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Mechanical and thermodynamical properties of matter in strong gravitational fields

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Abstract

In this thesis we study mechanical properties for compact stars and thermodynamical properties of thin matter shells in the context of general relativity. For compact stars made of perfect fluid with a small electrical charge we obtain the interior Schwarzschild limit, i.e., the limiting radius to mass relation. We also obtain the maximum mass that the star can attain. As for thin matter shells we use the same strategy for two different configurations: a shell in (3+1)-dimensional AdS (Anti de Sitter) spacetime and a rotating shell in (2+1)-dimensional AdS spacetime. We start by using the junction conditions to obtain the dynamical relevant quantities: rest mass, pressure and angular velocity. Then we use the first law of thermodynamics and we obtain the shell's entropy differential. The entropy is parametrized by a phenomenological function with free parameters. We also take the shells to their gravitacional radius and we obtain the Bekenstein-Hawking entropy of a black hole. Finally, we analyze the thermodynamic stability of the shell. For the (2+1)-dimensional spacetime the thermodynamic analysis is only done for the slolwy rotating limit.

Keywords

Compact stars, Schwarzschild limit, thin shell, thermodynamics, black hole, entropy

Resumo

Nesta tese estudam-se propriedades mecânicas de estrelas compactas e propriedades termodinâmicas de camadas finas de matéria, no âmbito da Relatividade Geral. Para estrelas compactas constituídas por um fluido perfeito com uma pequena carga eléctrica obteve-se o limite interior de Schwarzschild, que dá o limite mínimo da razão entre o raio da estrela e a sua massa. Obteve-se também a massa máxima que a estrela pode ter. No que toca às camadas finas de matéria, utilizou-se a mesma estratégia para duas configurações diferentes: uma camada num espaço-tempo de (3+1) dimensões AdS (anti de Sitter) e uma camada em rotação num espaço-tempo de (2+1) dimensões AdS. Inicialmente utilizam-se as condições de junção de forma a obter-se as quantidades dinâmicas: massa em repouso, pressão e velocidade angular. De seguida utiliza-se a primeira lei da termodinâmica para se obter o diferencial da entropia, sendo que a entropia é parametrizada por uma função fenomenológica com parâmetros livres. Além disso, colocam-se as camadas nos seus próprios raios gravitacionais e obtém-se a entropia de Bekenstein-Hawking de um buraco negro. Para finalizar analisa-se a estabilidade termodinâmica da camada. Para o problema a (2+1) dimensões a análise termodinâmica só é feita para o caso de baixa rotação.

Palavras Chave

Estrelas compactas, limite de Schwarzschild, camadas finas, termodinâmica, buraco negro, entropia

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Preface

The research of this thesis has been done at Centro Multidisciplinar de Astrofísica (CENTRA) in the Physics Department of Instituto Superior Técnico. The work was partially supported by Fundação para a Ciência e Tecnologia (FCT), through project Incentivo/FIS/UI0099/2014.

Chapters 3 to 5 were done in collaboration with Professor José P. S. Lemos. Chapter 2 was done in collaboration with Professor José P. S. Lemos, MSc. Gonçalo M. Quinta and Professor Vilson Zanchin. Chapter 6 was done in collaboration with Professor José P. S. Lemos, MSc. Gonçalo Quinta and PhD. Jorge Rocha.

Below are listed the works included in this thesis submitted and in preparation to be submitted soon in major international journals:

- José P. S. Lemos, Francisco J. Lopes, Gonçalo M. Quinta, Vilson Zanchin, *Compact stars with a small electric charge: the limiting radius to mass relation and the maximum mass for incompressible matter*, submitted (Chapter 2) [1].

- Francisco J. Lopes, José P. S. Lemos, *Entropy of thin shells in a (3+1)-dimensional asymptotically AdS spacetime and the Schwarzschild-AdS black hole limit*, in preparation (Chapters 3 and 5) [2].

- José P. S. Lemos, Francisco J. Lopes, Jorge Rocha, Gonçalo M. Quinta, *Dynamics of rotating thin shells in a (2+1)-dimensional asymptotically AdS spacetime*, in preparation (Chapter 4) [3].

- José P. S. Lemos, Francisco J. Lopes, *Entropy of slowly rotating thin shells in a (2+1)-dimensional asymptotically AdS spacetime and the BTZ black hole limit*, in preparation (Chapter 6) [4].

In adition there are works which have been published and will be published in books of abstracts and proceedings:

- Francisco J. Lopes, *Buchdahl limit for stars with a Schwarzschild interior and a small electrical charge: Analytical approach based on Misner's method to solve the TOV equation*, presented in VI Black Hole Workshop [5], book of abstracts (2013); also presented at ENEF 2013 (Encontro Nacional de Estudantes de Física) where took the 2nd prize to the best oral presentation.

- Francisco J. Lopes, *Compact stars: Radius to mass relation*, Fisica 14 - 19th Portuguese National Physics Conference [6], book of abstracts and to appear in proceedings (2014).

1

Introduction

The introduction that we are going to write is relative to our works [1–6].

1.1 General relativity and the strong field regime

Shortly after presenting the special theory of relativity in 1905, Einstein began to think how to describe gravity into a relativistic framework, which led him into a ten years crusade towards a new theory of gravitation. In 1915, Albert Einstein presented a geometric theory of gravitation, known as general relativity, or the general theory of relativity. This theory is the description of gravitation in modern physics and generalizes Newton's gravitation by interpretating gravity as a geometric property of spacetime. It is described by the Einstein field equations, a set of non-linear partial differential equations, which tell us that the spacetime is modeled by the distribuition of matter. Einstein used the equations linearized to make his first predictions. [7].

The field equations, also known as Einstein's field equations, are

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.1}$$

where Greek indices are spacetime indices running from 0 to d, with 0 as the time index and d as the number of spatial dimensions and where the velocity of light at vacuum equals one, $c = 1$. The Einstein tensor $G_{\mu\nu}$ is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric tensor, and R is the Ricci scalar. On the left side of Eq. (1.1) we have the Einstein tensor which is a purely geometric quantity, whereas on the right side we have the energy-momentum tensor, $T_{\mu\nu}$, which provides the information about the matter. G is the gravitational constant. At vacuum the equations reduce to

$$R_{\mu\nu} = 0. \tag{1.2}$$

In 1916, Schwarzschild presented the first non-trivial exact solution, for the gravitational field in vacuum outside a spherically symmetric mass, uncharged and without angular momentum [8].

General relativity predicts some results much different from those of classical physics, like gravitational time dilation [9], gravitational redshif of light [7], bending of light and gravitational lensing [10, 11], which all have been confirmed by experimental data. Among the relativistics theories of gravity it is the simplest one that is consistent with experiments. It also has some relevant astrophysical implications, such as the existence of black holes (see Sec. 1.3) and gravitational waves. In addition general relativy is used as a basis of nowadays most prominent cosmological models.

Strong field regime, as the name indicates, refers to a strong gravitational field. Basically, it is a gravitational field that causes large deviations from flat spacetime. Unfortunately we do not know if general relativity is valid in this regime, due to lack of experimental data. Nowadays motivations for alternatives to general relativity are mainly cosmological, due to problems like inflation, dark matter and dark energy. The sought alternatives theories must verify general relativity as a special case for sufficiently weak fields. These theories are based on the principle of least action, by choosing the appropriate lagragian density for gravity. Thus these theories modify the left-hand side of Eq. (1.1), by adding some extra terms. In this group of theories fall the following: scalar field theories, quasilinear theories, tensor theories, scalar-tensor theories, vector-tensor theories and bimetric theories [12].

Scalar theories try to describe the gravitational field using a scalar field, which must satisfy some field equation. Newton gravitation is a scalar field theory in the way that it is completely described by the potential ϕ which satisfies the Poisson equation. Indeed, the first attempts to create a relativistic theory of gravitation were scalar field theories like the two Nordström theories, whose the idea was to modify the Poisson's equation of Newtonian gravitation.

In quasilinear theories the physical metric is constructed algebraically from the Minkowski metric and matter variables. The most known quasilinear theory is the Whitehead's theory of gravitation, which makes the same predictions as general relativity for the four classical tests: perihelion shift, gravitational red shift, light bending and Shapiro time delay. However, this theory predicts a sidereal tide on Earth, caused by the gravitational field of the Milky Way which is in disagreement with experimental observations.[13]. There is another interpretation of Whitehead's theory that eliminates the unobserved sidereal tide effects, but predicts a new and unobserved effect, called the "Nordtvedt Effect", unverified up to date [14–16].

General relativy is an example of a tensor theory. In fact it is the most simple tensor theory. Others include theories like $f(R)$ gravity (a family of theories), Gauss-Bonnet gravity and Lovelock theory of gravity. Of these the $f(R)$ is the most popular and is based on the substitution of the Ricci scalar R for a function $f(R)$ in the lagragian density [17].

Scalar-tensor theories are theories that include both a scalar field and a tensor field to represent the gravitational interaction, like the Brans-Dicke theory [18]. On the other hand, vector-tensor theories include a tensor field and a vector field to represent the interaction.

Finally, bimetric theories contain both the normal tensor metric and a metric of constant curvature. They may also contain other scalar or vector fields.

1.2 Compact objects: continuous matter and thin shells

Compact objects is a term used in astrophysics to refer to the most dense objects in nature. We separate compact objects into two categories: continuous matter, i.e., compact stars, and thin matter shells. In this kind of objects it is imperative to use general relativity as the gravitation theory, since Newtonian theory of gravitation does not provide a realistic description because one can attain strong gravitacional fields.

White dwarfs, neutron stars and black holes are examples of compact stars. Also exotic stars, which are a compact star composed of something other than baryons as darkmatter. As the name indicates, compact stars are stellar objects with small volume for their mass, which means very high density. For instance, neutron stars typically have masses of the same order of magnitude as the sun's mass, $M \sim 1.4M_{\odot}$, in radii of only $R \sim 10$km. Thus we can have densitys $\rho \sim 10^{15}$g cm^{-3}, even larger than the nuclear ground state density, $\rho_0 \sim 10^{14}$g cm^{-3} [19].

These stars are formed at the endpoint of stellar evolution. A star loses energy through radiation, while shining. That energy comes from nuclear fusion in the interior of the star. When a star exhausts all its energy the gas pressure of the interior can no more support the weight of the star and it collapses to a denser object: the so-called compact star. Despite the fact that the compact star has no mechanism of energy production, it may still radiate the excess heat left from the collapse for many millions of years. Notwithstanding the fact that compact stars can achieve temperatures as high as $T \sim 10^{11}$K $\sim$ 10MeV, compact stars are cold in the sense that their temperature is small when compared to the baryon chemical potencial, $T \ll \mu$ (where we set the Boltzmann constant k_B equal to 1). The structure of compact stars implies that they can exist forever, because, although they radiate and loose energy, they do not depend on high temperatures to sustain nuclear reactions, since this type of mechanism does not take part of the star. Nearly 1% of the stars in the Milky Way are expected to be compact stars [20].

A possible way to simulate compact objects, while keeping the most important physical aspects, is using thin matter shells. A thin matter shell is a hypersurface which separates spacetime into two regions: the interior region and the exterior region. Due to the development of a singularity in spacetime there are some conditions that must be satisfied to ensure that the entire spacetime is a valid solution of Einstein's equations. These conditions are called junctions conditions (see Appendix A) [49]. In this kind of system, the material degrees of freedom of the shell are related to the gravitational degrees of freedom by the Einstein equations, which implies that the thermodynamics of the shell is acutely related to the structure of spacetime.

The usefulness of thin matter shells is also evident from the fact that we can take the black hole limit, i.e., the shells can be taken to their gravitational radius. Once we obtained the entropy from the thermodynamics approach, we can take the black hole limit and obtain the black hole entropy. Thus, the black hole thermodynamic properties can be attained using a much more simplified computation than the usual black hole mechanics. This idea was developed by Martinez [114] and is going to be applied in chapters 5 and 6 which will generalize the results from [114], by adding a negative cosmological constant and the results from [53] by considering slowly rotation, respectively.

1.3 Black holes

1.3.1 Astrophysical aspects

Although black holes are sometimes considered as compact stars, they are a different and special object. They are formed by the gravitational collapse of a massive star or a cluster of stars. As long as more mass is gathered, the star reaches its breaking point. The pressure is not sufficient to outweigh gravity and the star collapses. Black holes are objects of extreme density with such strong gravitational attraction that even light cannot escape. The boundary of the region from which nothing can escape is denominated event horizon.

Astronomers have discovered two types of astrophysical black holes, i.e., candidates to black holes: the stellar-mass black holes, with masses from 5 to 30 solar masses and supermassive black holes with masses from 10^6 to 10^{10} solar masses. We call them astrophysical black holes because there is very strong evidence that these objects are black holes, as predicted by theory, with event horizons. Cyg X-1, in the constellation of Cygnus, was the first stellar-mass black hole to be discovered [21, 22].

There are expected many millions of stellar-mass black holes in our galaxy, the Milky Way, and until now 24 of them have been confirmed by dynamical observations [23].

On the other hand, supermassive black holes are located typically at the center of galaxies. The reason expected to explain the mass growth of these black holes is rapid gas accretion, which is observable as active galactic nuclei as quasars. Active galatic nucleus is a compact region at the centre of a galaxy with a very high luminosity. If a given galaxy hosts an active galatic nucleus then it is called active galaxy. Still, quasars or quasi-stellar radio sources are the most energetic of the active galactic nuclei. Quasars also show a very high redshift, which by Hubble's law imply that thery are very distant in space and time from Earth. At the center of Milky way there is a supermassive black hole, the Sagittarius A*. It is a very bright radio source with a measured mass of $4 \times 10^6 M_\odot$ [24].

There is also evidence of another class of black holes, the intermediate-mass black hole, with masses between the stelar-mass black holes and supermassive black holes, that range from 10^2 to 10^5 solar masses. The more relevant evidence for these black holes comes from low-luminosity active galactic nuclei [25].

1.3.2 Classical aspects

The first solution of the field equations that predicted a black hole was the Schwarzschild's solution in 1916 [8] with the appropriate interpretation coming only in 1958 by Finkelstein [26]. A black hole has a singularity at its center where the curvature and thus the tidal forces become infinite. The next solution predicting a black hole was the Reissner-Nordström solution, for a charged, non-rotating, spherically symmetric body [27]. In 1963 Kerr presented the first solution for a rotating mass [28] and in 1965 it was generalized for the case of a rotating and electrically charged mass. [29, 30].

The Schwarzschild line element for a spherically symmetric, static, solution of the vacuum field equations, Eq. (1.2), is

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{1.3}$$

where t is the time coordinate, (r, θ, ϕ) are the spherical coordinates and m is the mass in the so-called interior region, i.e., such that $r < 2m$, in units $G = c = 1$. We can see that the line element from Eq. (1.3) has two singularities: $r = 2m$ and $r = 0$. The first, $r = 2m$, is removable, i.e., it is a coordinate singularity and has a physical interpretation: it is the event horizon of a black hole. Choosing appropriate coordinates leads us to an analytic extension of the Schwarzschild solution, like the advanced or retarded Eddington-Finkelstein coordinates. On the other hand, the singularity $r = 0$ is not removable and is called an intrinsic singularity and at this point the tidal forces go to infinity. One can go further and achieve the maximal analytic extension of the solution, by assuring that all geodesics emanating from an arbitrary point can be extended to infinite values of the affine parameter along the geodesic in both directions or either terminate on an intrinsic singularity [31]. The maximal extension of the Schwarzschild solution is the Kruskal solution [32]. This solution is composed of a wormhole which initially expands as a white hole and collapses in a black hole. The wormhole connects two asymptotically flat universes. It also shows that the region $r > 2m$ is static, whereas the interior region, $r < 2m$, is highly dynamic [31]. All black hole solutions of field equations in general relativity are fully characterized by three observable parameters: mass, angular momentum and electric charge. Every other information is lost throuht the black hole event horizon and is inacessible to any external observer. This result is known as "no-hair theorem" [7].

