R)$, as desired.

Conversely, suppose that $c(T_i) < c(R)$ for every $i \in R$. We will show that $P_T(G) \not\subset P_R(G)$ for every $T \subset R$, from where it will follow that $P_R(G) \in \text{Min}(J_G)$.

Let $T \subset R$ and pick $i \in R \setminus T$. Since $c(T_i) < c(R)$ and $c(T_i) = c(R) - k_i + 1$, it follows that $k_i \geq 2$. Recall that the connected components of $G_{[n]\setminus T_i}$ are $G_1', G_{k_i+1}, \cdots, G_{c(R)}$, where $V(G_1') = \bigcup_{i=1}^{k_i} V(G_i) \cup \{i\}$. Since $T \subset T_i$, that is, $\{n\} \setminus T_i \subset [n] \setminus T$, then $G_{[n]\setminus T}$ has one connected component $H$ containing $G_1'$ and in particular $V(H) \setminus R$ contains the subsets $V(G_1)$ and $V(G_2)$. Hence $V(H) \setminus R$ is not contained in any $V(G_k)$, with $k \in [c(R)]$. According to proposition 2.2.42, this means that $P_T(G) \not\subset P_R(G)$. $\square$

**Example 2.2.44.** Let $G$ be a cycle of length $n > 3$. Then, besides of the prime ideal $P_G(G)$, which is of height $n - 1$, the only other minimal prime ideals are the ideals $P_R(G)$ where $|R| = 1$ and no two elements $i, j \in R$ belong to the same edge of $G$. By lemma 2.2.25, each of these prime ideals has height $n$. In particular this implies that $J_G$ is not unmixed and therefore not Cohen-Macaulay, as we have shown before.
2.3 A special class of chordal graphs

A general classification of Cohen-Macaulay binomial edge ideals seems to be hopeless. We then restrict our attention to a special class of chordal graphs: the chordal graphs such that any two distinct maximal cliques intersect in at most one vertex. For this class of graphs, we can determine depth(S/Jc) and use that result to study the Cohen-Macaulayness of G.

Forests are a good example of such graphs.

**Proposition 2.3.1.** If G is a forest, then G is a chordal graph and any two distinct maximal cliques in G intersect in at most one vertex.

**Proof.** Since a forest has no cycles, then G is clearly chordal and all its maximal cliques have exactly two vertices. This means that any two distinct maximal cliques intersect in at most one vertex, as desired. □

We know a formula for dim(S/Jc) which depends only on the properties of the graph G and we would like to find a similar formula for depth(S/Jc). By corollary [2.2.34] one has depth(S/Jc) = n + ε whenever G is a graph on [n] with c connected components such that Jc is a Cohen-Macaulay ideal. This is also true for chordal graphs such that any two distinct maximal cliques intersect in at most one vertex.

So suppose now G is such a connected graph. By theorem [1.2.35] G is chordal if and only if Δ(G) is a quasi-forest. So let us consider a leaf order F1, ···, Fr on the facets of Δ(G). If r = 1, then G is a complete graph, so we will consider only r > 1.

**Proposition 2.3.2.** There exists a unique vertex i ∈ Fr such that i also belongs to some other facet of Δ(G).

**Proof.** Such i exists since G is connected. We will show that i is unique.

Pick j ∈ [r − 1] such that Fj is a branch of Fr. Pick k ∈ [r − 1] such that i ∈ Fk. Then i ∈ Fk ∩ Fr, and since Fk ∩ Fr ⊂ Fj ∩ Fr, then i ∈ Fj ∩ Fr. Since any two distinct facets of Δ(G) intersect in at most one vertex, it follows that Fj ∩ Fr = {i}. In particular, such vertex i is unique. □

**Proposition 2.3.3.** The ideal Q1 is the binomial edge ideal of the graph G′ which is obtained from G by replacing the facets F1, ···, Ft, Fr by the clique on the vertex set Fr ∪ (Uj=1 Fj).

**Proof.** Let R be a subset of [n] \ {i}. Let {j, k} ∈ E(G′) with j < k. If either {j, k} ∈ E(G) or {j, k} ∩ R ≠ ∅, then clearly xjyk − kxyj ∈ Pr(G). Suppose {j, k} ∉ E(G) and j, k ∉ R. Then j, k ∈ Fr ∪ (Uj=1 Fj) and in particular j, i, k is a path in G, and since i, j, k ∉ R, it follows that j, k lie in the same connected component of the induced subgraph of G in [n] \ R, hence xjyk − kxyj ∈ Pr(G). Since {j, k} ∈ E(G′) is arbitrary, Jc ⊂ Pr(G). Since C ⊂ [n] \ {i} is arbitrary, Jc′ ⊂ Q1.

It remains to show that Q1 ⊂ Jc′. Since G ⊂ G′, then Pr(G′) ⊂ Pr(G′) for every R ⊂ [n]. Let R be a subset of [n] with i ∉ R. Now we will show that Pr(G′) ⊂ Pr(G′) for every S ⊂ [n] \ {i}. To show this, it is enough to show that if j, k ∈ [n] \ (R ∪ {i}) lie in the same connected component of the induced subgraph of G′ in [n] \ R, then fjk ∈ Pr(G′).

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Let $P$ be a path in $G'_{[n]\setminus R}$ from $j$ to $k$ whose length is the smallest possible. Suppose that $P$ passes through $i$, that is, $P$ is of the form $j, \ldots, j', i, k', l, \ldots, k$. Then the vertices $j'$ and $k'$ belong to the clique $\bigcup_{r=1}^{m} F_{i_r}$ and so $\{j', k'\} \in E(G')$. But then $j, \ldots, j', k', \ldots, k$ is a path in $G'_{[n]\setminus R}$ from $j$ to $k$ whose length is smaller than the length of $P$, a contradiction. Hence $P$ does not pass through $i$ and in particular $j$ and $k$ lie in the same connected component of $G'_{[n]\setminus (R \cup \{i\})}$, thus $f_{jk} \in P_{R \cup \{i\}}(G')$.

Since $P_{R}(G') \subset P_{R \cup \{i\}}(G')$ for every $R \subset [n] \setminus \{i\}$, then $J_{G'} = \bigcap_{R \subset [n] \setminus \{i\}} P_{R}(G')$ and so it follows that $Q_{1} = \bigcap_{R \subset [n] \setminus \{i\}} P_{R}(G) \subset \bigcap_{R \subset [n] \setminus \{i\}} P_{R}(G') = J_{G'}$. \hfill $\square$

**Proposition 2.3.4.** The graph $G'$ is also a connected chordal graph on $[n]$ such that any two distinct maximal cliques intersect in at most one vertex.

**Proof.** Since $G$ and $G'$ are graphs on $[n]$ such that $G \subset G'$ and $G$ is connected, then $G'$ is also connected.

Let $C$ be a cycle in $G'$ of length greater than 3. If $C$ is already a cycle in $G$, there is nothing to prove. Otherwise, there exists $j, k \in V(C)$ such that $\{j, k\} \in E(C)$ but $\{j, k\} \notin E(G)$, hence $j$ and $k$ belong to distinct facets in the collection of facets $F_{i_1}, \ldots, F_{i_q}, F$, and in particular $j$ and $k$ are adjacent to $i$.

Suppose that $j$ and $k$ are the only vertices in $C$ which are adjacent to $i$. Then the edges of $G$ other than $\{j, k\}$ are edges in $G$ and so, by replacing the edge $\{j, k\}$ by the edges $\{j, i\}, \{i, k\}$, we get a cycle $C'$ in $G$ that must have a chord $\{j', k'\} \in E(G)$ such that $\{j', k'\} \neq \{j, k\}$ and $i \notin \{j', k'\}$. Then $j', k' \in V(C)$, $\{j', k'\} \in E(G')$ and $\{j', k'\} \notin E(C)$, therefore $\{j, k\}$ is a chord in $C$.

Suppose that there exists another vertex $l$ in $C$ which is adjacent to $i$. Then $j, k, l$ belong to the same maximal clique in $G'$. Now suppose without loss of generality that $k$ and $l$ are not adjacent vertices in the cycle $C$. But $\{k, l\} \in E(G')$ and so $\{k, l\}$ is a chord in $C$.

It remains to show that the clique $F_{r} \cup \bigcup_{j=1}^{q} F_{t_{j}}$ of $G'$ intersects any other maximal clique in at most one vertex. Let $k \in [r - 1] \setminus \{t_{1}, \ldots, t_{q}\}$. Then $F_{r} \cap F_{k} = \emptyset$ and so it is enough to show that $F_{k}$ intersects $\bigcup_{j=1}^{q} F_{t_{j}}$ in at most one vertex.

Suppose this is not the case. Since $F_{k}$ intersects each other maximal clique of $G$ in at most one vertex, then there exist two cliques in the collection $\{F_{t_{1}}, \ldots, F_{t_{q}}\}$, say $F_{t_{i}}$ and $F_{t_{j}}$, intersecting $F_{k}$ in distinct vertices. Let $F_{t_{i}} \cap F_{k} = \{j_{1}\}$ and $F_{t_{j}} \cap F_{k} = \{j_{2}\}$. Then the vertices $i, j_{1}, j_{2}$ form a clique in $G$ which intersects $F_{k}$ in two vertices but is not contained in it, a contradiction. \hfill $\square$

**Notation 2.3.5.** We denote $S_{i} = K'[\{x_{j}, y_{j} : j \neq i\}] \subset S$.

**Proposition 2.3.6.** The ideal $Q_{2}$ is such that $Q_{2} = (x_{i}, y_{i}) + J_{G''}$, where $G''$ is the restriction of $G$ to the vertex set $[n] \setminus \{i\}$.

**Proof.** If $R$ is a subset of $[n]$ containing $i$, then $(x_{i}, y_{i}) \in P_{R}(G)$. Hence $(x_{i}, y_{i}) \subset Q_{2}$. On other hand, since $G'' \subset G$, it follows that $J_{G''} \subset J_{G} \subset Q_{2}$, and so $(x_{i}, y_{i}) + J_{G''} \subset Q_{2}$.

It remains to show that $Q_{2} \subset (x_{i}, y_{i}) + J_{G''}$. Let $f \in Q_{2}$. Then $f \in P_{[i]}(G)$, and since $P_{[i]}(G) = (x_{i}, y_{i}) + P_{G}(G'')$ and $P_{G}(G'') \subset S_{i}$, then $f$ can be written in a unique way $f = g + h$ such that $g \in (x_{i}, y_{i})$ and $h \in S_{i}$.
Let \( R \) be a subset of \([n] \setminus \{i\}\). Then \( f \in P_{Ru(i)}(G) \), and since \( P_{Ru(i)}(G) = (x_i, y_i) + P_R(G'') \) and \( P_R(G'') \subset S_i \), it follows that \( h \in P_R(G'') \). Since \( R \subset [n] \setminus \{i\} \) is arbitrary, it follows that \( h \in J_G'' \), hence \( f \in (x_i, y_i) + J_G'' \), as desired. \( \square \)

**Corollary 2.3.7.** The ideal \( Q_2 \) is such that \( S/Q_2 \simeq S_i/J_G''. \)

**Proposition 2.3.8.** The graph \( G'' \) is a chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex. But it is not connected for it has \( q + 1 \) connected components.