1.3.3 Quantum aspects

Black holes also have a finite temperature, $T_{\rm H}$, and entropy [33–35]. This may seem strange since black holes as they appear in classical general relativity have thermodynamic temperature equal to absolute zero. However, when quantum effects arise, the black hole must have a non-null temperature [36]. They also emit radiation known as Hawking radiation. This radiation, which is also acknowledged as black hole evaporation, reduces the mass and the energy of the black hole and is caused by quantum effects. A simplified view of the process is that vacuum fluctuations create a particle-antiparticle pair near the event horizon. While one of the pair falls throught the event horizon and therefore in the black hole, the other escapes. To garantee the conservation of total energy the particle that entered the black hole must have a negative energy, so that the black hole reduces its mass in the process. As viewed by an observer outside of the black hole it seems that the black hole just emitted a particle.

If black holes did not have entropy, $S_{\rm BH}$, one would violate the second law of thermodynamic by throwing some mass into the black hole. Bekenstein proposed that the entropy was proportional to the area of its event horizon, by comparing the first law of thermodynamics to the energy conservation law of a black hole [33], also known as the first law of black hole dynamics. Besides the conservation of total energy, it is also supplemented by the conservation laws of total momentum, angular momentum, and charge. This law, for a rotating black hole with area A, mass m and angular momentum J, is stated as

$$dm = \frac{\kappa}{8\pi G} dA + \omega_{\rm BH} dJ \,, \tag{1.4}$$

where κ is the surface gravity and $\omega_{\rm BH}$ is the black hole angular velocity.

On the other hand the second law of black hole mechanics states that the black hole area never decreases ($\Delta A \geq 0$), similar to the second law of thermodynamics ($\Delta S \geq 0$). The second law of black hole mechanics is a mathematically rigorous consequence of general relativity, while the second law of thermodynamics is a law that stands for systems with many degrees of freedom. However, this relation between both laws proved to be of a fundamental nature [36]. This similarity led Bekenstein [33], in 1973, to propose that

$$dS_{\rm BH} = \frac{\kappa}{8\pi G T_{\rm H}} dA_{\rm BH} \,. \tag{1.5}$$

Bekenstein used a thermodynamic approach together with information theory to obtain the entropy up to a proportionality constant. Two years later Hawking [35], using the formalism of second quantization, found the black hole temperature, known as Hawking temperature

$$T_{\rm H} = \frac{\hbar\kappa}{2\pi} \,. \tag{1.6}$$

Thus, the black hole entropy, also known as Bekenstein-Hawking entropy is given by

$$S_{\rm BH} = \frac{1}{4} \frac{A_+}{l_{\rm p}^2} \,, \tag{1.7}$$

in units with $k_B = 1$ (to be used through this thesis), where $l_{\rm p} = \sqrt{G\hbar}$ is the Planck length, k_B is the Boltzmann constant, $\hbar$ is the Planck constant and A_+ is the area of the event horizon.

There are other ways to obtain the black hole entropy. Hawking also derived the black hole entropy, but from a path integral approach of quantum field theory in curved spacetime [37]. York [38] obtained the black hole entropy using the grand canonical ensemble. The last two approaches are: through quasi-black holes [39–41] and through thin matter shells [53, 114, 119]. In this work this approach of thin matter shells will be used.

Despite the fact that most of engineering problems which conciliate heat with work and energy can be solved with the theory of thermodynamics, one must remark that the full understaing of the macroscopic physics of a given system is not enough on a physical perspective. One needs to perceive the macroscopic behaviour of the system by understanding its microscopic dynamics, as one does on statistical mechanics where the laws of thermodynamics are obtained from first principles through the microscopic analysis of the system, concerning the notion of degree of freedom and its influence on the phase space. The entropy of a macroscopic system contains information about the degrees of freedom, since it is related to the number of ways in which the system can be formed. So if there is no microscopic theory the entropy can be used to find some clues about that theory. A black hole is an example of a system where we do not know the microscopic theory describing the system, despite the use of an expression for the entropy [33–35, 42]. This is because we need a theory for quantum gravity to describe the microscopics of a black hole. Whereas in statistical mechanics the degrees of freedom are well-known, for a black hole that is not true, since all the information is lost through the event horizon ("no-hair theorem" [7]). Thus, the black hole entropy quantifies the information of the system but does not clarify about the nature of the degrees of freedom, which can be gravitational or material. So we expect that a quantum theory of gravity should not create any type of distinction between material and gravitational degrees of freedom. However, this is still a theme for phenomenological studies. Quantum gravity becomes relevant at radii, $R \sim l_{\rm p} \sim 10^{-33}$cm. This kind of investigation may clear up some features of the thermodynamics of the gravitational field which can lead us to some aspects of a quantum theory of gravity. [43].

1.3.4 Black holes in other dimensions

The first and only black hole solution in (2+1)-dimensions was the Bañados-Teitelboim-Zanelli (BTZ) black hole, discovered in 1992, for spacetimes with negative cosmological constant. Then the interest in (2+1)-dimensional general relativity boosted.

It is also interesting to study black hole solutions in higher dimensions. This is because many quantum gravity theories require more than four dimensions, like String theory. Also the AdS/CFT correspondence relates the properties of d dimensional black holes with the proprerties of a quantum field theroy in $d-1$ dimensions. Finally the production of higher-dimensional black holes becomes a feasible possibility in future collides involving TeV-scale gravity and large extra dimensions [44–46].

1.4 Thesis outline

This thesis is organized as follows. Chapter 2 presents an analytical scheme to investigate the limiting radius to mass relation and the maximum mass of relativistic stars made of an incompressible fluid with a small electric charge, thus generalizing the Schwarzschild interior limit [104]. The investigation is carried out by using the hydrostatic equilibrium equation, i.e., the Tolman-Oppenheimer-Volkoff (TOV) equation [99, 106], together with the other equations of structure, with the further hypothesis that the charge distribution is proportional to the energy density and that the energy density is constant. Chapter 3 is dedicated to the analysis of the dynamics of a static thin matter shell in a (3+1)-dimensional asymptotically AdS (anti de Sitter) spacetime, while in Chapter 4 we do the same to a rotating thin matter shell in a (2+1)-dimensional spacetime with negative cosmological constant. For both chapters the strategy employed is the use of junction conditions (see Appendix A). In Chapter 5 we study the thermodynamics of the thin matter shell whose dynamics we analyze in Chapter 3. Chapter 6 follows the same line of work but with regard to the thin matter shell in the spacetime of Chapter 4, for the slowly rotating case. The approach for Chapter 5 and 6 is the following. We integrate the first law of thermodynamics to obtain the entropy of the thin shell, using its pressure, rest mass and angular velocity (for the rotating shell of Chapter 6), forthcoming from Chapters 3 and 4, respectively. The entropy is obtained up to a function of the gravitational radius of the shell. By choosing the most simple phenomenological one we arrive to an explicit expression for the entropy and then we analyze the thermodynamic stability of the shell [115].

2

Compact stars with a small electric charge: the limiting radius to mass relation and the maximum mass for incompressible matter

2.1 Introduction

Compact stars and their properties have been a theme of great relevance on several grounds. Chandrasekhar's celebrated work [73] on the maximum mass for white dwarfs advanced the way to the understanding of the nature and structure of compact stars. By using a cold equation of state in which the degeneracy electron pressure is the most relevant form of pressure for the support of a white dwarf against gravitational collapse, a radius-mass relation for these stars was deduced, from the non-relativistic electron regime in relatively large white dwarfs up to the relativistic electron regime in the most compact stars. He found that as the radius of the star approached zero the mass would go to a maximum value of $1.44\,M_{\odot}$, where $M_{\odot}$ is the sun's mass. This is the Chandrasekhar limit. It uses Newtonian gravitation. Landau [92] through heuristic arguments found that the mass limit for white dwarfs could be written as $M \sim M_{\rm pl}^3/m_n^2$ where m_n is the neutron or the proton mass and $M_{\rm pl}$ is the Planck mass, $M_{\rm pl} = \sqrt{\hbar c/G}$, with $\hbar$ being the Planck constant, G the Newton's constant of gravitation, and c the velocity of light, or, setting units such that $G = 1$ and $c = 1$, which we do from now on in this chapter, one has $M_{\rm pl} = \sqrt{\hbar}$. Putting in the numerical values for $M_{\rm pl}$ and m_n, the mass M of the star is about the Chandrasekhar mass limit $M \sim 1\ M_{\odot}$. Landau further deduced that the stars should have a radius of about $\lambda_e\, M_{\rm pl}/m_n$, where λ_e is the electron's Compton wavelength, $\lambda_e = \hbar/m_e$, m_e being the electron's mass, giving a radius of the order of 5000 km. He also found that there was another regime in which the star is composed of neutrons, supported by the degeneracy pressure of these particles, and has a maximum mass of about $M \sim M_{\rm pl}^3/m_n^2$. These neutron stars are much more compact with a radius $\lambda_n\, M_{\rm pl}/m_n$, λ_n being the neutron's Compton wavelength, giving about 10 km. For objects with a radius tending to zero one should use general relativity, rather than Newtonian gravitation. In general relativity a compact star can be defined neatly as a star that has a geometrical mass M (or, GM/c^2 if one restores G and c) somehow comparable to its radius R, i.e., $R/M \sim a$, with a a number not much bigger than 1, say of order of 10 or less. Whereas for an extended star like the Sun $R/M_{\odot} \sim 5 \times 10^5$, one has for a white dwarf $R/M_{\odot} \sim 3 \times 10^3$, and for a neutron star $R/M_{\odot} \sim 6$, showing that the latter is really compact. On a general relativistic basis, [99] worked out further the nature and structure of neutron stars. Working out on a stiff equation of state for star matter made of neutrons they found roughly the results of Landau, namely, the mass limit is about $1\ M_{\odot}$ and $R/M_{\odot} \sim 6$. This limit is called the Landau-Oppenheimer-Volkoff limit. Improvements have been made on these limits. Using a cold equation of state valid in the full range of highly compressed matter, the Harrison-Wheeler equation of state, the full set of equilibria in an R-M relation were found, in particular, the two maxima masses corresponding to the Chandrasekhar and Landau-Oppenheimer-Volkoff limits appear naturally [84]. See also [80, 82, 102] for further discussion on compact stars. These mass limits, as seen in the context of general relativity, appear because at these stages the energy associated to the pressure is so strong that its gravitating effect overwhelms the self support effect. Now, the properties of the stars get modified if either the constituent material is altered or on alternative theory of gravitation is used. For instance, it is believed that dark matter also inhabits the core of stars. The properties that dark matter can imprint on a star like the sun have been studied in [54, 55, 70, 71], and in compact stars in [66, 90, 91]. In addition, the structure of stars like the sun in alternative theories of gravity has been analyzed in [72], and the structure of neutron stars in those alternative theories has been analyzed in [83] with the conclusion that more massive stars than in general relativity can form.

In contrast to the fermion stars mentioned above, boson stars can have a wide range of mass limits, namely, $M \sim M_{\rm pl}^2/m_b$, $M \sim M_{\rm pl}^3/m_b^2$, or $M \sim M_{\rm pl}^4/m_b^3$, where m_b is the mass of the boson that makes up the star [89, 93, 103], see [87, 97] for reviews. These stars could have been formed in the beginning of the universe from the primordial gravitational collapse of the boson particles and have been proposed as alternatives to the usual compact objects [97], such as neutron stars and black holes, and also as part of the dark matter [105]. These stars can, in principle, be detected [74, 100, 108].

Now, the first compact star ever displayed in its full structure was a general relativistic star with a very stiff equation of state, a star made of an incompressible perfect fluid, i.e., $\rho(r) = $ constant, and isotropic pressure (where $\rho(r)$ is the energy density at the radius r) [104]. This interior Schwarzschild star solution is spherically symmetric and has a vacuum exterior. An incompressible equation of state is interesting from various aspects, since one can extract clean results and it also provides compactness limits. Furthermore, this incompressible fluid applies to both fermion and boson particles, as long as the fluid is at an incompressible state. As a drawback for such an equation of state, one can mention that the speed of sound through such a medium is infinite, but generically the overall structure is not majorly changed. Schwarzschild found that there was a limit, when the central pressure p_c goes to infinity and that the star's radius to the mass limit is $R/M = 9/4 = 2.25$, indeed a very compact star [104]. [107] and [98] rederived the Schwarzschild interior limit of $R/M = 9/4$ using the propitious Tolman-Oppenheimer-Volkoff (TOV) equation, a differential equation for the pressure profile as a function of the other quantities [99], see also [106]. In addition, [98] even found a maximum mass for a given density of the incompressible fluid, the Misner mass.

One can ask if the Schwarzschild limit can be modified, allowing for instance a lower R/M relation. As mentioned above, one way is to have some kind of repulsive matter or new field in the star. Another way is by resorting to some alternative theory of gravity. It is also well known that the introduction of new matter fields can be mimicked by modifications of the gravitational field. One example, even in Newtonian gravitation, is that the effects of the dark matter can be mimicked by modifying the gravitational field, e.g., in the MOND theories [56–58]. The same type of choice holds true in tensor theories of gravitation, since one can pass the excess of the gravitational field present in the left-hand side of the Einstein equations to their right hand side giving an effective energy-momentum tensor in a form of a new field, e.g., see [77]. Rather than introducing an alternative theory of gravitation we here opt to study the case in which we add a matter field to the existent matter. We consider matter with a small electric charge, introducing thus an additional electric field in addition to the usual matter and gravitational fields. This addition of an electric charge and an electric field to the Schwarzschild incompressible matter configurations brings insight to the configurations overall structure in more complex situations and its study in stars mimics other fields and possible alterations in the gravitational field.

The important quantity in knowing how much electric charge a star can support is the ratio of the mass m to the charge q of the main fundamental constituents of the star [80]. For normal matter the net electric matter in a star is utterly negligible as the ratio of the proton mass m_p to the proton charge e is $m_p/e = 10^{-18}$, giving thus $Q/M \simeq (m_p/e)^2 \simeq 10^{-36}$, where Q is the star's total charge [80], see also [85]. However, stars can contain some dark matter in their interior, and of the several dark matter fluid candidates some could be electrically charged. Indeed, natural candidates to compose the dark matter are supersymmetric particles. The lightest supersymmetric particles that make the bulk of dark matter should be neutral. One possible candidate is the neutralino [69], however, some of these particles could be electrically charged. The mass m to charge q ratio of these supersymmetric particles are much higher than the baryonic mass to charge ratio, indeed current supergravity theories indicate that some particles can have a ratio of one. For a $m/q \sim 0.1 - 0.3$ one has $Q/M \simeq 0.01 - 0.1$, a small but non-negligible electric charge. Thus, if dark matter populates the interior of stars, and some of it is made of electrically charged particles there is the possibility that stars have a tiny but non-negligible electric charge. In this case the radius-mass relations for the corresponding stars should get modifications and have an influence in the structure of the compact star.

That electric charge can influence the structure of a compact star was proposed earlier by [63] who wrote the appropriate TOV equation. Some electric compact configurations with an incompressible equation of state for the matter were studied numerically and the corresponding generalized Schwarzschild limit, i.e., central pressure going to infinity in these configurations, was analyzed. [61, 62, 75, 76]. Other equations of state for electrically charged matter, such as polytropic equations were used in [60, 78, 101], where star configurations and their structure were studied and the Schwarzschild electric limit for the given equation of state and for a given charge was considered. In particular, in [101] it was argued that upon gravitational collapse a star could retain, significantly, part of its electric charge. Other interesting equations of state were proposed and studied in [81, 96], and in [94] where electrically charged dust was studied. Electrically charged boson stars have been also studied and their properties analyzed [67, 88]. Bounds on the radius to mass relation for uncharged and charged stars have been put forward in [59, 68], respectively, see also [64, 79, 109]. It is also worth mentioning some work on charged Newtonian stars. That compact stars could exist was even noticed in the 18th century. A subset of these Newtonian compact stars are the dark stars of Mitchell, also mentioned later by Laplace, see [86]. The Chandrasekhar white dwarfs of very small radius, including the one with zero radius that gives the Chandrasekhar maximum mass, are also Newtonian compact stars, indeed the ones that have very small radii from the gravitational radius to zero radius provide an instance of the dark stars of Mitchell and Laplace. Of course these stars cannot exist in nature as for strong gravitational fields Newtonian gravitation is invalid. That compact Newtonian stars could be electrified was raised in [65] and further developed in [95]. Turning the table around, the real analogue of dark stars are the quasiblack holes considered in, e.g., [96].

In order to understand the effects of a small electric charge on a star, and in particular, on the interior Schwarzschild limit, we use an analytical scheme and investigate the limiting radius to mass relation and the maximum mass of relativistic compact stars made of an electrically charged incompressible fluid. The investigation is carried out using the hydrostatic equilibrium equation, i.e., the TOV equation, and the other structure equations, with the further hypothesis that the charge distribution is proportional to the energy density. The approach relies on Volkoff and Misner's method [98, 107] to solve the TOV equation. For zero charge one gets the interior Schwarzschild limit and under certain assumptions one gets the Misner mass. Our analysis for stars with a small electric charge shows that the maximum mass increases relatively to the uncharged case, whereas the minimum possible radius decreases, an expected effect since the new field is repulsive aiding the pressure to sustain the star against gravitational collapse.

The chapter is organized as follows. In Sec. 2.2 we give the general relativistic equations, the equations of structure for a static spherically symmetric configuration, the equations of state for energy density and charge density, and discuss the exterior spacetime and the boundary conditions. In Sec. 2.3 we review the pure, uncharged, interior Schwarzschild limit using the Volkoff and Misner's formalism to set the nomenclature. We also give the Misner mass. In Sec. 2.4 we study analytically the interior electric Schwarzschild limit using the Volkoff and Misner's formalism and give the electric correction to the Misner mass. In Sec. 2.5 we conclude. Sec. 2.6 is an appendix for the chapter, dedicated to the analysis of the behaviour of the perturbed pressure for small radius.

2.2 General relativistic equations

2.2.1 Basic equations

We are interested in analyzing highly compacted charged spheres as described by the Einstein-Maxwell equations with charged matter. In this chapter we set $G = 1$ and $c = 1$. The field equations are

$$G_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1}$$

$$\nabla_\nu F^{\mu\nu} = 4\pi J^\mu, \tag{2.2}$$

where Greek indices are spacetime indices running from 0 to 3, with 0 being a time index. The Einstein tensor $G_{\mu\nu}$ is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$, where $R_{\mu\nu}$ is the Ricci tensor $R_{\mu\nu}$, $g_{\mu\nu}$is the metric tensor, and R the Ricci scalar. The Faraday-Maxwell tensor $F_{\mu\nu}$ is defined in terms of an electromagnetic four-potential A_μ by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Equation (2.1) is the Einstein equation, stating the relation between the Einstein tensor and the energy-momentum tensor $T_{\mu\nu}$. $T_{\mu\nu}$ is written here as a sum of two terms,

$$T_{\mu\nu} = E_{\mu\nu} + M_{\mu\nu}\,. \tag{2.3}$$

$E_{\mu\nu}$ is the electromagnetic energy-momentum tensor, which is given in terms of the Faraday-Maxwell tensor $F_{\mu\nu}$ by the relation

$$E_{\mu\nu} = \frac{1}{4\pi}\left(F_\mu{}^\gamma F_{\nu\gamma} - \frac{1}{4}g_{\mu\nu}F_{\gamma\beta}F^{\gamma\beta}\right)\,. \tag{2.4}$$

$M_{\mu\nu}$ represents the matter energy-momentum tensor and we assume to be the energy-momentum tensor of a perfect fluid, namely,

$$M_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}, \tag{2.5}$$

with ρ and p being the energy density and the pressure of the fluid, respectively, and U_μ is the fluid four-velocity. Equation (2.2) is the Maxwell equation, stating the proportionality between the covariant derivative ∇_ν of the Faraday-Maxwell tensor $F_{\mu\nu}$ and the electromagnetic four-current J_μ. For a charged fluid, this current is given in terms of the electric charge density ρ_e by

$$J^\mu = \rho_e U^\mu \,. \tag{2.6}$$

The other Maxwell equation $\nabla_{[\alpha} F_{\beta\gamma]} = 0$, where [...] means antisymmetrization, is automatically satisfied.

2.2.2 Equations of structure

The line element for a static spherically symmetric spacetime is of the form

$$ds^2 = -B(r)\, dt^2 + A(r)\, dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{2.7}$$

where t, r, θ e ϕ are the usual Schwarzschild-like coordinates, and the metric potentials $A(r)$ and $B(r)$ are functions of the radial coordinate r only. The assumed spherical symmetry of the spacetime implies that the only nonzero components of a purely electrical Faraday-Maxwell tensor $F^{\mu\nu}$ are F^{tr} and F^{rt} with $F^{tr} = -F^{rt}$ and where F^{tr} is a function of the radial coordinate r alone, $F^{tr} = F^{tr}(r)$. The other components of $F^{\mu\nu}$ are identically zero. It is advantageous to define the total electric charge $q(r)$ inside a spherical surface labeled by the radial coordinate whose value is r by

$$q(r) = F^{tr} r^2 \sqrt{A(r)\, B(r)}\,. \tag{2.8}$$

I.e., one swaps F^{tr} for $q(r)$. It is also opportune to define a new quantity $m(r)$ in such a way that

$$\frac{1}{A(r)} = 1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}. \tag{2.9}$$

I.e., one swaps $A(r)$ for $m(r)$. The new function $m(r)$ represents the gravitational mass inside the sphere of radial coordinate r.

One of the Einstein equations can be substituted by the contracted Bianchi identity $\nabla_\mu T^{\mu\nu} = 0$, which gives

$$\frac{dB(r)}{dr} = \frac{B(r)}{p(r)+\rho(r)} \left[\frac{q(r)}{2\pi r^4}\frac{dq(r)}{dr} - 2\frac{dp(r)}{dr}\right], \tag{2.10}$$

a differential equation for B, q, and p.

Einstein equations also give a differential equation for $B(r)$ alone, i.e.,

$$\left(1 - \frac{2m(r)}{r} + \frac{q^2(r)}{r^2}\right)\left[1 + \frac{r}{B(r)}\frac{dB(r)}{dr}\right] = 1 + 8\pi r^2 \left[p(r) - \frac{q^2(r)}{8\pi r^4}\right]. \tag{2.11}$$

Now, we are ready to write the other three equations in a form we want to use. One finds that another of Einstein equations gives a differential equation for $m(r)$, i.e.,

$$\frac{dm(r)}{dr} = 4\pi\rho(r) r^2 + \frac{q(r)}{r}\frac{dq(r)}{dr}\,. \tag{2.12}$$

Since $m(r)$ represents the gravitational mass inside the sphere of radial coordinate r, Eq. (2.12) represents then the energy conservation as measured in the star's frame. The only non-vanishing component of the Maxwell equation (2.2) is given by

$$\frac{dq(r)}{dr} = 4\pi\rho_e(r)\, r^2\sqrt{1-\frac{2m(r)}{r}+\frac{q^2(r)}{r^2}}\,, \tag{2.13}$$

Finally, replacing Eq. (2.13) and the conservation equation (2.10) into Eq. (2.11) it yields

$$\frac{dp}{dr} = -(p+\rho)\frac{\left(4\pi pr + m/r^2 - q^2/r^3\right)}{(1-2m/r+q^2/r^2)} + \rho_e\,\frac{q/r^2}{\sqrt{1-2m/r+q^2/r^2}}\,. \tag{2.14}$$

where to simplify the notation we have dropped the functional dependence, i.e., $m(r) = m$, $q(r) = q$, $\rho(r) = \rho$, $p(r) = p$, and $\rho_e(r) = \rho_e$. Eq. (2.14) is the TOV equation modified by the inclusion of electric charge [63] (see also [60]). The system of equations (2.11)-(2.14) is the system we were looking for. We need now to specify the equation of state and the equation for the charge density profile.

2.2.3 Equation of state and the charge density profile

In the present model there are six unknown functions: $B(r)$, $m(r)$, $q(r)$, $\rho(r)$, $p(r)$, and $\rho_e(r)$; and just four equations: Eqs. (2.11), (2.12), (2.13), and (2.14). Additional relations are obtained from a model for the cold fluid, which furnishes relations among the pressure and the energy density. For an electrically charged fluid, a relation defining the electric charge distribution is also needed.

Here we assume an incompressible fluid, i.e.,

$$\rho(r) = \text{constant}\,. \tag{2.15}$$

So the energy density is constant along the whole star.

Following [101] (see also [60]), we assume a charge density proportional to the energy density,

$$\rho_e = \alpha\rho\,, \tag{2.16}$$

where, in geometric units, α is a dimensionless constant which we call the charge fraction. The charge density along the whole star is thus constant as well. Other equations for the charge distribution could be considered, as more charge concentration on the core, or more charge on the outer layers, see , e.g., [62, 75, 76].

We have now four equations: Eqs. (2.11), (2.12), (2.13), and (2.14); and four unknowns: $B(r)$, $m(r)$, $q(r)$, and $p(r)$, as ρ and ρ_e are given in (2.15) and (2.16), respectively. The resulting set of equations constitute the complete set of structure equations which, with some appropriate boundary conditions, can be solved simultaneously. We are not going to solve it, see previous paper [61]. Here we use this system of equations to find the Schwarzschild interior limit for the small charge case.

2.2.4 The exterior vacuum region to the star and the boundary conditions

The conditions at the center of the star are that $m(r=0) = 0$, $q(r=0) = 0$, and $A(r=0) = 1$ to avoid any type of singularities, and that $p(r=0) = p_c$, $\rho(r=0) = \rho_c$, and $\rho_e(r=0) = \rho_{ec}$, where p_c is the central pressure, ρ_c is the central energy density, and ρ_{ec} is the central charge distribution, the two latter having the same constant values throughout the star (see Eqs. (2.15-2.16)).

The interior solution is matched at the surface to the exterior Reissner-Nordström spacetime, with metric given by

$$ds^2 = -F(r)\, dT^2 + \frac{dr^2}{F(r)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) , \tag{2.17}$$

where

$$F(r) = 1 - 2M/r + Q^2/r^2 , \tag{2.18}$$

with the outer time T being proportional to the inner time t, and M and Q being the total mass and the total charge of the star, respectively.

At the surface of the star one has a vanishing pressure, i.e., $p(r = R) = 0$. The boundary conditions at the surface of the star are then $B(R) = 1/A(R) = F(R)$, $m(R) = M$, $q(R) = Q$, besides $p(R) = 0$.

An important quantity for the exterior metric is the gravitational or horizon radius r_+ of the configuration. The Reissner-Nordström metric, given through Eqs. (2.17)-(2.18), then has

$$r_+ = M + \sqrt{M^2 - Q^2} . \tag{2.19}$$

as the solution for its own gravitational radius.

2.3 The interior Schwarzschild limit: The zero charge case

2.3.1 Equations

Before we treat the small charge case analytically, we consider the exact Schwarzschild interior solution as given by [107] and displayed later in Misner's lectures [98]. For this we put $q = 0$ in Eqs. (2.11)-(2.14). Equation (2.11) is of no direct interest here, Eq. (2.12) gives

$$\frac{dm(r)}{dr} = 4\pi\rho(r)r^2 , \tag{2.20}$$

Eq. (2.13) is trivially satisfied in this case, and finally, Eq. (2.14) simplifies to

$$\frac{dp}{dr} = -(p+\rho)\frac{4\pi pr + m/r^2}{1 - 2m/r} . \tag{2.21}$$

Since, by equation (2.15), the density is constant we can integrate equation (2.20) obtaining

$$m(r) = \frac{4}{3}\pi\rho\, r^3 , \quad 0 \le r \le R , \tag{2.22}$$

where R is the star radius and we have imposed that there is no point mass in the center. Defining a characteristic length R_c as

$$R_c^2 = \frac{3}{8\pi\rho}, \tag{2.23}$$

we can rewrite the mass function, Eq. (2.22), as

$$m(r) = \frac{1}{2}\frac{r^3}{R_c^2} , \quad 0 \le r \le R . \tag{2.24}$$

Interchanging ρ and R_c as necessary and noting that $2\,\rho\, R_c^2 = \frac{3}{4\pi}$ we get from Eq. (2.21),

$$\frac{dp}{dr} = -\frac{(p+\rho)(3p+\rho)}{\rho}\frac{1}{2R_c^2}\frac{r}{1 - r^2/R_c^2} . \tag{2.25}$$

2.3.2 The interior Schwarzschild limit: The R and M relation and the minimum radius

Equation (2.25) is separable and can be integrated as

$$\int dp \, \frac{\rho}{(\rho+p)(\rho+3p)} = -\frac{1}{2} \int dr \, \frac{r}{R_c^2} \frac{1}{1 - r^2/R_c^2} \,, \tag{2.26}$$

with the boundary condition that the surface of the star R has zero pressure, i.e.,

$$p(R) = 0 \,. \tag{2.27}$$

Defining a new radial coordinate χ by

$$r = R_c \sin\chi \,, \tag{2.28}$$

Eq. (2.26) can be put in the form

$$\int dp \, \frac{\rho}{(\rho+p)(\rho+3p)} = -\frac{1}{2} \int d\left(\ln\cos\chi\right) . \tag{2.29}$$

subjected to the boundary condition

$$p(\chi_s) = 0 \,, \tag{2.30}$$

where χ_s is given through

$$R = R_c \, \sin\chi_s \,. \tag{2.31}$$

Integrating Eq. (2.29), subjected to the boundary condition (2.30), yields the pressure

$$p = \rho \frac{\cos\chi - \cos\chi_s}{3\cos\chi_s - \cos\chi} . \tag{2.32}$$

The central pressure, p_c is the pressure computed at zero radius $r = 0$, i.e., $\chi = 0$, so that

$$p_c = \rho \frac{1 - \cos\chi_s}{3\cos\chi_s - 1} \,. \tag{2.33}$$

This blows up,

$$p_c \to \infty \quad \text{when} \quad \cos\chi_s \to 1/3 \,. \tag{2.34}$$

This is equivalent to

$$\sin^2\chi_s = \frac{8}{9} \,. \tag{2.35}$$

Now, Eqs. (2.24) and (2.28) allow us to write

$$M = \frac{1}{2} R \, \sin^2\chi_s \,, \tag{2.36}$$

where $M \equiv m(R)$ is the star's total mass. Thus, Eqs. (2.35) and (2.36) yield

$$\frac{R}{M} = \frac{9}{4} \,. \tag{2.37}$$

Equation (2.37) is the Schwarzschild limit found by [104].

2.3.3 Misner mass bound

Following [98] we can also display a mass bound. Equation (2.24) gives

$$M = \frac{1}{2}\frac{R^3}{R_c^2}\,. \tag{2.38}$$

Eliminating R in Eqs. (2.37) and (2.38), and noting that $p_c \leq \infty$, one gets the mass bound

$$M \leq \frac{1}{2}\left(\frac{8}{9}\right)^{3/2} R_c\,. \tag{2.39}$$

To get a mass we have to have a density and thus an R_c. One can make sense of a constant density if one takes it as the density at which matter is almost incompressible and the pressure throughout the star is very high. If the fluid is an ideal gas this happens when the particles have relativistic velocities of the order 1. For fermions this happens when the Fermi levels are near the rest mass m_n of the fermions, neutrons say, while for bosons this means that the gas temperatures are of the order of the rest mass m_b of the particles. This gives, for both fermions and bosons, a density of one particle per cubic Compton wavelength. I.e., for a particle with mass m and Compton wavelength λ given by $\lambda = \hbar/m$ the density is $\rho \sim m^4/\hbar^3$. In the case of a star composed of neutrons, [98] obtains

$$M \leq 1.5 M_\odot\,, \tag{2.40}$$

where $M_\odot$ is the Sun's mass. This bound is similar to the Chandrasekhar limit $M_{\rm Chandrasekhar} = 1.44 M_\odot$, or the Oppenheimer-Volkoff mass, $M_{\rm OV} \simeq 1 M_\odot$, both found for equations of state different from the one used here and through totally different means.