**Proof.** Since \( G \) is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex and \( G'' \) is induced by \( G \), then \( G'' \) is also a chordal graph such that any two distinct maximal cliques intersect in at most one vertex.

We will now show that the cliques \( F_{t_1} \setminus \{i\}, \cdots, F_{t_q} \setminus \{i\}, F_r \setminus \{i\} \) all lie in distinct connected components of \( G'' \).

Suppose this is not the case and consider points \( u, v \) in distinct cliques of the collection \( F_{t_1} \setminus \{i\}, \cdots, F_{t_q} \setminus \{i\}, F_r \setminus \{i\} \) such that \( d(u, v) \) (as vertices of \( G'' \)) is the smallest possible. Consider a path \( u, \cdots, v \) in \( G'' \) of length \( d(u, v) \). Then \( i, u, \cdots, v, i \) is a cycle in \( G \), and since \( G \) is chordal, that cycle has a vertex \( w \) such that \( \{i, w\}, \{v, w\} \in E(G) \), and so the vertices \( i, v, w \) form a clique in \( G \). This clique intersects the clique of the collection \( F_{t_1}, \cdots, F_{t_q}, F_r \) which contains \( i \) and \( w \) in two vertices, a contradiction.

Hence \( G'' \) has at least \( q + 1 \) connected components. Suppose \( G'' \) has more than \( q + 1 \) connected components.

Consider vertices \( j_1, \cdots, j_{q+2} \) that lie in distinct connected components of \( G'' \). For \( k \in [q + 2] \) consider paths \( j_k, \cdots, j_k, i \) in \( G \). Then \( j_1', \cdots, j_k', F_r \cup (\bigcup_{j=1}^q F_{t_j}) \setminus \{i\} \) and so two of the \( j_1', \cdots, j_k' \), say \( j_1' \) and \( j_2' \) must lie on the same clique of the collection of cliques \( F_{t_1} \setminus \{i\}, \cdots, F_{t_q} \setminus \{i\}, F_r \setminus \{i\} \). But then \( \{j_1', j_2'\} \in E(G'') \) and so \( j_1, \cdots, j_1', j_2', \cdots, j_2 \) is a path from \( j_1 \) to \( j_2 \) in \( G'' \), a contradiction. \( \square \)

**Proposition 2.3.9.** The ideal \( Q_1 + Q_2 \) is such that \( S/(Q_1 + Q_2) \simeq S_i/J_H \), where \( H \) is the restriction of \( G' \) to the vertex set \([n] \setminus \{i\}\).

**Proof.** Since \( G'' \subset G' \), then \( J_{G''} \subset J_{G'} \), hence \( Q_1 + Q_2 = J_{G'} + ((x_i, y_i) + J_{G''}) = J_{G'} + (x_i, y_i) \). Since \( H \) is the restriction of \( G' \) to the vertex set \([n] \setminus \{i\}\), then \( J_{G'} + (x_i, y_i) = J_H + (x_i, y_i) \) and so the conclusion follows. \( \square \)

**Proposition 2.3.10.** The graph \( H \) is a connected chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex.

**Proof.** Since \( G' \) is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex and \( H \) is induced by \( G' \), then \( H \) is also a chordal graph such that any two distinct maximal cliques intersect in at most one vertex. It remains to show that \( H \) is connected. Let \( j, k \in [n] \setminus \{i\} \). Since \( G' \) is connected, then there exists a path in \( G' \) from \( j \) to \( k \). Let \( P \) be a path in \( G' \) from \( j \) to \( k \) whose length is the smallest
possible. Suppose that \( P \) passes through \( i \), that is, \( P \) is of the form \( j, \cdots, j', i, k, \cdots, k \). Then the vertices \( j' \) and \( k' \) belong to the clique \( F_r \cup \left( \bigcup_{j'=1}^{r} F_{1_j} \right) \) and so \( \{j', k'\} \in E(G') \). But then \( j, \cdots, j', k, \cdots, k \) is a path in \( G' \) from \( j \) to \( k \) whose length is smaller than the length of \( P \), a contradiction. Hence \( P \) does not pass through \( i \) and so \( P \) is a path in \( H \). Hence the assertion follows. \( \square \)

**Theorem 2.3.11.** Let \( G \) be a chordal graph on \([n]\) such that any two distinct maximal cliques intersect in at most one vertex. Then \( \text{depth}(S/J_G) = n + c \), where \( c \) is the number of connected components of \( G \).

**Proof.** As a consequence of lemma 0.5.15, we may assume that \( G \) is connected.

Let \( r \) be the number of maximal cliques of \( G \). If \( r = 1 \), then \( G \) is a complete graph and by corollary 2.2.17, \( J_G \) is a Cohen-Macaulay ideal, and by corollary 2.2.34, \( \text{depth}(S/J_G) = \text{dim}(S/J_G) = n + 1 \), as desired. Suppose \( r > 1 \).

We will use induction on \( n + r \). The base case is \( n + r = 2 \). But in this case \( r = 1 \). Suppose \( n + r > 2 \).

Since \( G' \) is a connected chordal graph on \([n]\) such that any two distinct maximal cliques intersect in at most one vertex and with less than \( r \) maximal cliques, by induction it follows that \( \text{depth}(S/J_{G'}) = n + 1 \), and since \( Q_1 = J_{G'} \), then \( \text{depth}(S/Q_1) = n + 1 \).

Since \( G'' \) is a chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex, with \( q + 1 \) connected components and with at most \( r \) maximal cliques, by induction it follows that \( \text{depth}(S_i/J_{G''}) = (n - 1) + (q + 1) = n + q \), and since \( S/Q_2 \cong S_i/J_{G''} \), then \( \text{depth}(S/Q_2) = n + q \).

Now consider the following exact sequence of \( S \)-modules:

\[
0 \rightarrow \frac{S}{Q_1} \rightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \rightarrow \frac{S}{Q_2} \rightarrow 0.
\]

By inequality (1) of proposition 0.3.23, \( \text{depth}(S/Q_1) \geq \min\{\text{depth}(S/Q_1 \oplus S/Q_2), \text{depth}(S/Q_2) + 1\} \), that is, \( n + 1 \geq \min\{\text{depth}(S/Q_1 \oplus S/Q_2), n + q + 1\} \), hence \( \text{depth}(S/Q_1 \oplus S/Q_2) \leq n + 1 \). On the other hand, by inequality (2), \( \text{depth}(S/Q_1 \oplus S/Q_2) \geq \min\{\text{depth}(S/Q_1), \text{depth}(S/Q_2)\} = \min\{n + 1, n + q\} = n + 1 \), hence \( \text{depth}(S/Q_1 \oplus S/Q_2) = n + 1 \).

Since \( H \) is a connected chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex and with at most \( r \) maximal cliques, by induction it follows that \( \text{depth}(S_i/J_H) = (n - 1) + 1 = n \), and since \( S/(Q_1 + Q_2) \cong S_i/J_H \), then \( \text{depth}(S/(Q_1 + Q_2)) = n \).

The decomposition \( J_G = Q_1 \cap Q_2 \) yields the following exact sequence of \( S \)-modules:

\[
0 \rightarrow \frac{S}{J_G} \rightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \rightarrow \frac{S}{Q_1 + Q_2} \rightarrow 0.
\]

By inequality (3) of proposition 0.3.23, \( \text{depth}(S/(Q_1 + Q_2)) \geq \min\{\text{depth}(S/J_G) - 1, \text{depth}(S/Q_1 \oplus S/Q_2)\} \), that is, \( n \geq \min\{\text{depth}(S/J_G) - 1, n + 1\} \), hence \( \text{depth}(S/J_G) - 1 \leq n \), that is, \( \text{depth}(S/J_G) \leq n + 1 \). On the other hand, by inequality (1), \( \text{depth}(S/J_G) \geq \min\{\text{depth}(S/Q_1 \oplus S/Q_2), \text{depth}(S/(Q_1 + Q_2)) + 1\} = \min\{n + 1, n + 1\} = n + 1 \), hence \( \text{depth}(S/J_G) = n + 1 \), as desired. \( \square \)

**Corollary 2.3.12.** If \( G \) is a forest on the vertex \([n]\) with \( c \) connected components, then \( \text{depth}(S/J_G) = n + c \).
This formula does not necessarily hold if $G$ has two maximal cliques intersecting in at least two vertices.

**Example 2.3.13.** If $G$ is the diamond with $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, then $G$ is a connected chordal graph with two maximal cliques: $\{1, 2, 3\}$ and $\{2, 3, 4\}$. These two cliques intersect in $\{2, 3\}$. Moreover, if $K = \mathbb{Q}$, then using Macaulay2 to compute $\text{depth}(S/J_G)$ we get that $\text{depth}(S/J_G) = 4 < n + c$ (where $n = 4$ and $c = 1$).

![Diagram of a diamond graph with vertices 1, 2, 3, 4.]

Since we have found a formula for $\text{depth}(S/J_G)$, where $G$ is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex, we now want to determine when $J_G$ is a Cohen-Macaulay binomial edge ideal. We can start by considering one of the simplest subclasses of chordal graphs such that any two distinct maximal cliques intersect in at most one vertex: paths. This case was solved in section 2.2. More precisely, by corollary 2.2.36, $J_G$ is a Cohen-Macaulay ideal whenever $G$ is a path.

**Proposition 2.3.14.** Let $G$ be a chordal graph on $[n]$ such that any two distinct maximal cliques intersect in at most one vertex. Suppose that $i$ lies in exactly $r$ maximal cliques $F_1, \cdots, F_r$ of $G$. Then $c(\{i\}) = r + c - 1$, where $c$ is the number of connected components of $G$.

**Proof.** We can assume that $G$ is connected and so we want to show that $c(\{i\}) = r$.

Suppose there exist $j, k \in [r]$ such that $j \neq k$ and the cliques $F_j \setminus \{i\}$ and $F_k \setminus \{i\}$ are contained in the same connected component of $G_{[n] \setminus \{i\}}$. Pick $u \in F_j \setminus \{i\}$ and $v \in F_k \setminus \{i\}$ such that $d(u, v)$ (as vertices of $G_{[n] \setminus \{i\}}$) is the smallest possible. Consider a path $u, \cdots, v, i$ in $G_{[n] \setminus \{i\}}$ from $u$ to $v$ of length $d(u, v)$. Then the only vertex on such path which lies in $F_k$ is $v$. On the other hand, $i, u, \cdots, v, i$ is a cycle in $G$, and since $G$ is chordal, that cycle has a vertex $w \notin F_k$ such that $\{i, w\}, \{v, w\} \in E(G)$ and so the vertices $i, v, w$ form a clique in $G$ which intersects $F_k$ in two vertices, a contradiction. Hence the cliques $F_j \setminus \{i\}$ and $F_k \setminus \{i\}$ are contained in distinct connected components of $G_{[n] \setminus \{i\}}$. Since $j, k \in [r]$ are arbitrary, it follows that $c(\{i\}) \geq r$.

It remains to show that $c(\{i\}) \leq r$. Let $u \in V(G) \setminus \{i\}$. Since $G$ is connected, there exists a path $u, \cdots, v, i$ from $u$ to $i$. Then $u$ lies in the same connected component of $G_{[n] \setminus \{i\}}$ as $v$. But $v$ is adjacent to $i$, therefore...
\( v \in F_j \setminus \{i\} \) for some \( j \in [r] \). Hence \( u \) must lie in the connected component of \( G_{[n]\setminus\{i\}} \) which contains the maximal clique \( F_j \setminus \{i\} \). Since \( u \in V(G) \) is arbitrary, it follows that \( c(\{i\}) \leq r \).

\[ \square \]

**Theorem 2.3.15.** Let \( G \) be a chordal graph on \( |v| \) such that any two distinct maximal cliques intersect in at most one vertex. The following conditions are equivalent:

1. \( J_G \) is unmixed.