2.4 The electrically charged interior Schwarzschild limit: The small charge case

2.4.1 Equations: perturbing with a small electric charge

2.4.1.A Expansion in the electric charge parameter α

We are going to solve equations (2.12), (2.13), and (2.14), treating the charge $q(r)$ as a small perturbation, thus assuming α small. To do so, we note that the solutions for the mass and the charge will be of the form

$$q(r) = q_1(r)\,, \tag{2.41}$$

$$m(r) = m_0(r) + m_1(r)\,, \tag{2.42}$$

where we are assuming that the non-perturbed charge is zero $q_0(r) = 0$, $m_0(r)$ is the mass of the uncharged star given by Eq. (2.20), or Eq. (2.22), and $q_1(r)$ and $m_1(r)$ are the perturbed small charge and mass functions to be determined. The pressure is also assumed to be given by the expansion

$$p(r) = p_0(r) + p_1(r)\,, \tag{2.43}$$

where p_0 is the pressure in the uncharged case, given by equation (2.25), or (2.32), and $p_1(r)$ is the perturbation induced in the pressure when a small charge is considered. Note that, while the boundary condition for the non-charged star was simply $p(R) = p_0(R) = 0$, the boundary condition for the charged star becomes

$$p_0(R) + p_1(R) = 0\,. \tag{2.44}$$

At this point, it will prove useful to introduce the dimensionless variable

$$x = \frac{r}{R_c}\,, \tag{2.45}$$

where R_c is the characteristic length defined in Eq. (2.23). The expressions for the mass, charge, and pressure in this new variable are generically defined as

$$m(x) = \frac{m(r)}{R_c}\,, \quad q(x) = \frac{q(r)}{R_c}\,, \quad p(x) = \frac{p(r)}{\rho}\,. \tag{2.46}$$

From Eq. (2.45) we defined x_s as the x at the surface, so that

$$x_s = \frac{R}{R_c}\,. \tag{2.47}$$

Accordingly, we define

$$m(x_s) = \frac{M}{R_c}\,, \quad q(x_s) = \frac{Q}{R_c}\,, \quad p(x_s) = \frac{p(R)}{\rho}\,. \tag{2.48}$$

2.4.1.B Calculation of the perturbed charge distribution q_1

Expanding Eq. (2.13) for small α, we get in the x variable that

$$\frac{dq_1}{dx} = \frac{3}{2}\,\alpha\frac{x^2}{\sqrt{1-x^2}}\,, \tag{2.49}$$

up to first order in α. Solving the above equation subject to the condition $q_1(0) = 0$ and expressing the solution in terms of the variable x, results in

$$q_1(x) = \frac{3}{4}\,\alpha\,\left(\arcsin x - x\sqrt{1-x^2}\right)\,. \tag{2.50}$$

2.4.1.C Calculation of non-perturbed and perturbed masses

The unperturbed mass m_0 can now be expressed simply as

$$m_0(x) = \frac{x^3}{2}\,. \tag{2.51}$$

One can also find an expression for m_1, namely,

$$m_1(x) = \frac{3}{8}\,\alpha^2\,\left(3x - x^3 - 3\sqrt{1-x^2}\arcsin x\right)\,. \tag{2.52}$$

Indeed, from Eqs. (2.12) and (2.42), it is clear that the equation for the perturbed mass m_1 is given by

$$\frac{dm_1}{dx} = \frac{q_1}{x}\frac{dq_1}{dx}\,. \tag{2.53}$$

Inserting Eq. (2.50) into Eq. (2.53), we can integrate it using the boundary condition that the total mass at the center of the star is $m(0) = 0$, which implies that $m_1(0) = 0$ since Eq. (2.22), or Eq. (2.51), satisfies $m_0(0) = 0$. Doing this, we are led to Eq. (2.52).

2.4.1.D Equations for the pressures, solution for the zeroth order pressure, and calculation of the perturbed pressure at the star's radius

(i) Equations for the pressures

To find the equations for the pressures $p_0(x)$ and $p_1(x)$, we begin by expressing Eq. (2.14) for the total pressure in terms of the variable x given in Eq. (2.45),

$$\frac{\mathrm{d}p}{\mathrm{d}x} = -\frac{(1+p(x))\left(3p(x)x/2 + m(x)/x^2 - q^2(x)/x^3\right)}{1-2m(x)/x + q^2(x)/x^2} + \\ + \frac{\alpha q}{x^2\sqrt{1-2m(x)/x+q^2(x)/x^2}} \,. \tag{2.54}$$

Now we can expanded the right side of the above equation in powers of α and retain the two lowest terms. By doing so, and using the expansion (2.43) on the left side of Eq. (2.54) and Eqs. (2.41)-(2.42) on the right hand side, we can equate the terms in equal powers of α, thus obtaining two differential equations. The first one, obtained from the 0th power in α is

$$\frac{dp_0(x)}{dx} = -\frac{(1+p_0(x))\left(3p_0(x)x/2 + m_0(x)/x^2\right)}{1-2m_0(x)/x}, \tag{2.55}$$

which is simply the differential equation for the unperturbed pressure. The second differential equation, to first order in α^2, is

$$\frac{dp_1}{dx} = \frac{\alpha q_1}{x^2\sqrt{1-2m_0/x}} - \frac{(1+p_0)(3p_0x/2+x/2)f_1}{(1-2m_0/x)^2} + \\ - \frac{p_1(3p_0x/2+x/2)+(1+p_0)(3p_1x/2+f_2)}{(1-2m_0/x)}, \tag{2.56}$$

which is the differential equation for the perturbed pressure p_1, where, again, to shorten equations we have dropped the dependence of variables p_1, p_0, m_1, m_0, and q_1 on x, and we have also defined the auxiliary functions $f_1 = f_1(x)$ and $f_2 = f_2(x)$ by

$$f_1(x) = \frac{2m_1(x)}{x} - \frac{q_1^2(x)}{x^2}\,, \tag{2.57}$$

and

$$f_2(x) = \frac{m_1(x)}{x^2} - \frac{q_1^2(x)}{x^3}\,. \tag{2.58}$$

Ultimately, we want to obtain an equation for the radius R for which the central pressure blows up. From Eq. (2.43), the central pressure is $p(0) = p_0(0) + p_1(0)$. In Sec. 2.6 we show that $p_1(0)$ is always finite. So we have to find a solution for the radius R at which $p_0(0)$ blows up.

(ii) Solution for the zeroth order pressure

We start by obtaining a solution for p_0. Since the boundary condition has changed relatively to the uncharged case, it is now given by Eq. (2.44), we cannot use a priori the form (2.32) for p_0. We use the solution to Eq. (2.55) without specifying any boundary condition. In the variable x, this means

$$p_0(x) = \frac{\sqrt{1-x^2}-A}{3A-\sqrt{1-x^2}}\,. \tag{2.59}$$

where A is an integration constant. To find out what this constant is, we insert the above equation into Eq. (2.44) and solve the resulting equation with respect to A. After expanding for the small charge parameter α, and so for small p_1, we are led to

$$A = \sqrt{1-x_s^2}\,(1+2p_1(x_s))\ , \tag{2.60}$$

up to first order in the perturbed quantities. Then, the expression for p_0, analogous to Eq. (2.32), becomes

$$p_0(x) = \rho \, \frac{\sqrt{1-x^2} - \sqrt{1-x_s^2}\,(1+2p_1(x_s))}{3\sqrt{1-x_s^2}\,(1+2p_1(x_s)) - \sqrt{1-x^2}}. \tag{2.61}$$

Since $p_0(x)$ depends on $p_1(x_s)$ in the denominator, we have to find $p_1(x_s)$, i.e., we have to calculate the perturbed pressure at the star's radius.

(iii) Calculation of the perturbed pressure at the star's radius

The equation for p_1, Eq. (2.56), cannot be solved analytically for all x. However, we are only interested in the value of p_1 at the surface of the star. At this particular radius it is possible to obtain the exact value of the perturbed pressure without ever solving Eq. (2.56). The reason for this is the fact that at the star's radius the pressure $p(x_s) = p_0(x_s) + p_1(x_s)$ is zero. Therefore, we can insert the boundary condition $p(x_s) = 0$ in the exact derivative of the pressure given by Eq. (2.54) and expand the resulting equation for small α, giving

$$\begin{aligned} \left.\frac{dp}{dx}\right|_{x=x_s} = \quad & -\frac{m_0(x_s)}{x_s^2(1-2m_0(x_s)/x_s)} + \\ & -\frac{m_0(x_s)}{1-2m_0(x_s)/x_s}\left(\frac{f_1(x_s)}{x_s^2} + \frac{f_2(x_s)}{m_0(x_s)}\right), \end{aligned} \tag{2.62}$$

up to first order, and where the auxiliary functions $f_1(x)$ and $f_2(x)$ are given by Eqs. (2.57) and (2.58), respectively. Using the expansion (2.43) on the left side of Eq. (2.62), one can clearly see that there is a compatibility condition which must be physically required, namely that, at the star surface, the first term on the right side of Eq. (2.62) must be equal to Eq. (2.55) and the second term equal to Eq. (2.56). Hence, we arrive at the two equations

$$\frac{3}{2}\,p_0^2(x_s) + 2\,p_0(x_s) = 0\,, \tag{2.63}$$

and

$$\frac{3}{2}\,p_1(x_s)p_0(x_s)x_s + p_0(x_s)f_2(x_s) = 0\,. \tag{2.64}$$

These equations give two different solutions, namely, $p_0(x_s) = 0$ and $p_1(x_s) = 0$, or $p_0(x_s) = -\frac{4}{3}$ and $p_1(x_s) = -\frac{2}{3}x_s^{-1}f_2(x_s)$. This latter solution does not satisfy the boundary condition (2.44) so the unique valid solution is given by

$$p_0(x_s) = 0\,, \tag{2.65}$$

and

$$p_1(x_s) = 0\,. \tag{2.66}$$

2.4.1.E Equation for the minimum radius

We see that the central pressure $p_0(0)$ given in Eq. (2.61) is divergent when the following condition holds,

$$3\sqrt{1-x_s^2}\,(1+2p_1(x_s)) = 1\,. \tag{2.67}$$

Expanding it in α^2 we arrive to

$$\frac{x_s}{m_0(x_s)} = \frac{9}{4} - \frac{9}{8}\,p_1(x_s)\,, \tag{2.68}$$

valid in first order in α^2. Using the expansion provided by Eq. (2.42), it can be shown that to first order in α^2, we have

$$\frac{x_s}{m(x_s)} = \frac{x_s}{m_0(x_s)} - x_s\,\frac{m_1(x_s)}{m_0^2(x_s)}\,. \tag{2.69}$$

Upon substituting Eq. (2.69) into Eq. (2.68) we conclude that

$$\frac{x_s}{m(x_s)} = \frac{9}{4} - \left(\frac{9}{8}p_1(x_s) + x_s\,\frac{m_1(x_s)}{m_0^2(x_s)}\right)\,. \tag{2.70}$$

Now, the minimum star radius R will not be just $\sqrt{8/9}\,R_c$ but will have corrections of order α^2. These corrections will induce changes of the order α^4 in Eq. (2.70). Thus, we can set $x_s = \sqrt{8/9}$ in Eq. (2.70), i.e.,

$$\frac{x_s}{m(x_s)} = \frac{9}{4} - \left(\frac{9}{8}p_1(x_s) + \sqrt{\frac{8}{9}}\,\frac{m_1(x_s)}{m_0^2(x_s)}\right)\,, \tag{2.71}$$

which is the equation we were looking for. Since $m_0(x_s)$, $m_1(x_s)$, and $p_1(x_s)$ can be taken directly from Eq. (2.51), Eq. (2.52), and Eq. (2.66), respectively, we can proceed to the final result. Indeed, using Eqs. (2.51), (2.52), (2.66) in (2.71), we obtain

$$\frac{x_s}{m_s} = \frac{9}{4} - 1.529\,\alpha^2\,, \tag{2.72}$$

up to order α^2.

In converting from the variable x back to r, we use

$$\frac{x_s}{m(x_s)} = \frac{R}{M}\,. \tag{2.73}$$

In addition, it can also be interesting to express α in terms of the total charge Q and mass M. The following relation valid for small q_1, or small α, can be found $\frac{Q}{M} = \frac{q_1(x_s)}{m_0(x_s)+m_1(x_s)} = \frac{q_1(x_s)}{m_0(x_s)}$ so that

$$\frac{q_1(x_s)}{m_0(x_s)} = \frac{Q}{M}\,, \tag{2.74}$$

up to order q_1. We can then express α in terms of the ratio Q/M. Using Eqs. (2.50) and (2.51) in Eq. (2.74), and solving the resulting equation for α, we obtain

$$1.641\,\alpha = \frac{Q}{M}\,. \tag{2.75}$$

2.4.2 The electric interior Schwarzschild limit: The R, M and Q relation for small charge

We are now in a position to calculate the desired ratio (2.72) in terms of the quantities R, M and Q and find the appropriate relation. Inserting Eq. (2.73) into Eq. (2.72) we find

$$\frac{R}{M} = \frac{9}{4} - 1.529\,\alpha^2\,, \tag{2.76}$$

which is one form of the interior Schwarzschild limit for small charge.

Inserting Eq. (2.75) into Eq. (2.76) we get

$$\frac{R}{M} = \frac{9}{4} - 0.568\,\frac{Q^2}{M^2}\,, \tag{2.77}$$

valid up to order Q^2/M^2. This is another form of the interior Schwarzschild limit for small charge.

We can also express the limit in terms of the horizon radius, r_+, for the Reissner-Nordström metric. The horizon radius is defined by Eq. (2.19), i.e., up to order Q^2/M^2 one has, $r_+ = M + \sqrt{M^2 - Q^2} = 2M\left(1 - \frac{1}{4}\frac{Q^2}{M^2}\right)$. So,

$$\frac{R}{r_+} = \frac{9}{8} - 0.003\frac{Q^2}{M^2}\,, \tag{2.78}$$

up to order Q^2/M^2. Equation (2.78) is yet another form of the interior Schwarzschild limit for small charge.

The electric interior Schwarzschild limit for small charge presented in various forms in Eqs. (2.76), (2.77), and (2.78) is the main result of this work. All the three forms of the electric interior Schwarzschild limit for small charge show that, in comparison with the uncharged case Eq. (2.37), the star can be more compact. In particular, Eq. (2.78) shows that in the charged case the radius of the star can be a bit nearer its own horizon.

In [61] these compact stars were studied numerically. An $R/M \times Q/M$ relation was given numerically for $0 \le Q/M \le 1$. For small charge, $Q/M \ll 1$, one can extract from the numerical calculations in [61] that $\frac{R}{M} \simeq 2.25 - 0.6\,\frac{Q^2}{M^2}$. This should be compared to our analytical calculation valid in first order of Q/M, given here in Eq. (2.77), i.e., $\frac{R}{M} = 2.25 - 0.568\,\frac{Q^2}{M^2}$. It shows that the numerical code used in [61] is compatible with the analytical calculation. In that work [61] it was also shown numerically that in the other extreme, namely, $Q/M = 1$, one would obtain a star at its own gravitational radius, $R/M = 1$, i.e., an (extremal) quasiblack hole.

2.4.3 A mass bound

We can adapt the mass bound from section 2.3.3 to the small charge case. Indeed, from Eq. (2.42) and the definition $M \equiv m(R)$, we have at the boundary

$$M = m_0(R) + m_1(R)\,. \tag{2.79}$$

Now, Eq. (2.51) at the boundary can be put in the form $m_0(R) = \frac{1}{2}\,R_c\,\sin^3\chi_s$. So, Eq. (2.79) yields

$$M = \frac{1}{2}R_c \sin^3\chi_s + m_1(R)\,. \tag{2.80}$$

Since $p_c \le \infty$, using equation (2.68) with $p_1(R) = 0$, one obtains $\sin^2\chi_s \le \frac{8}{9}$, and the bound for the non-perturbed mass m_0 is given by $m_0(R) = \frac{1}{2}R_c \sin^3\chi_s \le \frac{1}{2}\,\left(\frac{8}{9}\right)^{3/2} R_c$. In order to obtain the bound for m_1, we have to substitute the bound $\sin^2\chi_s \le \frac{8}{9}$ in equation (2.52). This is enough since we are only working up to order α^2. Thus, the mass bound for the small charge case is

$$M \le \frac{1}{2}\left(\frac{8}{9}\right)^{3/2} R_c\,\left(1 + 0.679\,\alpha^2\right)\,. \tag{2.81}$$

In the case of a compact star composed of neutrons in the incompressible state speckled with some charged particles, we obtain

$$M \le 1.5\,M_\odot\,\left(1 + 0.679\,\alpha^2\right)\,, \tag{2.82}$$

Comparing equation (2.82) with equation (2.40) we see that we can attain bigger mass on a charged star. This is expected since the electrostatic repulsion is opposite to the gravitational force, which means that we can put more mass on the star without it collapsing.