2. \( J_G \) is Cohen-Macaulay.

3. Each vertex of \( G \) lies in at most two distinct maximal cliques of \( G \).

**Proof.** The implication (2) \( \Rightarrow \) (1) follows from proposition \[\text{0.5.13}\] If \( J_G \) is unmixed, by corollary \[\text{2.2.33}\] one has \( \dim(S/J_G) = n + c \), where \( c \) is the number of connected components of \( G \). Theorem \[\text{2.3.11}\] tells us that \( \depth(S/J_G) = n + c \), therefore \( J_G \) is Cohen-Macaulay. Hence (1) and (2) are equivalent.

It remains to show (1) and (3) are equivalent. For this part we can assume that \( G \) is connected.

Suppose \( J_G \) is unmixed. Let us assume that there is a vertex \( i \) of \( G \) which lies in at least three distinct maximal cliques of \( G \). By proposition \[\text{2.3.14}\], \( c(\{i\}) \geq 3 \) and so, by lemma \[\text{2.2.25}\] one has \( \ht(P_{\{i\}}(G)) = n + 1 - c(\{i\}) \leq n - 2 \). Moreover, corollary \[\text{2.2.43}\] implies that \( P_{\{i\}}(G) \) is a minimal prime ideal of \( J_G \). But, by corollary \[\text{2.2.30}\] \( P_{\phi}(G) \) is also a minimal prime of \( J_G \) and \( \ht(P_{\phi}(G)) = n - 1 \), which contradicts the hypothesis that \( J_G \) is unmixed.

Suppose each vertex of \( G \) is the intersection of at most two maximal cliques. Let \( \{i_1, \ldots, i_r\} \) be the intersection vertices of the maximal cliques of \( G \) and let \( P_R(G) \in \Min(J_G) \). Suppose \( R \nsubseteq \{i_1, \ldots, i_r\} \). Pick \( i \in R \setminus \{i_1, \ldots, i_r\} \). By corollary \[\text{2.2.43}\] we have \( c(R \setminus \{i\}) < c(R) \). This implies that there exist two connected components of \( G_{[n]\setminus R} \), say \( H_1 \) and \( H_2 \), such that \( i \) is connected to \( H_1 \) and \( H_2 \). Let \( u \in V(H_1) \) and \( v \in V(H_2) \) such that \( \{i, u\}, \{i, v\} \in E(G) \). Since \( u \) and \( v \) lie in distinct connected components of \( G_{[n]\setminus R} \), then \( \{u, v\} \notin E(G) \), and since \( \{i, u\}, \{i, v\} \in E(G) \), then there exist two distinct maximal cliques in \( G \) containing \( \{i, u\} \) and \( \{i, v\} \), respectively, which must intersect in \( i \) and so \( i \in \{i_1, \ldots, i_r\} \), a contradiction. Hence \( R \subseteq \{i_1, \ldots, i_r\} \).

Now we show by induction on the cardinality of \( R \) that \( c(R) = |R| + 1 \). If \( R = \emptyset \), the result is obvious. Suppose \( R \neq \emptyset \). Pick \( i \in R \). By proposition \[\text{2.3.14}\] (with \( r = 2 \) and \( c = c(R \setminus \{i\}) \)), \( c(R) = c(R \setminus \{i\}) + 1 \). By induction, \( c(R \setminus \{i\}) = |R| \) and so \( c(R) = |R| + 1 \).

Hence, by lemma \[\text{2.2.25}\] \( \ht(P_R(G)) = |R| + (n - c(R)) = n - 1 \), and since \( P_R(G) \in \Min(J_G) \) is arbitrary, \( J_G \) is unmixed.

\[ \square \]

**Corollary 2.3.16.** If \( G \) is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex, then the Cohen-Macaulayness of \( J_G \) does not depend on the base field \( K \) chosen but only on the combinatorial data of the graph \( G \).
Example 2.3.17. If $G$ is the claw with $E(G) = \{\{1, 2\}, \{1, 3\}, \{1, 4\}\}$, then $G$ is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex, hence $\text{depth}(S/J_G) = 5$. But the vertex 1 is the intersection of three maximal cliques, hence $J_G$ is not Cohen-Macaulay.

![Diagram of a claw graph]

Example 2.3.18. If $G$ is such that $E(G) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{3, 6\}\}$, then $G$ is a chordal graph such that any two distinct maximal cliques intersect in at most one vertex. Since, each vertex of $G$ lies in at most two distinct maximal cliques of $G$, then $J_G$ is Cohen-Macaulay.

![Diagram of a chordal graph]

If $G$ is an arbitrary chordal graph, then it is not necessarily true that $J_G$ being unmixed implies $J_G$ being Cohen-Macaulay.

Example 2.3.19. Let $G$ be the graph as below. Then $G$ is a chordal graph with maximal cliques

$\{1, 2, 3\}, \{2, 3, 4\}, \{2, 3, 5\},$

therefore the edge $\{2, 3\}$ is contained in the three maximal cliques of $G$. Using Macaulay2 (it does not matter which field we consider) we get that $\text{Min}(J_G) = \{P_\emptyset(G), (x_2, x_3, y_2, y_3)\}$, and since $\dim(S/P_\emptyset(G)) = \dim(S/(x_2, x_3, y_2, y_3)) = 6$, it follows that $J_G$ is unmixed. However, $J_G$ is not Cohen-Macaulay. In fact, if $K = \mathbb{Q}$, then using Macaulay2 to compute $\text{depth}(S/J_G)$ we get that $\text{depth}(S/J_G) = 5$. 

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Corollary 2.3.20. Let $G$ be a forest on the vertex set $[n]$. The following conditions are equivalent:

1. $J_G$ is unmixed.
2. $J_G$ is a Cohen-Macaulay.
3. $J_G$ is a complete intersection.
4. Each connected component of $G$ is a path.

Proof. The implications $(3) \Rightarrow (2) \Rightarrow (1)$ follow from corollaries 0.5.10 and 0.5.13 while $(1) \Rightarrow (4)$ follows from implication $(1) \Rightarrow (3)$ of theorem 2.3.15. It remains to show the implication $(4) \Rightarrow (3)$.

By proposition 2.2.35 the binomial edge ideal of a path is a complete intersection, therefore the binomial edge ideal of a disjoint union of paths is also a complete intersection, hence the implication $(4) \Rightarrow (3)$ follows.

Example 2.3.21. We can also show that $J_G$ is not Cohen-Macaulay when $G$ is a claw using this corollary. In fact, $G$ is a tree which is not a path, hence the result follows.

2.4 Closed graphs

In the previous section we studied which chordal graphs such that any two distinct maximal cliques intersect in at most one vertex have a Cohen-Macaulay binomial edge ideal. In this section we will answer the same question, but for closed graphs. Recall that closed graphs were defined in section 2.2.

Let $G$ be a closed graph. Then the binomials $f_{ij}$ with $i < j$ and $\{i, j\} \in E(G)$ form a Gröbner basis for $G$ and so the square-free monomials $\text{in}_<(f_{ij}) = x_iy_j$ are the generators of $\text{in}_<(J_G)$, and so there exists a unique bipartite graph $H$ on the set of vertices $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$ whose monomial edge ideal is $\text{in}_<(J_G)$. The equality $\text{in}_<(J_G) = I(H)$ suggests the notation $H = \text{in}_<(G)$. Equivalently, $H = \text{in}_<(G)$ is the
bipartite graph on the set of vertices \( \{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\} \) such that \( \{x_i, y_j\} \in E(H) \) if and only if \( i < j \) and \( \{i, j\} \in E(G) \).

Since monomial edge ideals are easier to work with than binomial edge ideals, this construction will be used in this section to classify Cohen-Macaulay binomial edge ideals of closed graphs and in section 3.2 to determine the regularity of those ideals.

**Proposition 2.4.1.** If \( G \) is a claw, then \( G \) is not closed for any labelling of \( G \).

**Proof.** Suppose \( G \) is closed. Since \( G \) is a claw, one can assume that \( E(G) = \{\{i, j\}, \{i, k\}, \{i, l\}\} \) for some distinct positive integers \( i, j, k, l \). Suppose without loss of generality that \( i < j \). Since \( G \) is closed, \( \{i, j\}, \{i, k\} \in E(G) \) and \( \{j, k\} \notin E(G) \), then \( i > k \). Similarly, \( i > l \). But since \( G \) is closed, \( i > \max\{k, l\} \) and \( \{i, k\}, \{i, l\} \in E(G) \), it follows that \( \{k, l\} \in E(G) \), a contradiction. \( \square \)

![diagram](image)

**Proposition 2.4.2.** Any induced subgraph of a closed graph is closed as well.

**Proposition 2.4.3.** If \( G \) is a closed graph, then \( G \) is a claw-free (that is, none of its induced subgraphs is a claw) chordal graph.

**Proof.** Suppose \( G \) is closed. Let \( C \) be a cycle in \( G \) with length greater than 3. Pick the vertex \( i \in C \) such that \( i \leq j, \forall j \in C \). Let \( j \) and \( k \) be the two vertices in \( C \) which are adjacent to \( i \). Since \( i < \min\{j, k\} \) and \( \{i, j\}, \{i, k\} \in E(G) \), then \( \{j, k\} \) is also an edge of \( G \), hence it is a chord of \( C \).

Since \( G \) is closed, any induced subgraph of \( G \) is closed as well. But we have shown that claws are not closed for any of their \( 4! = 24 \) possible labellings, therefore \( G \) is claw-free. \( \square \)

**Example 2.4.4.** Cycles of length greater than 3 are not closed since they are not chordal in the first place.

**Example 2.4.5.** The graph with edges \( \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 4\}, \{2, 5\}, \{3, 6\} \) is a claw-free chordal graph, but is not closed. In fact, this graph is not closed for any of the \( 6! = 720 \) labels which can be considered.
A closed graph is chordal but may not belong to the class of chordal graphs we considered in section 2.3.

**Example 2.4.6.** If $G$ is the diamond with $E(G) = \{ \{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\} \}$, then $G$ is a closed graph with two maximal cliques: $\{1, 2, 3\}$ and $\{2, 3, 4\}$. These two cliques intersect in $\{2, 3\}$.

**Definition 2.4.7.** A path $i_0, \ldots, i_l$ is called directed if the sequence $i_0, \ldots, i_l$ is monotonic.

Directed paths are clearly closed graphs.

**Proposition 2.4.8.** A graph $G$ on $[n]$ is closed if and only if, for any two vertices $i \neq j$ on $G$, all paths of shortest length from $i$ to $j$ are directed.

**Proof.** Suppose that, for any two vertices $i \neq j$ on $G$, all paths of shortest length from $i$ to $j$ are directed.

Let $\{i, j\}, \{i, k\} \in E(G)$, with $i < \min\{j, k\}$ and $j \neq k$. Then $j, i, k$ is a path from $j$ to $k$ which is not directed, so it cannot be the shortest path. Hence $\{j, k\} \in E(G)$. Similarly it follows that if $\{i, k\}, \{j, k\} \in E(G)$, with $k > \max\{i, j\}$ and $i \neq j$, then $\{i, j\} \in E(G)$. This shows that $G$ is closed.
Conversely, assume that $G$ is closed. Let $i$ and $j$ be two distinct vertices and let $P$ be a path of shortest length from $i$ to $j$. Suppose $P$ is not directed. Then there exists a subpath $r, s, t$ of $P$ such that either $s < \min\{r, t\}$ or $s > \max\{r, t\}$. Then, since $G$ is closed and $r \neq t$, it follows that $\{r, t\} \in E(G)$. Replacing the subpath $r, s, t$ by $r, t$ we obtain a shorter path from $i$ to $j$, a contradiction.

As it happens with monomial edge ideals:

**Proposition 2.4.9.** The binomial edge ideal of a graph is Cohen-Macaulay if and only if each of the binomial edge ideals of its connected components is Cohen-Macaulay.

**Proof.** As in the proof for monomial edge ideals, one just needs to use corollary 0.5.16 and induction.

Thus it is enough to consider connected graphs. In section 2.1, we gave necessary and sufficient conditions for a bipartite graph $G$ without isolated vertices to be Cohen-Macaulay. If $G$ is a closed bipartite graph, it is easy to find out that $J_G$ is a Cohen-Macaulay ideal.

**Corollary 2.4.10.** A connected bipartite graph is closed if and only if it is a directed path.

**Proof.** A bipartite graph has no odd cycles. Since a closed graph is chordal and a connected chordal graph has a cycle of length 3 unless it is a tree, a connected closed bipartite graph must be a tree. If such tree is not a path, then there exists an induced subgraph which is a claw. Thus a connected closed bipartite graph must be a path, and since it is closed, it must be directed.

Conversely, if $G$ is a directed path, then $G$ is clearly closed.

**Corollary 2.4.11.** If $G$ is a closed bipartite graph, then $J_G$ is a Cohen-Macaulay ideal.

**Proof.** By proposition 2.4.9, we may assume that $G$ is connected. By corollary 2.4.10 $G$ is a path, and by corollary 2.2.36 $J_G$ is a Cohen-Macaulay ideal.

**Corollary 2.4.12.** If $G$ is a connected closed graph on $[n]$, then $\{i, i+1\} \in E(G)$ for all $i \in [n-1]$.

**Proof.** Since $G$ is connected, then in particular there exists a path in $G$ from $i$ to $i+1$. Since $G$ is closed, by proposition 2.4.8, the shortest path between $i$ and $i+1$ must be directed. But such path must necessarily be $i, i+1$ and so $\{i, i+1\} \in E(G)$.

A sufficient condition for a closed graph to have a Cohen-Macaulay binomial edge ideal is given by the following proposition:

**Proposition 2.4.13.** Let $G$ be a connected closed graph on $[n]$. Suppose that for every $i \leq j \leq k$ such that $\{i, j+1\}, \{j, k+1\} \in E(G)$, it follows that $\{i, k+1\} \in E(G)$. Then $J_G$ is a Cohen-Macaulay ideal.

**Proof.** By corollary 1.3.39, it is enough to show that $\text{in}_<(J_G)$ is a Cohen-Macaulay ideal.

Since $G$ is a closed graph, by theorem 2.2.7 it follows that $\text{in}_<(J_G)$ is generated by the monomials $x_i y_j$ with $\{i, j\} \in E(G)$ and $i < j$. Applying the automorphism $\varphi : S \to S$ which maps each $x_i$ to $x_i$ and
\( y_j \) to \( y_{j-1} \) for \( j > 1 \) and \( y_1 \) to \( y_n \), \( \text{in}_c(J_G) \) is mapped to the ideal generated by all monomials \( x_i y_j \) with \( i \leq j \) and \( \{i, j+1\} \in E(G) \). This ideal has all its generators in \( S' = K[x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}] \). Let \( I \subset S' \) be the ideal generated by these monomials. Then \( S'/\text{in}_c(J_G) \) is Cohen-Macaulay if and only if \( S'/I \) is Cohen-Macaulay. Note that \( I \) is the monomial edge ideal of the bipartite graph \( H \) on the vertex set \( \{x_1, \ldots, x_{n-1}\} \cup \{y_1, \ldots, y_{n-1}\} \) with \( \{x_i, y_j\} \in E(H) \) if and only if \( i \leq j \) and \( \{i, j+1\} \in E(G) \). By proposition 2.1.16, \( H \) is a Cohen-Macaulay graph if the following three conditions hold:

- For every \( i \in [n-1], \{x_i, y_i\} \in E(H) \).
- If \( i > j \), then \( \{x_i, y_j\} \notin E(H) \).
- If \( \{x_i, y_j\}, \{x_j, y_k\} \in E(H) \), then \( \{x_i, y_k\} \in E(H) \).

Suppose \( i \in [n-1] \). Since \( G \) is a connected closed graph, by corollary 2.4.12 it follows that \( \{i, i+1\} \in E(G) \) and so \( \{x_i, y_i\} \in E(H) \), as desired.

The second condition is trivially satisfied.

The third condition is equivalent to our assumption that whenever \( i \leq j \leq k \) are such that \( \{i, j+1\}, \{j, k+1\} \in E(G) \), it follows that \( \{i, k+1\} \in E(G) \).

We will later show that this sufficient condition turns out to be a necessary one.

**Example 2.4.14.** If \( G \) is a complete graph or a path (which we may assume that is directed), then \( G \) clearly satisfies the conditions of proposition 2.4.13, so \( J_G \) is a Cohen-Macaulay ideal, as we saw before.

**Example 2.4.15.** The paw \( G \) with \( E(G) = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{3, 4\}\} \) is such that \( J_G \) is Cohen-Macaulay for the same reasons.

**Proposition 2.4.16.** Let \( G \) be a connected closed graph on \([n]\) and let \( H \) be the induced subgraph of \( G \) on an interval \([a, b] \subset [n]\). Then \( H \) is also a connected closed graph.
Proof. Since $G$ is a closed graph and $H$ is an induced subgraph of $G$, then $H$ is also closed.

Since $G$ is a connected closed graph, then, by corollary 2.4.12 \{i, i + 1\} $\in E(G)$ for every $i \in [n - 1]$ and in particular \{i, i + 1\} $\in E(H)$ for every $i \in [a, b - 1]$. Hence $a, a + 1, \cdots, b - 1, b$ is a direct path in $H$ containing all its vertices and so $H$ is connected.

\[\Box\]

**Theorem 2.4.17.** Let $G$ be a connected graph on $[n]$. Then $G$ is closed if and only if all facets of $\Delta(G)$ are intervals on $[n]$.

\[\text{Proof.}\] Suppose there exists a labelling of $G$ such that all facets of $\Delta(G)$ are intervals on $[n]$. Let $\{i, j\}$ and $\{k, l\}$ be edges of $G$ with $i < j$ and $k < l$. If $i = k$ and $j \neq l$, then $\{i, j\}$ and $\{i, l\}$ belong to the same maximal clique, that is, a facet of $\Delta(G)$ which by assumption is an interval, and so $\{j, l\} \in E(G)$. Similarly one shows that if $j = l$ and $i \neq k$, then $\{i, k\} \in E(G)$. Thus $G$ is closed.

Suppose $G$ is closed. We will use induction on $n$. The case $n = 1$ is trivial, so assume $n > 1$.

Let $F$ be the union of $\{n\}$ with the set of vertices of $G$ which are adjacent to $n$ and let $k$ be the minimum of $F$. Then $F = [k, n]$. Indeed, if $j \in F \setminus \{n\}$, by corollary 2.4.12 one has $\{j, j + 1\} \in E(G)$ since $G$ is a connected closed graph. On the other hand, since $\{j, j + 1\}, \{j, n\} \in E(G)$ and $G$ is closed, it follows that either $j = n - 1$ or $\{j, n\} \in E(G)$. On the second case, $j + 1 \in F$. Since $j \in F \setminus \{n\}$ is arbitrary, starting with $j = k$ and proceeding with this algorithm until $j = n - 1$ we get that $F = [k, n]$.

Next observe that $F$ is a maximal clique of $G$, that is, a facet of $\Delta(G)$. First of all it is a clique, because if $i$ and $j$ are distinct elements of $F \setminus \{n\}$, then $\{i, n\}, \{j, n\} \in E(G)$ implies $\{i, j\} \in E(G)$, since $G$ is closed. Secondly, it is maximal since $\{j, n\} \notin E(G)$ for $j \notin F$.

Now let $H$ be an arbitrary facet of $\Delta(G)$. Since it was shown that $F$ is an interval, one can assume $H \neq F$. Since $H$ is a clique and $\{j, n\} \notin E(G)$ for every $j \in H \setminus F$, then $n \notin H$. Let $G'$ be the induced subgraph of $G$ on $[n - 1]$. By proposition 2.4.16, $G'$ is a closed connected graph. Since $H$ is a facet of $\Delta(G')$, then by induction $H$ is an interval on $[n - 1]$.

We now have four ways of describing connected closed graphs:

**Corollary 2.4.18.** Let $G$ be a connected graph. Then the following conditions are equivalent:

- $G$ is closed, that is, for all edges $\{i, j\}$ and $\{k, l\}$ with $i < j$ and $k < l$ one has $\{j, l\} \in E(G)$ if $i = k$ and $\{i, k\} \in E(G)$ if $j = l$.
- The generators $f_{ij}$ of $J_G$ form a quadratic Gröbner basis.
- For any two vertices $i \neq j$ on $G$, all paths of shortest length from $i$ to $j$ are directed.
- All facets of $\Delta(G)$ are intervals on $[n]$.

The last characterization of closed graphs mentioned will be essential to determine which closed graphs are Cohen-Macaulay.
Theorem 2.4.19. Let $G$ be a connected closed graph on $[n]$. Then the following conditions are equivalent:

1. $J_G$ is unmixed.
3. $\text{in}_<(J_G)$ is Cohen-Macaulay.
4. $G$ satisfies the condition that whenever $\{i, j + 1\}$ with $i \leq j$ and $\{j, k + 1\}$ with $j \leq k$ are edges of $G$, then $\{i, k + 1\}$ is an edge of $G$.
5. There exist integers $1 = a_1 < \cdots < a_r < a_{r+1} = n$ and a leaf order of the facets $F_1, \ldots, F_r$ of $\Delta(G)$ such that $F_i = [a_i, a_{i+1}]$ for every $i \in [r]$.

Proof. The implication (4) $\Rightarrow$ (3) is shown in the proof of proposition 2.4.13. The implications (3) $\Rightarrow$ (2) $\Rightarrow$ (1) are due to corollaries 1.3.39 and 0.5.13 respectively.

We now prove the implication (5) $\Rightarrow$ (4). Let $i \leq j \leq k$ be three vertices of $G$ such that $\{i, j + 1\}$ and $\{j, k + 1\}$ are edges of $G$. If $i = j$ or $j = k$, then the implication is obvious. Suppose $i < j < k$. Notice that $i$ and $j + 1$ belong to the same facet of $\Delta(G)$, let us say $F_i$. Since $i < j < j + 1$, then the only facet which contains $j$ is $F_i$. But then $k + 1$ must belong to $F_i$ as well since it is adjacent to $j$. Therefore $i$ and $k + 1$ are adjacent for they both belong to $F_i$.