2.5 Conclusions

In this chapter we have studied the interior Schwarzschild limit of spherically symmetric star configurations composed of a fluid with constant energy density ρ and with a small electrical charge distribution ρ_e proportional to ρ, $\rho_e = \alpha\rho$ with $\alpha \ll 1$. The exterior spacetime is described by the Reissner-Nordström metric. We have found through an analytical scheme that due to the electric charge distribution the limiting star configuration can have more mass and a smaller radius relatively to the limiting star with zero charge. This is expected since the electric charge distribution has a repulsive effect, adding to the pressure as a force that withstands the star. The analytical calculation obtained is in accord with the results from the numerical code used in [61]. For stars containing some type of dark matter in their interior there is the possibility that they possess a small but non-negligible electric charge, in which case our analytical formula is opportune and can be confronted quantitatively with observational data.

A related theme is the Buchdahl and the Buchdahl-Andréasson bounds. Buchdahl by imposing a simple set of assumptions, namely, the spacetime is spherically symmetric, the star is made of a perfect fluid, and the density is a nonincreasing function of the radius, found that the radius to mass relation is $R/M \geq 9/4$ [68]. Thus the Schwarzschild limit [104], i.e., the limiting R/M configuration when the central pressure goes to infinity, is an instance that saturates the Buchdahl bound. Following the line of reason of Buchdahl, [59] obtained a bound for the minimum radius of a star using the following energy condition $p + 2p_T \leq \rho$ where p_T is the tangential pressure, p is the radial pressure and ρ is the energy density. This bound, the Buchdahl-Andréasson bound, is given by $\frac{R}{M} \leq \frac{9}{\left(1+\sqrt{1+3Q^2/R^2}\right)^2}$. Retaining terms in first order in Q^2 one gets $\frac{R}{M} = \frac{9}{4} - 0.667\,\frac{Q^2}{M^2}$. Thus, our configurations of constant density and a charge distribution proportional to the energy density having $\frac{R}{M} = \frac{9}{4} - 0.568\,\frac{Q^2}{M^2}$ does not saturate the bound. This raises the question of whether there are other types of charged matter that can saturate the bound. One type is thin shells with an appropriate relation between surface energy density and surface pressure [59]. Are there continuous (non-thin-shell) distribution configurations that saturate the bound? It seems that, as the configurations analyzed here, the configurations studied in [62, 75, 76] do not saturate the bound. It remains to be seen if the electrically charged configurations analyzed in [81, 96] saturate the bound. For further study on bounds of electrically charged stars see [64, 79, 109].

2.6 Appendix: Behaviour of the perturbed pressure for small radius

Our goal is to compute the limit of infinite central pressure $p(r = 0)$ in this charged case, so it is important to analyze the behaviour of p_1 for small radius x, $x \sim 0$, which also means $\chi \sim 0$, and $r \sim 0$. We start by obtaining the perturbed charge, q_1, for small radius expanding equation (2.50), which gives

$$q_1(x) \sim \alpha\frac{x^3}{2}\,, \tag{2.83}$$

and the perturbed mass, m_1, from equation (2.52)

$$m_1(x) \sim \alpha^2 \frac{3}{20} x^5 \,. \tag{2.84}$$

The non-perturbed pressure, p_0, given by equation (2.61) becomes

$$p_0(x) \sim p_0(x=0) - \frac{\sqrt{1-x_s^2}}{3\sqrt{1-x_s^2}-1} x^2 \,. \tag{2.85}$$

Using equation (2.56) for the perturbed pressure, p_1, together with equations (2.83), (2.84) and (2.61) it gives

$$\frac{\mathrm{d}p_1}{\mathrm{d}x} \sim -p_1\left(3p_0(x=0)+2\right)x + \frac{1}{2}\alpha^2\, x \,. \tag{2.86}$$

Integrating equation (2.86) one obtains

$$p_1(x) \sim \frac{\alpha^2}{6p_0(x=0)+4} + \left(p_1(x=0) - \frac{\alpha^2}{6p_0(x=0)+4}\right) e^{-(3p_0(x=0)+2)x^2/2} \,. \tag{2.87}$$

So the unique way that the central pressure $p(x) = p_0(x) + p_1(x)$ blows up is $p_0(x=0)$ blowing up.

3

Dynamics of thin shells in a (3+1)-dimensional asymptotically AdS spacetime

3.1 Introduction

The study of anti-de Sitter (AdS) spacetimes, i.e., spacetimes with a negative cosmological constant, $\Lambda < 0$, is an important theme in general relativity field of sudy. Also, those spaces are used in theories of quantum gravity, formulated in the terms of the string theory or M-theory [110]. In this chapter we will focus in the (3+1)-dimensional AdS spacetime and we will study the dynamics of a particular self-gravitating system: the thin matter shell.

A thin matter shell is a hypersurface which separates spacetime into two regions, the interior region and the exterior region. Due to the development of a singularity in spacetime there are some conditions that must be satisfied to ensure that the entire spacetime is a valid solution of Einstein's equations: the junctions conditions (see Appendix A). Their appropriate use leads us to pressure and rest mass of the shell.

In this chapter we will adopt the following outline. In Sec. 3.2 we present the thin shell spacetime, whereas in Sec. 3.3 we arrive to the junction conditions, which lead to the pressure and rest mass of the shell. The no-trapped-surface condition and the weak and dominant energy conditions are also discussed in Sec. 3.4. At the end we conclude in Sec. 3.5.

3.2 The thin shell spacetime

In 3+1 dimensions, Einstein's equation with cosmological constant Λ is

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta} + \Lambda g_{\alpha\beta} \,, \tag{3.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, G_3 is the gravitacional constant in 2+1 dimensions, $T_{\alpha\beta}$ is the energy-momentum tensor and $g_{\alpha\beta}$ is the spacetime metric. We will choose units with the velocity of light equal to one, $c = 1$, so that G has units of the inverse of mass. The greek indices run from $0, 1, 2, 3$, with 0 begin the time index. As we want to work in an AdS spacetime, with negative cosmological constant, we define the AdS length l as

$$-\Lambda = \frac{1}{l^2} \,. \tag{3.2}$$

We assume a Schwarzschild AdS spacetime divided in two regions by a spherically symmetric thin matter shell. The existence of the shell implies that certain conditions must be satisfied in order the hypersuperface Σ that describes the evolution of the shell have a well defined geometry. The formalism from [49, 111–113] gives two junction conditions, one for the induced metric h_{ab}, Eq. (A.16), on that hypersuperface Σ and other for the extrinsic curvature of Σ, K_{ab} in terms of a shell's stress tensor S_{ab}, Eq. (A.35).

For a spherically symmetric (3+1)-dimensional spacetime one can write the line element on the form

$$ds^2 = -F_{\pm}(r)dt^2 + \frac{dr^2}{F_{\pm}(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{3.3}$$

In the outer region, i.e. for $r > R$, where R is the shell's radius and r is the radial Schwarzschild solution usual coordinate, the metric is given by the Schwarzschild AdS line element, Eq. (3.3) with

$$F_{+}(r) = 1 - \frac{2mG}{r} + \frac{r^2}{l^2} \,, \tag{3.4}$$

where $l^2 = -1/\Lambda$ and m is the Arnowitt-Deser-Misner (ADM) mass.

In the inner region the metric is simply given by equation (3.3) with

$$F_-(r) = 1 + \frac{r^2}{l^2} \,. \tag{3.5}$$

An observer comoving with the shell describes Σ with parametric equations $r = R(\tau)$ and $t = T(\tau)$ where τ is his proper time. Taking advantage of the spherical symmetry of the shell we can describe Σ using the coordinates (τ,θ,ϕ), which yields the following line element for the induced metric

$$ds^2_\Sigma = -d\tau^2 + R^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \,. \tag{3.6}$$

3.3 The thin shell gravitational junction conditions

Applying the first junction condition, Eq. (A.15), leads to

$$-F_\pm \dot{T}^2 + \frac{\dot{R}^2}{F_\pm} = -1 \,, \tag{3.7}$$

and the dot denotes differentiation with respect to τ.

With the line element for the hypersurface Σ, Eq. (3.3), the non-null components of the extrinsic curvature are

$$K^{\tau}_{\pm\tau} = \frac{\dot{\beta}_\pm}{\dot{R}} \,, \tag{3.8}$$

$$K^{\theta}_{\pm\theta} = K^{\phi}_{\pm\phi} = \frac{\beta_\pm}{R} \,, \tag{3.9}$$

where $\beta_\pm = \sqrt{F_\pm(r) + \dot{R}^2}$. Using the second junction condition, equation (A.35) one obtains the non-null components of the stress-energy tensor of the shell

$$S^\tau_\tau = \frac{\beta_+ - \beta -}{4\pi G R} \,, \tag{3.10}$$

$$S^\theta_\theta = S^\phi_\phi = \frac{\beta_+ - \beta_-}{8\pi G R} + \frac{\dot{\beta}_+ - \dot{\beta}_-}{8\pi G \dot{R}} \,. \tag{3.11}$$

Since we assume the shell to be a perfect fluid, the stress-energy tensor is of the form $S^a_b = (\sigma + p)u^a u_b + p h^a_b$, where σ is the surface density and p is the superficial pressure. Defining the shell rest mass as $M = 4\pi R^2 \sigma$ and imposing that the shell is static, so that $\dot{R} = \ddot{R} = 0$, gives

$$M = \frac{R}{G}\left(\sqrt{1 + \frac{R^2}{l^2}} - \sqrt{1 + \frac{R^2}{l^2} - \frac{2mG}{R}}\right) \,, \tag{3.12}$$

$$p = \frac{1}{8\pi G R}\left(\sqrt{1 + \frac{R^2}{l^2} - \frac{2mG}{R}} - \sqrt{1 + \frac{R^2}{l^2}} + \frac{\frac{mG}{R} + \frac{R^2}{l^2}}{\sqrt{1 + \frac{R^2}{l^2} - \frac{2mG}{R}}} - \frac{\frac{R^2}{l^2}}{\sqrt{1 + \frac{R^2}{l^2}}}\right) \,. \tag{3.13}$$

We can solve Eq. (3.12) for the ADM mass m

$$m = M\sqrt{1 + \frac{R^2}{l^2}} - \frac{GM^2}{2R} \,. \tag{3.14}$$

In our treatment it will be useful to define the redshift

$$k = \sqrt{F_+} = \sqrt{\left(1 + \frac{R^2}{l^2}\right)\left(1 - \frac{r_+}{R}\frac{1 + r_+^2/l^2}{1 + R^2/l^2}\right)}\,, \tag{3.15}$$

where r_+ is the gravitacional radius of the shell, defined as the positive root of F_+. Now we are able to write M and p in terms of k and R, which are simply given by

$$M = \frac{R}{G}\left(\sqrt{1 + \frac{R^2}{l^2}} - k\right)\,, \tag{3.16}$$

$$p = \frac{\sqrt{1 + R^2/l^2} - k}{16\pi G R k \sqrt{1 + R^2/l^2}}\left(1 + \frac{3R^2}{l^2} - k\sqrt{1 + \frac{R^2}{l^2}}\right)\,. \tag{3.17}$$

3.4 Energy Conditions

At this point we have the mechanics of the static shell completely described. However we must impose some mechanical constraints on the shell. The first one is that there are no trapped surfaces, which gives

$$R \geq r_+\,. \tag{3.18}$$

From Eq. (3.15) one can see that at $r = r_+$, $k = 0$, so that pressure, Eq. (3.17), blows up. This implies that the all results derived are absence of physical meaning for $R < r_+$. After that we consider the energy conditions. The weak energy condition enforces M and p to be non-negative, what is automatically satisfied. Additionally the dominant energy condition $p \leq \sigma = \dfrac{M}{4\pi R^2}$ leds to

$$k \geq \frac{1 + 3R^2/l^2}{5\sqrt{1 + R^2/l^2}}\,, \tag{3.19}$$

or equivalently

$$R(1 + R^2/l^2) - \frac{R(1 + 3R^2/l^2)^2}{25(1 + R^2/l^2)} \geq r_+(1 + r_+^2/l^2)\,. \tag{3.20}$$

From this we obtain a relation R/r_+ which is function of l. R/r_+ is plotted in function of R/l in Fig. 3.1 and we have $R/r_+ \geq 25/24 \simeq 1.042$ for all l. At $l \to \infty$, i.e., $R/l = 0$ we recover $R/r_+ = 25/24$ as in [114].

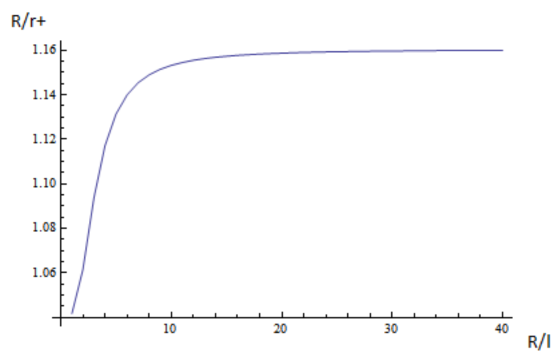

Figure 3.1: R/r_+ as function of R/l in order that the dominant energy condition $p \leq \sigma$ is obeyed.

3.5 Conclusions

In this chapter we have studied the dynamics of a thin matter shell composed of a perfect fluid in a (3+1)-dimensional AdS spacetime. Through the junction conditions we have arrived to the rest mass and the pressure of the shell. We also have analyzed the weak and dominant energy conditions of the system. Finally, the results of this chapter will be useful for study the thermodynamics of this system, which will be done in Chapter 5

4

Dynamics of rotating thin shells in a (2+1)-dimensional asymptotically AdS spacetime

4.1 Introduction

Sometimes it is interesting to reduce the spatial dimension to 2 and study general relativiy in (2+1)-dimensions. This decrease in dimensionality reduces the degrees of freedom and simplifies the calculations, although maintaining the essential physical features. The interest in (2+1)-dimensional spacetimes suffered an increment after the discovery of a black hole solution in spacetimes asymptotically AdS, the Bañados-Teitelbom-Zanelli (BTZ) black hole [47] [48].

In this chapter we are going to study the dynamics of a thin matter shell (see Appendix A) in a (2+1)-dimensional rotating AdS spacetime, i.e., a ring dividing two vacuum regions, the interior region and the exterior region. The exterior spacetime is BTZ and thus asymptotically AdS and inside the ring the spacetime is flat AdS. Our approach to the dynamics of the shell will be equivalent to the ones [51], [50] and [49], for a (3+1)-dimensional rotating shell.

In this chapter we will adopt the following outline. In Sec. 4.2 we present the thin shell spacetime. In Sec. 4.3 we present the junction conditions and we obtain the linear energy density, pressure and angular velocity of the shell, in the slowly rotating limit, whereas in Sec. 4.4 we arrive to the same quantities for the rotating general case. Finally we conclude in Sec. 4.5.

4.2 The thin shell spacetime

In 2+1 dimensions, Einstein's equation is

$$G_{\alpha\beta} = 8\pi G_3 T_{\alpha\beta} + \Lambda g_{\alpha\beta}\,, \tag{4.1}$$

where $G_{\alpha\beta}$ is the Einstein tensor, G_3 is the gravitacional constant in 2+1 dimensions, Λ is the cosmological constant, $T_{\alpha\beta}$ is the energy-momentum tensor and $g_{\alpha\beta}$ is the spacetime metric. We will choose units with the velocity of light equal to one, $c = 1$, so that G_3 has units of the inverse of mass. The greek indices run from $0, 1, 2$, with 0 begin the time index. Since we want to work in an AdS spacetime, with negative cosmological constant, we define the AdS length l as in Eq. (3.2)

We consider a timelike shell with radius R, i.e., a ring, in a (2+1)-dimensional spacetime. This ring divides spacetime into two regions, an inner region and an outer one.