It remains to show the implication (1) $\Rightarrow$ (5). Since $G$ is closed, then by theorem 2.4.17 $\Delta(G)$ has facets $F_1, \ldots, F_r$ where each facet is an interval. We may order the intervals $F_i = [a_i, b_i]$ such that $1 = a_1 < \cdots < a_r < n$. Then $1 < b_1 < \cdots < b_r = n$ and since $G$ is connected, it follows that $a_{i+1} \leq b_i$ for every $i \in [r - 1]$.

Let $i \in [r - 1]$ and $R = [a_{i+1}, b_i]$. Suppose $c(R) = 1$. Since $G$ is closed, then $G_{[n] \setminus R}$ is also closed, and since $c(R) = 1$, by corollary 2.4.12 it follows that $\{a_{i+1} - 1, b_i + 1\} \subseteq E(G)$. Let $j \in [r]$ such that $[a_j, b_j]$ is a facet of $\Delta(G)$ containing both $a_{i+1} - 1$ and $b_i + 1$. Then $a_j < a_{i+1}$, therefore $j \leq i$. On the other hand, $b_j > b_i$, therefore $j > i$, a contradiction. Hence $c(R) \geq 2$. Since $G$ is a connected closed graph, by proposition 2.4.16 $G_{[1, a_{i+1} - 1]}$ and $G_{[b_i + 1, n]}$ are connected graphs and so $c(R) = 2$, therefore $\text{ht}(P_R(G)) = n + |R| - c(R) = n + (b_i - a_{i+1} + 1) - 2 = n + (b_i - a_{i+1}) - 1$. Since $c(R) = 2$, then to show $P_R(G) \in \text{Min}(J_G)$, by corollary 2.2.43 it is enough to show that $G_{([n] \setminus R) \cup \{j\}}$ is connected for every $j \in R$.

Let $j \in R$. Since we have seen that $G_{[1, a_{i+1} - 1]}$ and $G_{[b_i + 1, n]}$ are connected graphs, then to show that $G_{([n] \setminus R) \cup \{j\}}$ is connected it is enough to show that $j$ is adjacent to $a_i \in [1, a_{i+1} - 1]$ and $b_i + 1 \in [b_i + 1, n]$. But $j \in [a_{i+1}, b_i] \subseteq [a_i, b_i] \cap [a_{i+1}, b_{i+1}]$ and the conclusion follows since both $[a_i, b_i]$ and $[a_{i+1}, b_{i+1}]$ are cliques in $G$.

Since both $P_G(G)$ and $P_R(G)$ are minimal primes of $J_G$, our assumption implies that $n + (b_i - a_{i+1}) - 1 = n - 1$, that is, $b_i = a_{i+1}$. Putting $a_{r+1} = b_r = n$ we get that $F_i = [a_i, a_{i+1}]$ for every $i \in [r]$. It is clear from this that $F_1, \ldots, F_r$ is a leaf order.

Corollary 2.4.20. If $G$ is a closed graph, then the Cohen-Macaulayness of $J_G$ does not depend on the base field $K$ chosen but only on the graph $G$. 

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Corollary 2.4.21. If $G$ is a closed graph such that $J_G$ is a Cohen-Macaulay ideal, then $G$ is a chordal graph such that every two distinct maximal cliques intersect in at most one vertex.

Example 2.4.22. If $G$ is the diamond with $E(G) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, then $G$ is a connected closed graph with two maximal cliques: $\{1, 2, 3\}$ and $\{2, 3, 4\}$. These two cliques intersect in two vertices, 2 and 3, hence $J_G$ is not a Cohen-Macaulay ideal.
Chapter 3

Regularity of binomial edge ideals

Castelnuovo-Mumford regularity is one of the most fundamental invariants in Commutative Algebra and Algebraic Geometry. In fact, already in the late 19th century this invariant was present, a long time before it was properly defined. One of its first hidden appearances may be found in Castelnuovo’s work.

However, a proper definition of Castelnuovo-Mumford regularity was only given in 1966 by Mumford, who called it Castelnuovo regularity. In fact, Mumford defined the notion of being $m$-regular in the sense of Castelnuovo for a coherent sheaf of ideals over a projective space and a given integer $m$. More precisely, a sheaf of ideals over a projective space is called $m$-regular if, for every $i > 0$, the $i$-th Serre cohomology group of the $(m-i)$-fold twist of this sheaf vanishes. The smallest $m$ such that the sheaf of ideals in question is $m$-regular is what today is usually called the Castelnuovo-Mumford regularity of such sheaf.

Castelnuovo-Mumford regularity also found much interest in Commutative Algebra. In 1982, Ooishi defined the Castelnuovo-Mumford regularity of a graded module in terms of certain local cohomology modules. His definition essentially corresponds to Mumford’s definition via the Serre-Grothendieck correspondence between local cohomology and sheaf cohomology. In 1984, Eisenbud and Goto made explicit the link between this “algebraic” Castelnuovo-Mumford regularity of a graded module over a polynomial ring and its minimal free resolution. In fact, in section 0.6 we defined the Castelnuovo-Mumford regularity of a graded module in terms of its minimal free resolution.

As Woodroofe wrote in [14], the Castelnuovo-Mumford regularity of an ideal $I$ is one of the main measures of the complexity of $I$. In this chapter, the regularity of binomial edge ideals will be studied.

In section 3.1 we state some results and conjectures about regularity bounds of binomial edge ideals.

In section 3.2 we show that, when $G$ is a closed graph, the regularity of $J_G$ is given by the lengths of its induced paths.

In section 3.3 we show that the conjectures stated in section 3.1 hold not only for closed graphs but also for the class of graphs considered in section 2.3.
3.1 Regularity bounds

As in the previous chapter, \( S = K[x_1, \cdots, x_n, y_1, \cdots, y_n] \) will be a polynomial ring. On section 0.6 we defined the Castelnuovo-Mumford regularity of a graded \( S \)-module and stated some of its properties.

It is easy to determine the regularity of \( J_G \) when \( G \) is a path, for in this case \( J_G \) is a complete intersection.

**Proposition 3.1.1.** If \( G \) is a path of length \( l \), then \( \text{reg}(S/J_G) = l \).

**Proof.** Suppose without loss of generality that \( E(G) = \{\{1, 2\}, \cdots, \{l, l+1\}\} \). Then by proposition 2.2.35 \( J_G = (f_{12}, f_{23}, \cdots, f_{l,l+1}) \) is a complete intersection. Using corollary 0.6.14 with \( M = S \) (and so \( \text{reg}(M) = 0 \)) and \( k_1 = \cdots = k_l = 2 \) we get the desired result.

Let \( G \) be a graph with connected components \( G_1, \cdots, G_c \). If \( S_i \) is the polynomial ring in the indeterminates indexed by the vertex set of \( G_i \) for \( i \in [c] \), then, as a consequence of lemma 0.6.15 and induction, \( \text{reg}(S/J_G) = \sum_{i=1}^c \text{reg}(S_i/J_{G_i}) \). Hence to study the regularity of binomial edge ideals, we only need to consider connected graphs. Now suppose \( G \) is a connected graph on \([n]\) and let \( l \) be the length of the longest induced path of \( G \). The equality \( \text{reg}(S/J_G) = l \) does not necessarily hold.

**Example 3.1.2.** If \( G \) is the graph as below, then \( G \) is not a closed graph and the largest induced path in \( G \) has length 4. If \( K = \mathbb{Q} \), then using Macaulay2 to compute \( \text{reg}(S/J_G) \) we get that \( \text{reg}(S/J_G) = 6 \).

![Graph Image]

However, in [16], Matsuda and Murai showed the following theorem:

**Theorem 3.1.3.** If \( G \) is a graph on \([n]\), then \( l \leq \text{reg}(S/J_G) \leq n - 1 \), where \( l \) is the length of the longest induced path of \( G \).

**Proof.** The proofs of these two inequalities are shown in [16] and they go beyond the techniques used in this dissertation. More precisely, in these proofs there are considered gradings on \( S \) other than the usual one. In fact, the usual grading is an \( \mathbb{N} \)-grading while the gradings considered to show that \( \text{reg}(S/J_G) \geq l \) and that \( \text{reg}(S/J_G) \leq n - 1 \) are \( \mathbb{N}^n \)-gradings and \( \mathbb{N}^{2n} \)-gradings, respectively.

**Corollary 3.1.4.** Let \( G \) be a graph on \([n]\) with connected components \( G_1, \cdots, G_c \). Then

\[
l_1 + \cdots + l_c \leq \text{reg}(S/J_G) \leq n - c,
\]

where, for \( i \in [c] \), \( l_i \) is the length of the largest induced path of \( G_i \).
Proof. Just combine the equality \( \text{reg}(S/J_G) = \sum_{i=1}^{c} \text{reg}(S_i/J_{G_i}) \) with the inequalities \( l_i \leq \text{reg}(S_i/J_{G_i}) \leq |V(G_i)| - 1 \) where \( i \in [c] \).

In section 3.2 it will be shown that the equality \( \text{reg}(S/J_G) = l_1 + \cdots + l_c \) turns out to be true whenever \( G \) is a closed graph.

**Corollary 3.1.5.** If \( G \) is a connected graph such that \( \text{reg}(S/J_G) = 1 \), then \( G \) is a complete graph.

**Proof.** Suppose by contradiction that \( G \) is not complete. Then \( G \) has an induced path of length 2, hence \( \text{reg}(S/J_G) \geq 2 \). In section 3.2 we will see that the converse of this corollary holds.

In [15], Madani and Kiani conjectured that if \( G \) is a graph with \( r \) maximal cliques, then \( \text{reg}(S/J_G) \leq r \). Note that it is not always true that \( r \leq n - c \).

**Example 3.1.6.** If \( G \) is the cycle of length 4, then \( n = 4 \), \( c = 1 \) and \( r = 4 \), hence \( r > n - c \).

**Proposition 3.1.7.** Let \( G \) a graph with connected components \( G_1, \ldots, G_c \) and with \( r \) maximal cliques. Then \( l_1 + \cdots + l_c \leq r \), where, for \( i \in [c] \), \( l_i \) is the length of the largest induced path in \( G_i \).

**Proof.** Since \( r = r_1 + \cdots + r_c \), where, for \( i \in [c] \), \( r_i \) is the number of maximal cliques in \( G_i \), then it is enough to show this when \( G \) is connected.

Let \( P \) be a path in \( G \) of length greater than \( r \). Then two of the edges of \( P \), say \( \{u, v\}, \{u', v'\} \), with \( v' \notin \{u, v\} \), must be contained in the same maximal clique of \( G \) and in particular \( \{v, v'\} \in E(G) \) and so \( P \) is not an induced path of \( G \).

Hence, once proven that \( \text{reg}(S/J_G) = l_1 + \cdots + l_c \) when \( G \) is a closed graph, it will follow that the Madani-Kiani conjecture holds for closed graphs. We will show in section 3.3 that it also turns out to be true for the class of graphs considered in section 2.3.

In [16], Matsuda and Murai conjectured that if \( G \) is a graph on \( [n] \), then \( \text{reg}(S/J_G) = n - 1 \) if and only if \( G \) is a path. Once proven that \( \text{reg}(S/J_G) = l_1 + \cdots + l_c \) when \( G \) is a closed graph, it will follow that this conjecture holds for closed graphs. As with the Madani-Kiani conjecture, in section 3.3 it will also be shown that this conjecture turns out to be true for the class of graphs considered in section 2.3.

While in this dissertation the Matsuda-Murai conjecture was meant to be presented as a conjecture, it happened to be fully shown this year, on April 6th, in [18] by Madani and Kiani. Since the partial proof for chordal graphs such that any two maximal cliques intersect in at most one vertex is an interesting proof which...
uses some nice results on the combinatorial data of these graphs, I decided to keep it in the dissertation, not forgetting to indicate a reference for the full proof of the Matsuda-Murai conjecture.

### 3.2 Regularity of binomial edge ideals of closed graphs

If $G$ is a closed graph, then by theorem 1.3.38, $\text{reg}(S/J_G) \leq \text{reg}(S/\text{in}_<(J_G))$. But $\text{in}_<(J_G)$ is the monomial edge ideal of a bipartite graph and so results on the regularity of monomial edge ideals of bipartite graphs can be used to study the regularity of binomial edge ideals of closed graphs.

**Lemma 3.2.1.** Let $G$ be a connected closed graph on $[n]$. Then the bipartite graph $H = \text{in}_<(G)$ is weakly chordal.

**Proof.** Let $C$ be a cycle in $\overline{H}$ of length greater than 4. The induced subgraphs of $\overline{H}$ on the sets $\{x_1, \cdots, x_n\}$ and $\{y_1, \cdots, y_n\}$ are complete graphs. Since $C$ is a cycle of length greater than 4, then it contains at least three vertices from one of those two complete graphs, therefore it has a chord.

Since $H$ is a bipartite graph on $\{x_1, \cdots, x_n\} \cup \{y_1, \cdots, y_n\}$, then a cycle $C$ in $H$ of length greater than 4 has set of vertices $\{x_{i_1}, y_{j_1}, \cdots, x_{i_k}, y_{j_k}\}$, where $k \geq 3$. We make the convention that $i_{k+1} = i_1$ and $j_{k+1} = j_1$. We divide the proof that $C$ has a chord in the following steps:

1. Check that, for $l \in [k]$, $\max\{i_l, i_{l+1}\} < j_l$ and $\{i_l, j_l\}, \{i_{l+1}, j_l\}, \{i_l, i_{l+1}\}, \{j_l, j_{l+1}\} \in E(G)$.

2. Consider the case when there exists $l \in [k]$ such that $i_l < j_{l+1} < j_l$. This case is easy to solve. The other case is when, for every $l \in [k]$, either $j_{l+1} \leq i_l$ or $j_l < j_{l+1}$ holds.

3. For the remaining case, pick $t \in [k]$ such that $i_t$ is the maximum of the sequence $i_1, \cdots, i_k, i_{k+1} = i_1$ and start by checking that one of two following inequalities hold: $i_{t+1} < j_{t+1} \leq i_t < j_t$ or $i_{t+1} < i_t < j_t < j_{t+1}$.

4. By last, check that any of those two inequalities will imply that $C$ has a chord.

For the first step, let $l \in [k]$. Then $\{x_{i_l}, y_{j_l}\} \in E(H)$, that is, $i_l < j_l$ and $\{i_l, j_l\} \in E(G)$. On the other hand, $\{x_{i_{l+1}}, y_{j_l}\} \in E(H)$, that is, $i_{l+1} < j_l$ and $\{i_{l+1}, j_l\} \in E(G)$. Let $l \in [k]$. Since $G$ is closed, $\max\{i_l, i_{l+1}\} < j_l$ and $\{i_l, j_l\}, \{i_{l+1}, j_l\} \in E(G)$, it follows that $\{i_l, i_{l+1}\} \in E(G)$. Since $G$ is closed, $i_{l+1} < \min\{j_l, j_{l+1}\}$ and $\{i_l, j_l\}, \{i_{l+1}, j_{l+1}\} \in E(G)$, it follows that $\{j_l, j_{l+1}\} \in E(G)$.

For the second step, let $l \in [k]$ such that $i_l < j_{l+1} < j_l$. Since $G$ is closed, $i_l < j_{l+1} < j_l$ and $\{i_l, j_l\}, \{j_{l+1}, j_l\} \in E(G)$, it follows that $\{i_l, j_{l+1}\} \in E(G)$ and so $\{x_{i_l}, y_{j_{l+1}}\} \in E(H)$, hence $\{x_{i_l}, y_{j_{l+1}}\}$ is a chord in $C$.

For the third step we know that, for every $l \in [k]$, either $j_{l+1} \leq i_l$ or $j_l < j_{l+1}$ holds. In the first case, the inequalities $i_l < j_l$ and $i_{l+1} < j_{l+1}$ imply

$$i_{l+1} < j_{l+1} \leq i_t < j_t. \tag{3.1}$$
In the second case, the inequality \( \max\{i_t, i_{t+1}\} < j_t \) implies the inequality

\[
\max\{i_t, i_{t+1}\} < j_t < j_{t+1}. \tag{3.2}
\]

Pick \( t \in [k] \) such that \( i_t \) is the maximum of the sequence \( i_1, \cdots, i_k, i_{k+1} = i_t \). Since this sequence cannot have two consecutive equal terms, then \( \max\{i_{t-1}, i_{t+1}\} < i_t \). Since \( i_{t-1} < i_t \), the inequality (3.1) with \( l = t - 1 \) cannot hold, and so the inequality (3.2) with \( l = t - 1 \), that is, \( \max\{i_t, i_{t+1}\} < j_{t-1} < j_t \), holds. Moreover, \( i_{t-1} < i_t \) implies \( i_{t-1} < i_t < j_{t-1} < j_t \). On the other hand, one of the two inequalities (3.1) and (3.2), with \( l = t \), must hold. The first inequality is \( i_{t+1} < j_{t+1} \leq i_t < j_t \). Since \( i_{t+1} < i_t \), the second inequality implies that \( i_{t+1} < i_t < j_t < j_{t+1} \).

Now we end with the fourth and final step.

If \( i_{t+1} < j_{t+1} \leq i_t < j_t \) holds, we get the chain \( i_{t+1} < j_{t+1} \leq i_t < j_t - 1 < j_t \). Since \( G \) is closed, \( i_{t+1} < j_{t-1} < j_t \) and \( \{i_{t+1}, j_t\}, \{j_{t-1}, j_t\} \in E(G) \), it follows that \( \{i_{t+1}, j_t\} \in E(G) \) and so \( \{x_{i_{t+1}}, y_{j_{t-1}}\} \in E(H) \), hence \( \{x_{i_{t+1}}, y_{j_{t-1}}\} \) is a chord in \( C \).

If \( i_{t+1} < i_t < j_{t+1} \) holds, since \( G \) is closed, \( i_{t+1} < j_{t+1} < i_t \) and \( \{i_{t+1}, j_t\}, \{i_t, j_{t+1}\} \in E(G) \), it follows that \( \{i_t, j_{t+1}\} \in E(G) \) and so \( \{x_{i_t}, y_{j_{t+1}}\} \in E(H) \), hence \( \{x_{i_t}, y_{j_{t+1}}\} \) is a chord in \( C \).

Corollary 3.2.2. Let \( G \) be a closed graph on \([n]\) and let \( H = \text{in}_< (G) \). Then

\[
\text{reg}(K[x_1, \cdots, x_n, y_1, \cdots, y_n]/I(H)) = \text{indmatch}(H).
\]

Proof. By lemma 3.2.1 \( H \) is a weakly chordal bipartite graph. By theorem 2.1.9 the result follows. □

Proposition 3.2.3. Let \( G \) be a connected closed graph on \([n]\) and \( H = \text{in}_< (G) \). Then \( \text{indmatch}(H) = l \), where \( l \) is the length of the longest induced path of \( G \).

Proof. First we show that \( \text{indmatch}(H) \geq l \). Let \( i_0, \cdots, i_l \) be an induced path in \( G \) of length \( l \). If such path is not directed, then one can suppose without loss of generality that there exists \( t \in [l - 1] \) such that \( \max\{i_{t-1}, i_{t+1}\} < i_t \). But then, since \( \{i_{t-1}, i_t\}, \{i_t, i_{t+1}\} \in E(G) \) and \( i_{t-1} \neq i_{t+1} \), it follows that \( \{i_{t-1}, i_{t+1}\} \in E(G) \), which contradicts the fact that \( i_0, \cdots, i_l \) is an induced path in \( G \). Hence \( i_0, \cdots, i_l \) is a directed path and in particular we can assume that \( i_0 < \cdots < i_l \) and so the edges \( \{x_{i_0}, y_{i_1}\}, \cdots, \{x_{i_{l-1}}, y_{i_l}\} \) form an induced subgraph of \( H \), which is an induced matching.

We now show that \( \text{indmatch}(H) \leq l \). Let \( \text{indmatch}(H) = m \). Then \( H \) has \( m \) pairwise disjoint edges \( \{x_{i_1}, y_{j_1}\}, \cdots, \{x_{i_m}, y_{j_m}\} \) that form an induced subgraph of \( H \). We may assume that \( i_1 < \cdots < i_m \). To show the desired inequality we construct an induced path of length \( m \) in \( G \). We now divide the proof in the following steps:

1. Show that one can suppose without loss of generality that if \( t \in [n] \) and \( s \in [m] \) are such that \( t < i_s \) and \( \{x_t, y_s\} \in E(H) \), then \( \{x_t, y_{j_1}\}, \cdots, \{x_{i_{s-1}}, y_{j_{s-1}}\}, \{x_t, y_{j_s}\}, \{x_{i_{s+1}}, y_{j_{s+1}}\}, \cdots, \{x_{i_m}, y_{j_m}\} \) is not an induced matching of \( H \).
2. Show that the inequality \( i_1 < j_1 \leq i_2 < j_2 \leq \cdots < j_{m-1} \leq i_m < j_m \) holds.

3. We will show that \( i_1, \ldots, i_m, j_m \) is an induced path in \( G \). First, we show that \( \{i_s, j_m\} \notin E(G) \) for every \( s \in [m-1] \) and that \( \{i_s, i_{s'}\} \notin E(G) \) whenever \( s' > s + 1 \).

4. Finally, we show that \( \{i_s, i_{s+1}\} \in E(G) \) for every \( s \in [m-1] \). Suppose \( \{i_s, i_{s+1}\} \notin E(G) \) for some \( s \in [m-1] \). First we show that \( \{j_s, i_{s+1}\} \notin E(G) \).

5. Secondly, consider \( j \) to be the smallest \( t \) such that \( \{t, i_{s+1}\} \in E(G) \). Show that \( j > j_s \) and, from there, get a contradiction. We conclude that \( \{i_s, i_{s+1}\} \in E(G) \) for every \( s \in [m-1] \).

For the first step, let \( s \in [m] \). Since \( \{x_i, y_j\} \in E(H) \), then \( i_s < j_s \) and \( \{i_s, j_s\} \in E(G) \).

Let \( i'_1 \) be the smallest integer \( t \) such that \( \{x_t, y_j\} \in E(H) \) and \( \{x_t, y_j\} \in E(G) \) is an induced matching of \( H \). Then \( i'_1 \) is well defined and \( i'_1 \leq i_1 \). Moreover, if \( t < i'_1 \) is such that \( \{x_t, y_j\} \in E(H) \), then \( \{x_t, y_j\} \in E(G) \) is not an induced matching.