The exterior metric is given by the BTZ line element [47]

$$\begin{aligned} ds_+^2 = g^+_{\alpha\beta} dx_+^\alpha dx_+^\beta = &- \left(\frac{r^2}{l^2} - 8G_3 m\right) dt_+^2 \\ &+ \frac{dr^2}{\left(\frac{r^2}{l^2} - 8G_3 m + \frac{16J^2G_3^2}{r^2}\right)} \\ &- 8G_3 J dt_+ d\phi + r^2 d\phi^2, \quad r \geq R\,, \end{aligned} \tag{4.2}$$

written in exterior coordinate system $x_+^\alpha = (t_+, r, \phi)$, where J is the shell's angular momentum and m is the Arnowitt-Deser-Misner (ADM) mass. On the other hand, the interior region is flat AdS spacetime with metric given by

$$ds_-^2 = g^-_{\alpha\beta} dx^\alpha dx^\beta = -\frac{\rho^2}{l^2}\, dt_-^2 + \frac{l^2}{\rho^2} d\rho^2 + \rho^2 d\psi^2 \quad \rho \leq R\,, \tag{4.3}$$

written in interior coordinate system $x^\alpha_- = (t_-, \rho, \psi)$.

4.3 The thin shell gravitational junction conditions: slowly rotating limit

We start by analyzing the dynamics of this rotating shell by taking the slowly rotating limit. This means we will work consistently to first order in J, the exterior spacetime angular momentum. The exterior metric, Eq. (4.2), becomes

$$\begin{aligned} ds_+^2 = g^+_{\alpha\beta}dx^\alpha dx^\beta = &-\left(\frac{r^2}{l^2} - 8G_3 m\right) dt_+^2 \\ &+ \frac{dr^2}{\left(\frac{r^2}{l^2} - 8G_3 m\right)} \\ &- 8G_3 J dt_+ d\phi + r^2 d\phi^2, \quad r \leq R\,. \end{aligned} \tag{4.4}$$

The induced metric, as viewed from the exterior region, is obtained by setting $r = R$ in Eq. (4.4), which gives

$$ds_\Sigma^2 = -\left(\frac{R^2}{l^2} - 8G_3 m\right) dt_+^2 - 8G_3 J dt_+ d\phi + R^2 d\phi^2\,. \tag{4.5}$$

One can remove the off-diagonal term of the induced metric through a transformation to a co-rotating frame. Accordingly we define a new polar coordinate ψ by

$$\psi = \phi - \Omega t_+\,, \tag{4.6}$$

which makes the induced metric, Eq. (4.5), diagonal if Ω is choosen to be

$$\Omega = \frac{4G_3 J}{R^2}\,. \tag{4.7}$$

Note that this polar coordinate ψ is precisely the interior polar coordinate, as in Eq. (4.3). Thus, in coordinates (t, ψ), where we chose $t \equiv t_+$, the induced metric is written as

$$ds_\Sigma^2 = -\left(\frac{R^2}{l^2} - 8G_3 m\right) dt^2 + R^2 d\psi^2\,. \tag{4.8}$$

The induced metric, as viewed from the interior region, is obtained by setting $\rho = R$ in Eq. (4.3), which gives

$$ds_\Sigma^2 = -\frac{R^2}{l^2}\, dt_-^2 + R^2 d\psi^2\,. \tag{4.9}$$

Applying the first junction condition, Eq. (A.15), leads to

$$\left(\frac{R^2}{l^2} - 8G_3 m\right) dt^2 = \frac{R^2}{l^2}\, dt_-^2\,, \tag{4.10}$$

so that we can write the interior metric as

$$ds_-^2 = -\frac{\rho^2}{l^2}\frac{\left(R^2/l^2 - 8G_3 m\right)}{R^2/l^2}\, dt^2 + \frac{l^2}{\rho^2} d\rho^2 + \rho^2 d\psi^2\,. \tag{4.11}$$

Now we are in conditions to compute the extrinsic curvature as viewed from both sides of the shell, given by Eq. (A.32). As seen from the exterior region

$$K_t^t = \frac{-R}{l^2\sqrt{-8G_3m + \frac{R^2}{l^2}}} , \tag{4.12}$$

$$K_\psi^t = \frac{-4G_3J/R}{\sqrt{-8G_3m + \frac{R^2}{l^2}}} , \tag{4.13}$$

$$K_t^\psi = \frac{4G_3J\sqrt{-8G_3m + \frac{R^2}{l^2}}}{R^3} , \tag{4.14}$$

$$K_\psi^\psi = \frac{\sqrt{-8G_3m + \frac{R^2}{l^2}}}{R} . \tag{4.15}$$

As seen from the shell's interior

$$K_t^t = K_\psi^\psi = \frac{1}{l} , \tag{4.16}$$

are the only non-vanishing components. Therefore the components of the surface stress-energy tensor are

$$S_t^t = \frac{-1}{8\pi G_3 l}\left(1 - \frac{l}{R}\sqrt{-8G_3m + \frac{R^2}{l^2}}\right) , \tag{4.17}$$

$$S_\psi^t = \frac{-J\sqrt{-8G_3m + \frac{R^2}{l^2}}}{2\pi R^3} , \tag{4.18}$$

$$S_t^\psi = \frac{J}{2\pi R\sqrt{-8G_3m + \frac{R^2}{l^2}}} , \tag{4.19}$$

$$S_\psi^\psi = \frac{-1}{8\pi G_3 l}\left(-1 + \frac{R}{l}\frac{1}{\sqrt{-8G_3m + \frac{R^2}{l^2}}}\right) . \tag{4.20}$$

This is not the surface stress-energy tensor of a perfect fluid so we will attempt to write it in the perfect fluid form

$$S^{ab} = \lambda u^a u^b + p\left(h^{ab} + u^a u^b\right) , \tag{4.21}$$

where h^{ab} is the induced metric from either outer or inner regions, λ is a surface density, p is a surface pressure and u^a is a velocity field. Recognizing that the shell must move rigidly in the ψ direction with an uniform angular velocity ω implies that the velocity vector is expressed as

$$u^a = \gamma(t^a + \omega\psi^a) . \tag{4.22}$$

Then one obtains

$$\gamma = \sqrt{\frac{-1}{-8G_3m + \frac{R^2}{l^2}}} , \tag{4.23}$$

$$\lambda = -S_t^t = \frac{1}{8\pi G_3 l}\left(1 - \frac{l}{R}\sqrt{-8G_3m + \frac{R^2}{l^2}}\right) , \tag{4.24}$$

$$p = S^{\psi}_{\psi} = \frac{1}{8\pi G_3 l}\left(-1 + \frac{R}{l}\frac{1}{\sqrt{-8G_3 m + \frac{R^2}{l^2}}}\right) . \tag{4.25}$$

At this point is useful to introduce the gravitacional radius r_+ and the Cauchy radius r_- of the shell, with subsequent explicit expression

$$r_{\pm} = 2l\sqrt{G_3 m \pm \sqrt{G_3^2 m^2 - \frac{J^2 G_3^2}{l^2}}} . \tag{4.26}$$

From transformation of Eq. (4.21) one also obtains

$$\omega = \frac{-S^{\psi}_{t}}{S^{\psi}_{\psi} - S^{t}_{t}} = \frac{r_-}{r_+ l} - \frac{r_- r_+}{l R^2} . \tag{4.27}$$

The shell's angular velocity as measured in the nonrotating frame is $\Omega_{shell} = \omega + \Omega$, given by

$$\Omega_{shell} = \frac{r_-}{l r_+} , \tag{4.28}$$

which coincides with the BTZ spacetime angular velocity [52]. Also, in the slowly rotating limit, λ and p, are equal to the ones of the non-rotating BTZ thin matter shell [53]. We define the shell's rest mass as

$$M = 2\pi R \lambda , \tag{4.29}$$

so that, using Eq. (4.24)

$$M = \frac{R}{4G_3 l}\left(1 - \frac{l}{R}\sqrt{-8G_3 m + \frac{R^2}{l^2}}\right) . \tag{4.30}$$

Therefore we can solve Eq. (4.30) for the ADM mass m

$$m = -2G_3 M^2 + \frac{R}{l} M . \tag{4.31}$$

It will be also useful to define the quantity

$$k = \sqrt{\left(1 - \frac{r_+^2}{R^2}\right)\left(1 - \frac{r_-^2}{R^2}\right)} = \sqrt{\left(1 - \frac{r_+^2}{R^2}\right)} , \tag{4.32}$$

up to order J^2. Now we are in position to write M and p in terms of k and R

$$M = \frac{R}{4G_3 l}\left(1 - k\right) , \tag{4.33}$$

$$p = \frac{1}{8\pi G_3 l}\left(\frac{1}{k} - 1\right) . \tag{4.34}$$

4.3.1 Energy conditions

At this point we have the dynamical problem solved, however we need to impose some mechanical constraints and energy conditions on the shell. The first constraint is that the shell must be outside any trapped surface in order for the spacetime defined to make sense. This constraint yields

$$R \geq r_+ , \tag{4.35}$$

and means the shell is outside its own gravitational radius. The next constraint is the weak energy condition, which is automatically satisfied since we impose λ and p non-negative. Futhermore the dominant energy condition, $p \leq \lambda$, leads to the condition

$$\frac{r_+}{R} = 0 . \tag{4.36}$$

4.4 The thin shell gravitational junction conditions: exact solution

Now we analyze exactly the dynamic of this rotating shell. The induced metric, as viewed from the exterior region, is obtained by setting $r = R$ in Eq. (4.2), which gives

$$ds^2_\Sigma = -\left(\frac{R^2}{l^2} - 8G_3 m\right) dt_+^2 - 8G_3 J dt_+ d\phi + R^2 d\phi^2 \,. \tag{4.37}$$

To remove the off-diagonal term in Eq. (4.37), we go to a co-rotating frame, defining the new polar coordinate ψ by

$$\psi = \phi - \Omega t_+ \,, \tag{4.38}$$

which makes the induced metric, Eq. (4.37), diagonal if Ω is choosen to be

$$\Omega = \frac{4G_3 J}{R^2} \,. \tag{4.39}$$

Therefore, in coordinates (t, ψ), where we chose $t \equiv t_+$, the induced metric is written as

$$ds^2_\Sigma = -\left(\frac{R^2}{l^2} - 8G_3 m + \frac{16J^2 G_3^2}{R^2}\right) dt^2 + R^2 d\psi^2 \,. \tag{4.40}$$

On the other hand the induced metric, as viewed from the interior region, is obtained by setting $\rho = R$ in Eq. (4.3), which gives

$$ds^2_\Sigma = -\frac{R^2}{l^2} dt_-^2 + R^2 d\psi^2 \,. \tag{4.41}$$

Applying the first junction condition, Eq. (A.15), yields

$$\left(\frac{R^2}{l^2} - 8G_3 m + \frac{16J^2 G_3^2}{R^2}\right) dt^2 = \frac{R^2}{l^2} dt_-^2 \,, \tag{4.42}$$

so that we can write the interior metric as

$$ds^2_- = -\frac{\rho^2}{l^2} \frac{\left(R^2/l^2 - 8G_3 m + \dfrac{16J^2 G_3^2}{R^2}\right)}{R^2/l^2} dt^2 + \frac{l^2}{\rho^2} d\rho^2 + \rho^2 d\psi^2 \,. \tag{4.43}$$

Using Eq. (A.32) the components of the extrinsic curvature are

$$K^t_t = \frac{-R}{l^2\sqrt{-8G_3 m + \dfrac{R^2}{l^2} + \dfrac{16J^2 G_3^2}{R^2}}} \left(1 - \frac{16J^2 G_3^2}{R^2}\right) \,, \tag{4.44}$$

$$K^t_\psi = \frac{-4G_3 J/R}{\sqrt{-8G_3 m + \dfrac{R^2}{l^2} + \dfrac{16J^2 G_3^2}{R^2}}} \,, \tag{4.45}$$

$$K^\psi_t = \frac{4G_3 J\sqrt{-8G_3 m + \dfrac{R^2}{l^2} + \dfrac{16J^2 G_3^2}{R^2}}}{R^3} \,, \tag{4.46}$$

$$K^\psi_\psi = \frac{\sqrt{-8G_3 m + \dfrac{R^2}{l^2} + \dfrac{16J^2 G_3^2}{R^2}}}{R} \,. \tag{4.47}$$

As seen from the shell's interior the only non-vanishing components are

$$K^t_t = K^\psi_\psi = \frac{1}{l} \,. \tag{4.48}$$

Using the second junction condition, Eq. (A.35), the components of the surface stress-energy tensor are

$$S^t_t = \frac{-1}{8\pi G_3 l}\left(1 - \frac{l}{R}\sqrt{-8G_3 m + \frac{R^2}{l^2} + \frac{16J^2G_3^2}{R^2}}\right) , \tag{4.49}$$

$$S^t_\psi = \frac{-J\sqrt{-8G_3 m + \frac{R^2}{l^2} + \frac{16J^2G_3^2}{R^2}}}{2\pi R^3} , \tag{4.50}$$

$$S^\psi_t = \frac{J}{2\pi R\sqrt{-8G_3 m + \frac{R^2}{l^2} + \frac{16J^2G_3^2}{R^2}}} , \tag{4.51}$$

$$S^\psi_\psi = \frac{1}{8\pi G_3 l}\left(-1 + \frac{R}{l}\frac{1}{\sqrt{-8G_3 m + \frac{R^2}{l^2} + \frac{16J^2G_3^2}{R^2}}}\right) . \tag{4.52}$$

We will attempt to write the surface stress-energy tensor in a perfect fluid form as done in Eqs. (4.21)-(4.22). Also the shell must move rigidly in the ψ direction with an uniform angular velocity ω implying that the velocity vector is expressed as

$$u^a = \gamma(t^a + \omega\psi^a) . \tag{4.53}$$

Therefore, one obtains

$$\lambda = \frac{M}{2\pi R} = \frac{1}{8\pi G_3 l}(1-k) + \frac{r_+^2 r_-^2}{R^4}\frac{\left(1 - R^2/r_+^2\right)}{8\pi G_3 lk} , \tag{4.54}$$

$$p = \frac{1}{8\pi G_3 l}\left[\frac{1}{k}\left(1 - \frac{r_+^2 r_-^2}{R^4}\right) - 1\right] + \frac{r_-^2}{R^2}\frac{\left(R^2 - r_+^2\right)}{8\pi G_3 lk\left(r_+^2 - r_-^2\right)}\left(-\frac{2r_-^2 r_+^2}{R^4} + \frac{r_+^2 + r_-^2}{R^2}\right) , \tag{4.55}$$

$$\omega = \frac{r_-}{r_+ l} - \frac{r_- r_+}{lR^2} , \tag{4.56}$$

where Eq. (4.32) was used. The shell's angular velocity measured in the nonrotating frame is $\Omega_{shell} = \omega + \Omega$, given by

$$\Omega_{shell} = \frac{r_-}{lr_+} , \tag{4.57}$$

which coincides with the BTZ spacetime angular velocity [52].

4.4.1 Energy conditions

Finally we need to impose some mechanical constraints and energy conditions on the shell. The constraint that the shell must be outside any trapped surface yields

$$R \geq r_+ . \tag{4.58}$$

The weak energy condition is automatically satisfied since we impose λ and p non-negative.