Let \( i'_2 \) be the smallest integer \( t \) such that \( \{x_t, y_j\} \in E(H) \) is an induced matching of \( H \). Then \( i'_2 \) is well defined and \( i'_2 \leq i_2 \). Moreover, if \( t < i'_2 \) is such that \( \{x_t, y_j\} \in E(H) \), then \( \{x_t, y_j\} \in E(G) \) is not an induced matching.

Proceeding with this construction, we define \( i'_3, \ldots, i'_m \) such that \( \{x_i', y_j\}, \ldots, \{x_{i_m}, y_{j_m}\} \) is an induced matching of \( H \) and if \( t \in [n] \) and \( s \in [m] \) are such that \( t < i'_s \) and \( \{x_t, y_j\} \in E(H) \), then \( \{x_t, y_j\} \in E(G) \) is not an induced matching of \( H \). To simplify notations, we replace \( i'_1, \ldots, i'_m \) by \( i_1, \ldots, i_m \) in the affirmation we just made. That is, \( \{x_i, y_j\}, \ldots, \{x_{i_m}, y_{j_m}\} \) is an induced matching of \( H \) and if \( t \in [n] \) and \( s \in [m] \) are such that \( t < i_s \) and \( \{x_t, y_j\} \in E(H) \), then \( \{x_t, y_j\} \in E(G) \) is not an induced matching of \( H \).

For the second step, suppose there exists \( s \in [m-1] \) such that \( j_s > i_{s+1} \). Since \( G \) is closed, then by theorem \( 2.4.17 \), all facets of \( \Delta(G) \) are intervals, and since \( \{i_s, j_s\} \in E(G) \), it follows that \( [i_s, j_s] \) is a face in \( \Delta(G) \), and since \( i_s < i_{s+1} < j_s \), it follows in particular that \( \{i_{s+1}, j_s\} \in E(G) \), hence \( \{i_{s+1}, y_{j_s}\} \in E(H) \).

But then the edges \( \{x_i, y_j\}, \{x_{i+1}, y_{j+1}\} \) do not form an induced matching of \( H \), a contradiction. Hence \( j_s \leq i_{s+1} \) for every \( s \in [m-1] \).

Then the following inequality holds: \( i_1 < j_1 \leq i_2 < j_2 \leq \cdots < j_{m-1} \leq i_m < j_m \).

Now consider the third step. Suppose that there exists \( s \in [m-1] \) such that \( \{i_s, j_m\} \in E(G) \). Since \( i_s < j_m \), then \( \{x_i, y_j\} \in E(H) \), hence the edges \( \{x_i, y_j\}, \{x_{i_m}, y_{j_m}\} \) do not form an induced matching of \( H \), a contradiction.

Suppose there exists \( s, s' \in [m] \) such that \( s' > s + 1 \) and \( \{i_s, i_{s'}\} \in E(G) \). Since all facets of \( \Delta(G) \) are intervals, then \( [i_s, i_{s'}] \) is a face in \( \Delta(G) \), and since \( i_s < j_s \leq i_{s+1} \leq i_{s'} \leq j_{s'} \leq i_{s'} \), it follows in particular that \( \{i_{s+1}, j_s\} \in E(G) \), and since \( i_s < j_{s'} \), then \( \{x_i, y_{j_{s'-1}}\} \in E(H) \). But then the edges \( \{x_i, y_{j_s}\}, \{x_{i_{s'-1}}, y_{j_{s'-1}}\} \) do not form an induced matching of \( H \), a contradiction.
For the fourth step, let \( s \in [m - 1] \) such that \( \{i_s, i_{s+1}\} \notin E(G) \). Since \( j_s \leq i_{s+1} \) and \( \{i_s, j_s\} \in E(G) \), then \( j_s \neq i_{s+1} \), therefore \( j_s < i_{s+1} \) and so \( i_s < j_s < i_{s+1} \). Suppose \( \{j_s, i_{s+1}\} \in E(G) \). We will show that, in this case,

\[
\{x_{i_1}, y_{j_1}\}, \ldots, \{x_{i_s}, y_{j_s}\}, \{x_{i_s}, y_{i_{s+1}}\}, \{x_{i_{s+1}}, y_{j_{s+1}}\}, \ldots, \{x_{i_m}, y_{j_m}\}
\]

is an induced matching of \( H \) with \( m + 1 \) edges, contradicting \( \text{indmatch}(H) = m \). We just have to show that \( x_{j_s} \) is adjacent to none of the vertices \( y_{j_1}, \ldots, y_{j_m} \) and that \( y_{i_{s+1}} \) is adjacent to none of the vertices \( x_{i_1}, \ldots, x_{i_m} \).

Suppose there exists \( q \in [m] \) such that \( \{x_{i_q}, y_{i_{s+1}}\} \in E(H) \). Then \( i_q < i_{s+1} \) and \( \{i_q, i_{s+1}\} \in E(G) \), and by the third step \( q = s \). But our initial assumption was that \( \{i_s, i_{s+1}\} \notin E(G) \). Therefore \( y_{i_{s+1}} \) is adjacent to none of the vertices \( x_{i_1}, \ldots, x_{i_m} \).

Suppose there exists \( q \in [m] \) such that \( \{x_{j_q}, y_{i_{s+1}}\} \in E(H) \). Then \( i_q < i_{s+1} \) and \( \{i_q, i_{s+1}\} \in E(G) \), and by the third step \( q = s \). But our initial assumption was that \( \{i_s, i_{s+1}\} \notin E(G) \). Therefore \( y_{i_{s+1}} \) is adjacent to none of the vertices \( x_{i_1}, \ldots, x_{i_m} \).

We will now show that

\[
\{x_{i_1}, y_{j_1}\}, \ldots, \{x_{i_s}, y_{j_s}\}, \{x_{i_s}, y_{i_{s+1}}\}, \{x_{i_{s+1}}, y_{j_{s+1}}\}, \ldots, \{x_{i_m}, y_{j_m}\}
\]

is an induced matching of \( H \) with \( m + 1 \) edges, contradicting \( \text{indmatch}(H) = m \) (this contradiction implies that \( \{i_s, i_{s+1}\} \in E(G) \) for every \( s \in [m - 1] \)). We just have to show that \( x_j \) is adjacent to none of the vertices \( y_{j_1}, \ldots, y_{j_m} \) and that \( y_{i_{s+1}} \) is adjacent to none of the vertices \( x_{i_1}, \ldots, x_{i_m} \).

Suppose there exists \( q \in [m] \) such that \( \{x_{i_q}, y_{i_{s+1}}\} \in E(H) \). Then \( i_q < i_{s+1} \) and \( \{i_q, i_{s+1}\} \in E(G) \), and by the third step \( q = s \). But our initial assumption was that \( \{i_s, i_{s+1}\} \notin E(G) \). Therefore \( y_{i_{s+1}} \) is adjacent to none of the vertices \( x_{i_1}, \ldots, x_{i_m} \).

Suppose there exists \( q \in [m] \) such that \( \{x_{j_q}, y_{i_{s+1}}\} \in E(H) \). Then \( j_q < j_s < j_q \), therefore \( q > s \). Since \( G \) is closed, then all facets of \( \Delta(G) \) are intervals, and since \( \{j_q, j_q\} \in E(G) \),
it follows that \([j, j_q]\) is a face in \(\Delta(G)\). Since \(j < i_{s+1} \leq i_q < j_q\), in particular \(\{i_{s+1}, j_q\} \in E(G)\), thus \(\{x_{i_{s+1}}, y_{j_q}\} \in E(H)\), and since \(\{x_{i_1}, y_{j_1}\}, \ldots, \{x_{i_m}, y_{j_m}\}\) is an induced matching of \(H\), it follows that \(q = s + 1\). But in this case \(\{x_{i_1}, y_{j_1}\}, \ldots, \{x_{i_s}, y_{j_s}\}\) \(\{x_{i_{s+1}}, y_{j_{s+1}}\}, \ldots, \{x_{i_m}, y_{j_m}\}\) is an induced matching of \(H\), contradicting the fact that \(i_{s+1}\) is the smallest \(t\) such that

\[
\{x_{i_1}, y_{j_1}\}, \ldots, \{x_{i_s}, y_{j_s}\}, \{x_{i_t}, y_{j_{s+1}}\}, \ldots, \{x_{i_m}, y_{j_m}\}\]

is an induced matching of \(H\).

**Corollary 3.2.4.** Let \(G\) be a connected closed graph on \([n]\). Then \(\text{reg}(S/J_G) = l\), where \(l\) is the length of the longest induced path of \(G\).

**Proof.** The inequality \(\text{reg}(S/J_G) \geq l\) follows from theorem 3.1.3.

On the other hand, theorem 1.3.38 implies that \(\text{reg}(S/J_G) \leq \text{reg}(S/\text{in}_{<}(J_G))\). By corollary 3.2.2 \(\text{reg}(S/I(H)) = \text{indmatch}(H)\), and by proposition 3.2.3 \(\text{indmatch}(H) = l\). Since \(I(H) = \text{in}_{<}(J_G)\), it follows that

\[
\text{reg}(S/J_G) \leq \text{reg}(S/\text{in}_{<}(J_G)) = \text{reg}(S/I(H)) = \text{indmatch}(H) = l.
\]

**Corollary 3.2.5.** If \(G\) is a complete graph, then \(\text{reg}(S/J_G) = 1\).

**Proof.** Just recall that complete graphs have no induced paths of length greater than 1.

**Theorem 3.2.6.** Let \(G\) be a closed graph on \([n]\) with the connected components \(G_1, \ldots, G_c\). Then \(\text{reg}(S/J_G) = l_1 + \cdots + l_c\), where, for \(i \in [c]\), \(l_i\) is the length of the longest induced path of \(G_i\).

**Proof.** This is a direct consequence of lemma 0.6.15 and induction.

**Corollary 3.2.7.** Let \(G\) be a closed graph. Then the regularity of the ideals \(J_G\) and \(\text{in}_{<}(J_G)\) does not depend on the base field \(K\) chosen but only on the graph \(G\).

**Corollary 3.2.8.** Both the Matsuda-Murai conjecture and the Madani-Kiani conjecture hold for closed graphs.

**Example 3.2.9.** If \(G\) is a complete graph, then \(G\) is a closed graph whose induced paths have length 1, hence \(\text{reg}(S/J_G) = 1\).
3.3 Partial results on two conjectures

We know a formula for \( \text{reg}(S/J_G) \), where \( G \) is a closed graph, which only depends on the combinatorial properties of \( G \). But what happens to \( \text{reg}(S/J_G) \) when \( G \) is not closed? As it was done in section 2.3, we will consider the class of chordal graphs such that any two distinct maximal cliques intersect in at most one vertex.

If \( G \) is such a connected graph, then it is not necessarily true that \( \text{reg}(S/J_G) = l \), where \( l \) is the length of the largest induced path in \( G \).

**Example 3.3.1.** If \( G \) is the graph as below, then \( G \) is a tree whose largest induced path in \( G \) has length 4. If \( K = \mathbb{Q} \), then using Macaulay2 to compute \( \text{reg}(S/J_G) \) we get that \( \text{reg}(S/J_G) = 6 \).

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{tree_graph.png}}
\end{array}
\]

However, in this section we will show that both the Matsuda-Murai conjecture and the Madani-Kiani conjecture hold for this class of graphs. Recall that, in section 3.2, we have seen that both the Matsuda-Murai conjecture and the Madani-Kiani conjecture hold for closed graphs.

So now let \( G \) be a connected chordal graph such that any two distinct maximal cliques intersect in at most one vertex. We will define the graphs \( G', G'', H \) and the ideals \( Q_1, Q_2 \) as in section 2.3 and we will use the results about these graphs and ideals which were proven there.

**Theorem 3.3.2.** Let \( G \) be a chordal graph on \( [n] \) such that any two distinct maximal cliques intersect in at most one vertex. Then \( \text{reg}(S/J_G) \leq r \) where \( r \) is the number of maximal cliques of \( G \).

**Proof.** As a consequence of lemma 0.6.15 we may assume that \( G \) is connected.

If \( r = 1 \), then \( G \) is a complete graph and corollary 3.2.5 implies \( \text{reg}(S/J_G) = 1 \). Suppose \( r > 1 \).

We will use induction on \( n + r \). The base case is \( n + r = 2 \). But in this case \( r = 1 \). Suppose \( n + r > 2 \).

Since \( G' \) is a connected chordal graph on \( [n] \) such that any two maximal cliques intersect in at most one vertex and with at most \( r - 1 \) maximal cliques, by induction it follows that \( \text{reg}(S/J_{G''}) \leq r - 1 \), and since \( Q_1 = J_{G''} \), then \( \text{reg}(S/Q_1) \leq r - 1 \).

Since \( G'' \) is a chordal graph on \( [n] \setminus \{i\} \) such that any two distinct maximal cliques intersect in at most one vertex, which is not connected and with at most \( r \) maximal cliques, by induction it follows that \( \text{reg}(S_i/J_{G''}) \leq r \), and since \( S/Q_2 \cong S_i/J_{G''} \), then \( \text{reg}(S/Q_2) \leq r \).

Since \( \text{reg}(S/Q_1) \leq r - 1 \) and \( \text{reg}(S/Q_2) \leq r \), by corollary 0.6.12 it follows that \( \text{reg}(S/Q_1 \oplus S/Q_2) \leq r \).

Since \( H \) is a connected chordal graph on \( [n] \setminus \{i\} \) such that any two distinct maximal cliques intersect in at most one vertex and with at most \( r - 1 \) maximal cliques, by induction it follows that \( \text{reg}(S_i/J_H) \leq r - 1 \), and since \( S/(Q_1 + Q_2) \cong S_i/J_H \), then \( \text{reg}(S/(Q_1 + Q_2)) \leq r - 1 \).
Recall the following exact sequence of $S$-modules:

$$0 \rightarrow \frac{S}{J_G} \rightarrow \frac{S}{Q_1} \oplus \frac{S}{Q_2} \rightarrow \frac{S}{Q_1 + Q_2} \rightarrow 0.$$

By inequality (1) of proposition 0.6.11, $\text{reg}(S/J_G) \leq \max\{\text{reg}(S/Q_1 \oplus S/Q_2), \text{reg}(S/(Q_1 + Q_2)) + 1\} \leq r,$ as desired. □

Hence the Madani-Kiani conjecture holds for chordal graphs such that any two distinct maximal cliques intersect in at most one vertex.

**Proposition 3.3.3.** Let $G$ be a chordal graph on $[n]$ without isolated vertices such that any two distinct maximal cliques intersect in at most one vertex. Then $G$ has at most $n - 1$ maximal cliques.

**Proof.** We may assume that $G$ is connected and we will use induction on $n$. If $n = 2$, the result is obvious. Suppose $n > 2$.

Let $K$ be the largest clique on $G$. Given a vertex $v$ in $K$, we define $G_v$ as the induced subgraph of $G$ such that $V(G_v) = \{v\} \cup \{u \in V(G) : \text{there exists a path from } u \text{ to } v \text{ whose only vertex in } K \text{ is } v\}$. Since $G$ is connected, it follows that $[n] = \bigcup_{v \in V(K)} V(G_v)$.

The graphs $G_v$ are pairwise disjoint. In fact, suppose that there exists a vertex $u$ not in $K$ and vertices $v, v'$ in $K$ such that $u \in G_v \cap G_{v'}$. Let $u, \ldots, v$ and $u, \ldots, v'$ be paths in $G$ whose only vertices in $K$ are $v$ and $v'$, respectively. Then $u, \ldots, v, v', \ldots, u$ is a cycle in $G$ whose only vertices in $K$ are $v$ and $v'$. Since $G$ is chordal, that cycle has a vertex $w$ not in $K$ such that $\{v, w\}, \{v', w\} \in E(G)$ and so the vertices $v, v', w$ form a clique in $G$ which is not contained in $K$ but intersects it in two vertices, a contradiction.

Let $v, v'$ be distinct vertices in $K$. Let $u \in V(G_v) \setminus \{v\}$ and $u' \in V(G_{v'}) \setminus \{v'\}$. Let $u, \ldots, v$ and $u', \ldots, v'$ be paths in $G$ whose only vertices in $K$ are $v$ and $v'$, respectively. If $\{u, u'\} \in E(G)$, then $u, \ldots, v, v', \ldots, u'$ is a cycle in $G$ whose only vertices in $K$ are $v$ and $v'$. Since $G$ is chordal, that cycle has a vertex $w$ not in $K$ such that $\{v, w\}, \{v', w\} \in E(G)$ and so the vertices $v, v', w$ form a clique in $G$ which is not contained in $K$ but intersects it in two vertices, a contradiction. Hence $\{u, u'\} \notin E(G)$.

In particular, every clique in $G$ not contained in $K$ must be a clique in some $G_v$, and since $G$ is a connected graph, such $G_v$ must have vertices other than $v$. Let $C = \{v \in V(K) : \text{not } V(G_v) \subseteq \{v\}\}$.

Let $v \in C$. Since $G_v$ is an induced subgraph of $G$, it follows that $G_v$ is also a chordal graph whose maximal cliques intersect in at most one vertex and by induction $G_v$ has at most $|V(G_v)| - 1$ maximal cliques.

Hence $G$ has at most $\sum_{v \in C} (|V(G_v)| - 1) + 1$ maximal cliques. Since $|V(G_v)| - 1 = 0$ whenever $v \in V(K) \setminus C$, then $\sum_{v \in C} (|V(G_v)| - 1) + 1 = \sum_{v \in V(K)} (|V(G_v)| - 1) + 1 = \sum_{v \in V(K)} |V(G_v)| - |V(K)| + 1 = n - |V(K)| + 1$, which is smaller than $n - 1$ since $|V(K)| \geq 2$. □

**Corollary 3.3.4.** Let $G$ be a connected chordal graph on $[n]$ such that any two distinct maximal cliques intersect in at most one vertex. If $G$ has $n - 1$ maximal cliques, then $G$ is a tree.
Proof. As in the proof of proposition 3.3.3, \( G \) has at most \( n - |V(K)| + 1 \) maximal cliques. But \( n - |V(K)| + 1 \leq n - 1 \), and so such inequality must be an equality, that is, \( |V(K)| = 2 \), and since \( K \) is the largest clique in \( G \), \( G \) has no cycles of length 3, and since \( G \) is chordal, \( G \) must be a tree. \( \square \)

Corollary 3.3.5. Let \( G \) be a connected chordal graph on \([n]\) such that any two distinct maximal cliques intersect in at most one vertex. If \( \text{reg}(S/J_G) = n - 1 \), then \( G \) is a path.

Proof. We will use induction on \( n \). If \( n = 2 \), the result is obvious. Suppose \( n > 2 \).

Since \( \text{reg}(S/J_G) = n - 1 \), by theorem 3.3.2 it follows that \( G \) has at least \( n - 1 \) maximal cliques. Moreover, by proposition 3.3.3 it follows that \( G \) has exactly \( n - 1 \) maximal cliques. Hence, by corollary 3.3.4 it follows that \( G \) is a tree.

Since \( G' \) is a graph with less maximal cliques that \( G \), then \( G' \) has at most \( n - 2 \) maximal cliques and so theorem 3.3.2 implies \( \text{reg}(S/J_{G'}) \leq n - 2 \), and since \( Q_1 = J_{G'} \), then \( \text{reg}(S/Q_1) \leq n - 2 \).

Since \( G'' \) is a chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex, then by proposition 3.3.3 \( G'' \) has at most \( n - 2 \) maximal cliques and so theorem 3.3.2 implies \( \text{reg}(S_i/J_{G''}) \leq n - 2 \), and since \( S/Q_2 \cong S_i/J_{G''} \), then \( \text{reg}(S/Q_2) \leq n - 2 \).

Consider again the following exact sequence of \( S \)-modules:

\[
0 \to \frac{S}{J_G} \to \frac{S}{Q_1} \oplus \frac{S}{Q_2} \to \frac{S}{Q_1 + Q_2} \to 0.
\]

By inequality (1) of proposition 0.6.11, \( n - 1 = \text{reg}(S/J_G) \leq \max\{\text{reg}(S/Q_1 \oplus S/Q_2), \text{reg}(S/(Q_1 + Q_2)) + 1\} \leq \max\{n - 2, \text{reg}(S/(Q_1 + Q_2)) + 1\} \). From the inequality \( n - 1 \leq \max\{n - 2, \text{reg}(S/(Q_1 + Q_2)) + 1\} \) it follows that \( \text{reg}(S/(Q_1 + Q_2)) + 1 \geq n - 1 \), that is, \( \text{reg}(S/(Q_1 + Q_2)) \geq n - 2 \).

Since \( S/(Q_1 + Q_2) \cong S_i/J_H \), where \( H \) is a connected chordal graph on \([n] \setminus \{i\}\) such that any two distinct maximal cliques intersect in at most one vertex, then by theorem 3.1.3 it follows that \( \text{reg}(S/(Q_1 + Q_2)) = n - 2 \), and by induction \( H \) must be a path. This implies that the clique on \( \left( F_i \cup \bigcup_{j=1}^q F_j \right) \setminus \{i\} \) has two vertices and so it follows that \( q = 1 \). Let \( i, j, k \) be the three vertices in \( F_i \cup F_k \). Then the graph \( G \) can be obtained from the path \( H \) by adding the vertex \( i \) and by replacing the edge \( \{j, k\} \) by the edges \( \{i, j\} \) and \( \{i, k\} \). Hence \( G \) is a path. \( \square \)

Hence the Matsuda-Murai conjecture holds for chordal graphs such that any two distinct maximal cliques intersect in at most one vertex.

Example 3.3.6. Let \( G \) be a connected chordal graph on \( 4 \) vertices such that any two distinct maximal cliques intersect in at most one vertex. If \( G \) is a complete graph, then \( \text{reg}(S/J_G) = 1 \). If \( G \) is a path of length 3, then \( \text{reg}(S/J_G) = 3 \). In the remaining cases, the length of the largest induced path of \( G \) is 2, hence theorem 3.1.3 implies that \( \text{reg}(S/J_G) \geq 2 \). On the other hand, corollary 3.3.5 implies that \( \text{reg}(S/J_G) < 3 \) and so \( \text{reg}(S/J_G) = 2 \).
This year, on April 6th, Madani and Kiani presented a full proof for the Matsuda-Murai conjecture in [18].
Bibliography