Futhermore the dominant energy condition, $p \leq \lambda$ is satisfied if

$$k \in [k_-, k_+] , \tag{4.59}$$

with $k_\pm = 1 \pm \sqrt{1-a}$ and

$$a = \left(1 - \frac{r_+^2 r_-^2}{R^4}\right) + \frac{r_-^2}{R^2}\left[\left(1 - r_+^2/R^2\right) + \frac{R^2 - r_+^2}{r_+^2 - r_-^2}\left(-\frac{2r_-^2 r_+^2}{R^4} + \frac{r_+^2 + r_-^2}{R^2}\right)\right] \tag{4.60}$$

4.5 Conclusions

In this chapter we have obtained the dynamics of a thin matter shell in a (2+1)-dimensional rotating AdS spacetime. By using the junction conditions we have arrived to the rest mass, pressure and angular velocity of the shell. First we have done it for the slowly rotating limit and after that we have obtained the exact solution. We also have analyzed the weak and dominant energy conditions of the system. Note that, because of rotation, the shell's surface energy-stress tensor was not in the form of a perfect fluid. We overcame this difficulty by attempting to write the tensor in a perfect fluid form like in [49–51]. The results of this chapter will be useful for the study of thermodynamics on this system, which will be the theme of work in Chapter 6.

5

Entropy of thin shells in a (3+1)-dimensional asymptotically AdS spacetime and the Schwarzschild-AdS black hole limit

5.1 Introduction

In this chapter we are interested in the (3+1)-dimensional AdS spacetime. In particular we are going to focus our attention in the thermodynamics of a particular gravitating system: a thin matter shell, of which dynamics have been studied in chapter 3.

Once we obtained the entropy from the thermodynamics approach, we can take the black hole [40] limit, i.e., the shell can be taken to its gravitational radius, obtaining the black hole entropy. Thus, the black hole thermodynamic properties can be attained using a much more simplified computation than the usual black hole mechanics. This idea was developed by Martinez [114] and is going to be applied throughout this chapter, which will generalize the result from [114], by adding a negative cosmological constant.

The Schwarzschild AdS black hole is a semi classical system, spherically symmetric, with a Bekenstein-Hawking entropy $S_{\rm BH} = \frac{1}{4}\frac{A_{\rm h}}{l_{\rm p}}$, where $A_{\rm h}$ is the horizon area, $A_{\rm h} = 4\pi r_+^2$, r_+ is the horizon radius and $l_{\rm p}$ is the Planck length given by $l_{\rm p} = \sqrt{G\,\hbar}$, with G being the gravitational constant and $\hbar$ Planck's constant. We will use units $c = k_B = 1$, where c is the vacuum speed of light and k_B is the Boltzmann constant. That black hole has a Hawking temperature given by $T_{\rm H} = \hbar\frac{1+3r_+^2/l^2}{4\pi r_+}$ [116].

We will adopt the following outline. Sec. 5.2 is dedicated to the thermodynamics generics of the shell, to present the entropy representation and the state variables shell's proper mass, M, and area, A. In Sec. 5.3 we show the pressure equation of state in terms of the proper mass and radius of the shell, and we obtain the temperature equation of state. In Sec. 5.4 we obtain shell's entropy up to an arbitrary function of the gravitational radius. We also push the shell to its gravitacional radius and equate that function to the inverse Hawking temperature. Then we found that the entropy is precisely Bekenstein-Hawking entropy of the Schwarzschild AdS black hole. In Sec. 5.5 we take a phenomenological expression for that arbitrary function which allows us to compute the shell's entropy. After that we analyze the thermodynamic stability and we conclude in Sec. 5.6.

5.2 Thermodynamics and stability conditions for the thin shell

Proceeding to the computation of the entropy of the shell, we assume the entropy S to be a function of the shell's rest mass M and area $A = 4\pi R^2$, i.e.,

$$S = S(M, A)\,. \tag{5.1}$$

Note that the number of particles N is considered constant. We also consider the assumption that the shell is a hot shell, i.e., it has a well defined local temperature T. Thus the first law of thermodynamics can be written as

$$TdS = dM + pdA\,. \tag{5.2}$$

It is far-reaching to note that the temperature plays the role of an integration factor, which implies the following integrability condition

$$\left(\frac{\partial\beta}{\partial A}\right)_M = \left(\frac{\partial\beta p}{\partial M}\right)_A\,, \tag{5.3}$$

for the sake of guarantee that the differential dS is exact and where we define the inverse temperature $\beta = 1/T$. To find the entropy S one needs two equations of state, which are

$$p = p(M, A)\,, \tag{5.4}$$

$$\beta = \beta(M, A)\,. \tag{5.5}$$

There is then the achievability of studying the local intrinsic stability of the shell, which is assured if the following inequalities hold simultaneously (see [115] [119])

$$\left(\frac{\partial^2 S}{\partial A^2}\right)_M \leq 0\,, \tag{5.6}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_A \leq 0\,, \tag{5.7}$$

$$\left(\frac{\partial^2 S}{\partial M \partial A}\right)^2 - \left(\frac{\partial^2 S}{\partial A^2}\right)\left(\frac{\partial^2 S}{\partial M^2}\right) \leq 0\,. \tag{5.8}$$

5.3 The two equations of state: equations for the pressure and temperature

5.3.1 The two independent thermodynamic variables

Although the first law of thermodynamics, Eq. (5.2), is simpler when expressed using the area A, we are going to use it in terms of the shell's radius R, related to A by

$$R = \sqrt{\frac{A}{4\pi}}\,. \tag{5.9}$$

At this point our thermodynamics variables are (R, M).

5.3.2 The pressure equation of state

The pressure equation of state, $p(M, R)$, is given through Eq. (3.17)

$$p(M,R) = \frac{\sqrt{1 + R^2/l^2} - k(M,R)}{16\pi G R k(M,R)\sqrt{1+R^2/l^2}}\left(1 + \frac{3R^2}{l^2} - k(M,R)\sqrt{1 + \frac{R^2}{l^2}}\right)\,, \tag{5.10}$$

which is equivalent to

$$p(M,R) = \frac{GM^2}{16\pi R^2\left(R\sqrt{1+R^2/l^2} - GM\right)} + \frac{R^2}{l^2}\frac{M/\sqrt{1+R^2/l^2}}{8\pi R^2\left(\sqrt{1+R^2/l^2} - GM/R\right)}\,. \tag{5.11}$$

5.3.3 The temperature equation of state

Futhermore, to determine the temperature equation of state Eq. (5.5), we apply the integrability condition Eq. (5.3), which, using the variables (r_+, R), turns into

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+} = \beta\frac{1 + R^2/l^2 - k^2}{2Rk^2} + \beta\frac{R}{k^2 l^2}\,, \tag{5.12}$$

which the analytic solution is found to be

$$\beta(r_+, R) = b(r_+)k(r_+, R)\,. \tag{5.13}$$

Therefore we obtain the Tolman's formula for the temperature of a body in curved spacetime [120]. $b(r_+)$ is an arbitrary function of the gravitacional radius r_+ and is also the inverse temperature of the shell if its radius is infinite $b(r_+) = \beta(r_+, \infty)$. This function remains arbitrary unless we specify the configurations for the shell's matter fields. It is important to note that b is a function of (M, R), through the variable $r_+ = r_+(M, R)$.

5.4 Entropy of the thin shell

At this point we have assembled all we demand to compute the entropy of the shell. First we note that

$$p = -\frac{1}{8\pi R}\left(\frac{\partial M}{\partial R}\right)_{r_+}, \tag{5.14}$$

from Eqs. (3.15), (3.16) and (3.17). Thus, changing variables from (M, A) to (r_+, R), Eq. (5.2) becomes

$$dS = \beta(r_+, R)\frac{1}{2k(r_+, R)G}\left(1 + \frac{3r_+^2}{l^2}\right). \tag{5.15}$$

Combining Eqs. (5.15) and (5.13) gives the differential of the entropy, only in terms of r_+

$$dS = \frac{b(r_+)}{2G}\left(1 + \frac{3r_+^2}{l^2}\right) dr_+ . \tag{5.16}$$

This implies that

$$S = S(r_+), \tag{5.17}$$

i.e., the entropy S is a function of r_+ alone. However S is a function of (M, R), with the functional dependence being through $r_+(M, R)$.

In order to obtain a specific expression for the entropy, we need to specify the function $b(r_+)$.

5.4.1 The temperature equation of state and the entropy

Let us specify a particular function for $b(r_+)$ of the shell, given by the following dependence

$$b(r_+) = \gamma\frac{r_+}{1 + 3r_+^2/l^2}, \tag{5.18}$$

with γ being some constant with units of inverse mass times inverse radius. This constant is determined by the properties of the shell itself. Inserting this result, Eq. (5.18), in Eq. (5.16) yields the entropy differential for that particular shell

$$dS = \frac{\gamma}{2G} r_+ dr_+ . \tag{5.19}$$

Then the entropy of the shell is

$$S = \frac{\gamma}{4G} r_+^2 + S_0, \tag{5.20}$$

where S_0 is an integration constant. When the shell vanishes ($M = r_+ = 0$) it seems physically reasonable to assume that the entropy must be zero, which implies that $S_0 = 0$.

5.4.2 The black hole limit

Straightaway we can take the black hole limit $R \to r_+$ like in [53, 114, 119]. As the shell is drawins near its gravitacional radius, we have to consider quantum fields, whose backreaction will diverge unless we take the temperature of the shell at infinity, $b(r_+)^{-1}$, to be equal to the Hawking temperature

$$T_{\rm H} = \frac{\hbar}{4\pi}\frac{1+3r_+^2/l^2}{r_+}\,, \tag{5.21}$$

so that

$$b(r_+) = \frac{1}{T_{\rm H}} = \frac{4\pi}{\hbar}\frac{r_+}{1+3r_+^2/l^2}\,, \tag{5.22}$$

which means that γ depends only on fundamental constants

$$\gamma = \frac{4\pi}{\hbar}\,. \tag{5.23}$$

Implanting this $b(r_+)$ in Eq. (5.16) and integrating gives the entropy of the shell in the black hole limit

$$S(r_+) = \frac{\pi r_+^2}{l_p^2} = \frac{A_+}{4l_{\rm p}^2}\,, \tag{5.24}$$

where $l_{\rm p} = \sqrt{G\hbar}$ is the Planck length. One can observe that Eq. (5.24) is precisely the Bekenstein-Hawking entropy for a black hole, forasmuch as $A_+ = A_{\rm h}$ is the black hole horizon area.

5.5 Another specific equation of state for the temperature of the thin matter shell: entropy and stability

5.5.1 The temperature equation of state and the entropy

For the sake of gathering an expression for the entropy, one have to specify a suitable equation of state for $b(r_+)$. The simplest ansatz is the following power law

$$b(r_+) = 2G\eta r_+^a\,. \tag{5.25}$$

Inserting Eq. (5.25) into Eq. (5.16) and integrating yields

$$S = \eta r_+^{a+1}\left(\frac{1}{a+1} + \frac{3r_+^2}{l^2}\frac{1}{a+3}\right) + S_0\,, \tag{5.26}$$

for $a > -1$ and where S_0 is a constant. It is reasonable to assume that a zero mass shell must possess zero entropy, so that $a > -1$ and $S_0 = 0$.

5.5.2 The stability conditions for the specific temperature ansatz

After that we analyze the shell's stability through Eqs. (5.6)-(5.8). The first inequality, Eq. (5.6), gives

$$a \le 0 \vee \left(a > 0 \wedge \frac{2ak^2}{1+3r_+^2/l^2} \le \frac{r_+}{R}\right)\,. \tag{5.27}$$

The next condition, Eq. (5.7), yields

$$\left(\sqrt{1+R^2/l^2} - k\right)\left(-\frac{3r_+}{R} + a\left(\sqrt{1+R^2/l^2}-k\right)\frac{l^2}{3r_+^2}\right. \tag{5.28}$$
$$\left. + a\frac{2R^2}{3r_+^2\sqrt{1+R^2/l^2}}\right) - \frac{2R^2}{l^2}\frac{Rr_+}{l^2}\frac{1}{(1+R^2/l^2)^{3/2}} \le 0\,.$$

Finally, from Eq. (5.8) one obtains the following inequation

$$
\begin{aligned}
&\left(\frac{2ak^2}{1+3r_+^2/l^2}-\frac{r_+}{R}\right)\frac{GM}{2R}\left[-\frac{2R^2}{l^2}\frac{Rr_+}{l^2}\frac{1}{(1+R^2/l^2)^{3/2}}\right.\\
&\left.-\frac{3r_+}{R}\frac{GM}{R}+a\left(\frac{GM}{R}+\frac{2R^2/l^2}{\sqrt{1+R^2/l^2}}\right)\frac{GM}{R}\frac{1}{1+3r_+^2/l^2}\right]\\
&-\left[\left(\frac{GM}{R}+\frac{R^2/l^2}{\sqrt{1+R^2/l^2}}\right)\left(\frac{r_+}{R}+\frac{ak}{1+3r_+^2/l^2}\frac{GM}{R}\right)\right.\\
&\left.+\frac{ak}{1+3r_+^2/l^2}\frac{GM}{R}\frac{R^2/l^2}{\sqrt{1+R^2/l^2}}\right]^2\geq 0\,,
\end{aligned}
\tag{5.29}
$$

where $GM/R=\sqrt{1+R^2/l^2}-k$, by Eq. (3.16).

5.6 Conclusions

In this chapter we have generalized the Martinez [114] work on the thermodynamics of a self-gravitating (3+1)-dimensional thin matter shell by adding a negative cosmological constant. Inside the shell the spacetime is described by the AdS metric, outside is given by the Schwarzschild AdS metric, and it is asymptotically AdS. We have obtained the rest mass and pressure using the junction conditions in Chapter 3, which led to a differential for the entropy, by using the first law of thermodynamics. We have found the inverse temperature $b(r_+)$ and the entropy to be functions of r_+ alone. However this r_+ is itself a function of the thermodynamic variables M and R.

We have choosed an ansatz for the thermal equation of state, which led to Eq. (5.26) for the entropy of the shell. The complexity in the thermodynamic stability analysis, Eqs. (5.27)-(5.29) forbid the possibility of obtaining a range for the parameter a and for R, as in [114].

In the case where the shell was pushed to its gravitacional radius and where the inverse temperature was equal to the inverse of the Hawking temperature we found that the entropy was equal to the Bekenstein-Hawking entropy of a Schwarzschild AdS black hole. One should note that the Hawking temperature is a quantum result that was inserted by hand on the entropy, which was obtained only by thermodynamics of spacetime, without specifying the material degrees of freedom.

6

Entropy of slowly rotating thin shells in a (2+1)-dimensional asymptotically AdS spacetime and the BTZ black hole limit

6.1 Introduction

In this chapter we will focus in the thermodynamics of a slowly rotating thin matter shell in a (2+1)-dimensional asymptotically AdS spacetime, of which dynamics have been studied in chapter 4.

We intend to find the shell's entropy, study its thermodynamic stability, and take the black hole[53, 114, 119] limit, i.e., when the shell is taken to its gravitational radius and obtain the corresponding entropy. We will adopt the method developed by Martinez [114] and obtain the BTZ black hole, [47], thermodynamic properties with a much more straightforward computation than the usual black hole mechanics.

The BTZ black hole is a semi classical system with a Bekenstein-Hawking entropy $S_{\rm BH} = \frac{1}{4}\frac{A_{\rm h}}{l_{\rm p}^2}$, where $A_{\rm h}$ is the horizon area, $A_{\rm h} = 4\pi r_+^2$, r_+ is the horizon radius and $l_{\rm p}$ is the Planck length given by $l_{\rm p} = \sqrt{G\,\hbar}$, with G being the gravitational constant and $\hbar$ Planck's constant. In this chapter we use units $c = k_B = 1$, where c is the vacuum speed of light and k_B is the Boltzmann constant. The black hole has a Hawking temperature given by $T_{\rm H} = \dfrac{\hbar}{2\pi l^2}\dfrac{r_+^2 - r_-^2}{r_+}$ and an angular velocity $\omega_{\rm BH} = \dfrac{r_-}{l r_+}$ [52].

This chapter is organized as follows. Sec. 6.2 presents the thermodynamics generics of the shell, the entropy representation and the state variables which are shell's proper mass, M, area, A and shell's angular momentum J. In Sec. 6.3 we give the pressure equation of state in terms of the thermodynamic variables. We also obtain the temperature equation of state and the angular velocity for the first law of thermodynamics. In Sec. 6.4 we arrive to shell's entropy with an arbitrary function of the gravitational radius to be specify. Then we take the black hole limit and we arrive to the Bekenstein-Hawking entropy of the BTZ black hole. In Sec. 6.5 we prescribe a phenomenological expression for the arbitrary function which leads us to a complete expression of the shell's entropy. With that arbitrary function specified we analyze the thermodynamic stability. Finally we conclude in Sec. 6.6.

6.2 Thermodynamics and stability conditions for the thin shell

At this moment we have to consider that the shell as a temperature T as measured locally, i.e., is a hot shell, and has an entropy S.

In the formalism stated in [115], the entropy S of a system is given in terms of the state independent variables, which we must designate, in order to obtain the complete thermodynamics description of the system. The natural choice of variables is (M, A, J), i.e, the proper local mass M, the ring perimeter A and the angular momentum J, so that

$$S = S(M, A, J)\,. \tag{6.1}$$

With this variables, we can write the first law of thermodynamics as

$$TdS = dM + p\,dA - \omega dJ\,, \tag{6.2}$$

where T, p and ω are the temperature, the pressure and the angular velocity, respectively. For the sake of finding S, one has to perceive the equations of state for the quantities that appear in Eq. (6.2) and that are not thermodynamic variables, i.e.,

$$p = p(M, A, J)\,, \tag{6.3}$$

$$\omega = \omega(M, A, J)\,, \tag{6.4}$$

and

$$\beta = \beta(M, A, J)\,, \tag{6.5}$$

where $\beta = 1/T$ is the inverse temperature.

In order to Eq. (6.2) for dS to be an exact differential, there are three integrability condition to be satisfied, as follow (see [115, 119])

$$\left(\frac{\partial \beta}{\partial A}\right)_{M,J} = \left(\frac{\partial \beta p}{\partial M}\right)_{A,J} . \tag{6.6}$$

$$\left(\frac{\partial \beta}{\partial J}\right)_{M,A} = -\left(\frac{\partial \beta \omega}{\partial M}\right)_{A,J} . \tag{6.7}$$

$$\left(\frac{\partial \beta p}{\partial J}\right)_{M,A} = -\left(\frac{\partial \beta \omega}{\partial A}\right)_{M,J} . \tag{6.8}$$

There is also the possibility of studying the local intrinsic thermodynamical stability of the shell, which is guaranteed as long as the entropy of the system stays in a maximum. Following [115], if the inequalities

$$\left(\frac{\partial^2 S}{\partial M^2}\right)_{A,J} \leq 0\,, \tag{6.9}$$

$$\left(\frac{\partial^2 S}{\partial A^2}\right)_{M,J} \leq 0\,, \tag{6.10}$$

$$\left(\frac{\partial^2 S}{\partial J^2}\right)_{M,A} \leq 0\,, \tag{6.11}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial A^2}\right) - \left(\frac{\partial^2 S}{\partial M \partial A}\right)^2 \geq 0\,, \tag{6.12}$$

$$\left(\frac{\partial^2 S}{\partial A^2}\right)\left(\frac{\partial^2 S}{\partial J^2}\right) - \left(\frac{\partial^2 S}{\partial A \partial J}\right)^2 \geq 0\,, \tag{6.13}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial J^2}\right) - \left(\frac{\partial^2 S}{\partial M \partial J}\right)^2 \geq 0\,, \tag{6.14}$$

$$\left(\frac{\partial^2 S}{\partial M^2}\right)\left(\frac{\partial^2 S}{\partial J \partial A}\right) - \left(\frac{\partial^2 S}{\partial M \partial A}\right)\left(\frac{\partial^2 S}{\partial M \partial J}\right) \geq 0\,, \tag{6.15}$$

are satisfied, then stability is guarantee.

6.3 The three equations of state: equation for the pressure, equation for the temperature and equation for angular velocity

6.3.1 The three independent thermodynamic variables

We define shell's rest mass M as

$$M = 2\pi R\lambda \,, \tag{6.16}$$

where λ is given by Eq. (4.24).

Second, we note that

$$A = 2\pi R \,, \tag{6.17}$$

so that the variables perimeter A and the radius R can be directly swapped as independent variables. At this point we choose our thermodynamics variables as (M, R, J).

6.3.2 The pressure equation of state

By combining Eqs. (4.25) and (4.24) we arrive to the seeked equation of state (6.3)

$$p(M, R, J) = p(M, R) = \frac{1}{8\pi G_3 l}\left(\frac{1}{1 - \dfrac{4G_3 M l}{R}} - 1\right) \,, \tag{6.18}$$

where A has been swapped with R, by Eq. (6.16). This equation follows exclusively from the spacetime structure and does not depend on the matter composing the shell. This can be observed since the equation is derived only by using gravitational considerations and the junction conditions on the ring, without ever specifying the matter fields. It is the same as obtained in [53].

6.3.3 The temperature equation of state

Now we turn our attention to the other equation of state, Eq. (6.5), for $\beta(M, R, J)$. This equation is obtained from the first integrability condition, Eq. (6.6), by changing from variables (M, A, J) to (R, r_+, r_-), yielding

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+, r_-} = \frac{\beta}{R k^2} \,, \tag{6.19}$$

with $k = k(R, r_+, r_-) = k(R, r_+)$, as in Eq. (4.32). The analytic solution of this equation is

$$\beta(R, r_+) = \frac{R}{l}\, k(R, r_+)\, b(r_+) \,, \tag{6.20}$$

where $b(r_+)$ is an arbitrary function of the gravitational radius r_+, up to order J^2.

The function $b(r_+)$ is the Tolman relation for the temperature in a (2+1)-dimension gravitational system [120]. It has units of inverse temperature and is the inverse temperature of the shell if located at radius R such that $R = \sqrt{r_+^2 + l^2}$.

Although b depends only on r_+, it is forced to depend on the state variables (M, R, J) through the functions $r_+(m(M, R, J) \simeq r_+(m(M, R))$. Also the integrability condition does not specify b, which is expected, as discussed in [114] (see also [118]). To yield a precise form for b one must specify the matter fields of the shell.

6.3.4 The angular velocity equation of state

For the sake of finding the equation of state for the angular velocity, Eqs. (6.6)-(6.8) can be combined to obtain

$$\frac{1}{2\pi}\left(\frac{\partial\omega}{\partial R}\right)_{r_+,r_-}+\omega\left(\frac{\partial p}{\partial M}\right)_{R,J}+\left(\frac{\partial p}{\partial J}\right)_{M,R}=0\,, \tag{6.21}$$

where we mixed terms in variables (R, M, J) and (R, r_+, r_-) for computational simplicity.

The solution of this differential equation is given by

$$\omega(R, r_+, r_-) = \frac{\omega_0(r_+, r_-)l}{Rk} - \frac{r_+r_-}{R^3k}\,, \tag{6.22}$$

where $\omega_0(r_+, r_-)$ is an arbitrary function. Recalling the dynamics we obtained the angular velocity of Eq. (4.27), which is precisely the angular velocity from Eq. (6.22) blueshifted, i.e., divided by $R/l\,k$ and with $\omega_0(r_+, r_-)$ fixed by

$$\omega_0(r_+, r_-) = \frac{r_-}{lr_+}\,. \tag{6.23}$$

This is analog to the Tolman relation for the temperature [120]. In this case we pull the angular velocity from infinite, which is the result from dynamics, Eq. (4.27), to the shell's radius by blueshift.

6.4 Entropy of the thin shell

Now we have perceived all we need to find the entropy S. Inserting Eq. (6.16) and the three equations of state, Eqs. (6.18), (6.20) and (6.22) into Eq. (6.2) gives the differential of the entropy

$$dS = \frac{b(r_+)r_+}{4Gl^2}dr_+\,. \tag{6.24}$$

Integration of Eq. (6.24) yields the entropy, $S(r_+)$ and introduces an integration constant, S_0. The entropy is a function of r_+ alone, so it is a special function of (M, R) given by $S = S(r_+(M, R))$, for slowly rotation. To get a specific expression for the entropy we need to prescribe the function $b(r_+)$.

6.4.1 The temperature equation of state and the entropy

Let us specify the following dependence for $b(r_+)$

$$b(r_+) = \gamma l^2\frac{r_+}{r_+^2 - r_-^2} \simeq \gamma\frac{l^2}{r_+}\,, \tag{6.25}$$

with γ being some constant with units of inverse mass times inverse radius, which is determined by the properties of the shell . Inserting this result, Eq. (6.25), in Eq. (6.24) leads us to the entropy differential for that particular shell

$$dS = \frac{\gamma}{4G}dr_+\,. \tag{6.26}$$

Therefore the entropy of the shell is

$$S = \frac{\gamma}{4G}r_+ + S_0\,, \tag{6.27}$$

where S_0 is an integration constant. It seems physically reasonable to impose that the entropy must be zero when the shell vanishes ($M = 0$ and $J = 0$ so that $r_+ = 0$), which implies that $S_0 = 0$.

6.4.2 The black hole limit

Straight off we can take the black hole limit $R \to r_+$ like [114], [53] and [119]. As the shell gets near its gravitacional radius, we have to take in account quantum fields, whose backreaction will diverge if the temperature of the shell at infinity, $b(r_+)^{-1}$, is not equal to the Hawking temperature

$$T_{\rm H} = \frac{\hbar}{2\pi l^2}\frac{r_+^2 - r_-^2}{r_+} \simeq \frac{\hbar}{2\pi l^2} r_+ \,, \tag{6.28}$$

so that

$$b(r_+) = \frac{1}{T_{\rm H}} = \frac{2\pi l^2}{\hbar}\frac{1}{r_+} \,, \tag{6.29}$$

which implies that γ only depends on fundamental constants

$$\gamma = \frac{2\pi}{\hbar} \,. \tag{6.30}$$

Fixing this $b(r_+)$ gives the entropy of the shell in the black hole limit

$$S(r_+) = \frac{\pi r_+}{2 l_{\rm p}^2} = \frac{A_+}{4 l_p^2} \,, \tag{6.31}$$

where $l_{\rm p} = \sqrt{G\hbar}$ is the Planck length. One can notice that Eq. (6.31) is precisely the Bekenstein-Hawking entropy for a black hole, forasmuch as $A_+ = A_{\rm h} = 2\pi r_+$ is the black hole horizon perimeter.

6.5 Another specific equation of state for the temperature of the thin matter shell: entropy and stability

6.5.1 The temperature equation of state and the entropy

In order to gather an expression for the entropy, we have to specify an equation of state for $b(r_+)$. The simplest one is the following power law

$$b(r_+) = 4G\alpha l^2 \frac{r_+^a}{l_{\rm p}^{a+2}} \,, \tag{6.32}$$

where α and a are constants. Combining Eq. (6.32) with Eq. (6.24) and integrating gives

$$S = \frac{\alpha}{a+2}\left(\frac{r_+}{l_{\rm p}^{a+2}}\right)^{a+2} + S_0 \,, \tag{6.33}$$

for $a \neq -2$ and where S_0 is an integration constant. However, when the shell's mass vanishes we assume that the entropy is zero, which fixes $a > -2$ and $S_0 = 0$.

6.5.2 The stability conditions for the specific temperature ansatz

Now we analyze the shell's stability through Eqs. (6.9)-(6.15). In this slowly rotating approximation only Eqs. (6.9)-(6.11) are relevant. The first inequality, Eq. (6.9), implies

$$\frac{a}{a+1} R^2 \leq r_+^2 \,, \tag{6.34}$$

which combined with the no-trapped surface condition, Eq. (4.35), gives

$$\frac{a}{a+1} R^2 \leq r_+^2 \leq R^2 \,. \tag{6.35}$$

This restricts the interval of a to

$$-1 \leq a < \infty \,. \tag{6.36}$$

Now the second condition, Eq. (6.10), yields

$$r_+^2 \geq 2R^2 \left(1 - \sqrt{1 - \frac{r_+^2}{R^2}}\right) \,, \tag{6.37}$$

which means that R obeys

$$\frac{R}{r_+} \to \infty \,, \tag{6.38}$$

since $r_+ > 0$.

The next condition,Eq. (6.12), turns on the following inequality

$$a + 1 \leq 0 \,. \tag{6.39}$$

The other thermodynamic stability conditions, Eqs. (6.12)-(6.15), are superfluous. Combining Eqs. (6.36) and (6.39) gives

$$a = -1 \,. \tag{6.40}$$

Finally this thermodynamic system, with temperature given by Eq. (6.32) and entropy given by Eq. (6.33), is stable if obeys

$$a = -1 \quad \text{and} \quad \frac{R}{r_+} \to \infty \tag{6.41}$$

with $r_+ > 0$. Therefore the thermodynamic stability condition for the shell, Eq. (6.41), are more restrictive than the no-trapped-surface condition Eq. (4.35) and is the same as the dominant energy condition Eq. (4.36).

6.6 Conclusions

We have generalized the Lemos and Quinta [53] work on the thermodynamics of a self-gravitating (2+1)-dimensional thin matter shell by considering rotation, yet slowly rotation. Inside the shell the spacetime is described by the BTZ metric without matter and outside is given by the BTZ metric with slowly rotation, thus being asymptotically AdS. Using the rest mass, the pressure and the angular velocity obtained using the junction conditions in Chapter 4, we arrived to a differential for the entropy, by using the first law of thermodynamics. Both shell's entropy and inverse temperature $b(r_+)$ are approximately functions of r_+ alone. However this r_+ is itself a function of the thermodynamic variables M and R. Futhermore, we have found that the angular velocity for the first law of thermodynamics is the angular velocity from dynamics blueshifted, because the first is defined on the shell and the second is defined at infinite radius. Therefore the arbitrary function that appears in the angular velocity obtained from thermodynamics is fixed by the result from dynamics.

With the choosen ansatz for the thermal equation of state, we were lead to Eq. (5.26) for the entropy of the shell and we analyzed the shell's stability. The inverse temperature b has been chosen to be proportional to a power law of the form r_+^a, for some number a. The shell is stable if $a = -1$ and it must be placed at infinity $R \to \infty$.

We also took the shell to its own gravitacional radius, which by quantum arguments fixed the inverse temperature equal to the inverse of the Hawking temperature and we found that the entropy was equal to the Bekenstein-Hawking entropy of a BTZ black hole.

7

Conclusions

In this thesis we tracked down some mechanical properties of compact stars and thin matter shells, as well as some thermodynamic properties of thin matter shells which lead us to the black hole entropy. On the subject of compact stars we have obtained the interior Schwarzschild limit of spherically symmetric star configurations composed of a fluid with constant energy density and with a small electrical charge distribution proportional to the energy density. This analysis is opportune since stars containg some type of dark matter may possess a small but non-negligble electric charge (see chapter 2). We have found that the limiting star configuration can have more mass on a smaller radius due to the eletric charge distribution, which is an expected result from its repulsive effect. The numerical code used in [61] is compatible with our analytical calculation. Futhermore our configurations of constant density and a charge distribution proportional to the energy density do not saturate the Buchdahl-Andréasson bound [59]. This raises the question of whether or not there are other types of charged matter that can saturate the bound. Thin shells with an appropriate relation between surface energy density and surface pressure do saturate the bound [59]. Are there continuous matter (non-thin-shell) distributions that saturate the bound? This question is an open problem to solve.

As for thin matter shells the procedure for the two AdS spacetimes studied was the following. We imposed junction conditions on the shell which gives the dynamical quantities for the shell to remain static on the radial coordinate. Inserting those quantities in the first law of thermodynamics led us to the entropy for the thin matter shell up to an arbitrary function of the shell's gravitational radius. The matter contained in the shell specifies this function. We also take the shell to its gravitational radius which fixes the temperature to be the Hawking temperature and we recover the Bekenstein-Hawking entropy. This result is not trivial since nothing tells us that a thin shell system placed at its own gravitational radius has the black hole entropy. This seems to show some evidence that the degrees of freedom of a black hole are situated at its event horizon. Finally we studied the thermodynamic stability of the shells.

In chapter 3 we have analyzed the dynamics of a thin matter shell composed of a perfect fluid in a (3+1)-dimensional AdS spacetime. We obtained the rest mass and the pressure of the shell, using junction conditions. We also analyzed the weak and dominant energy conditions of the system.

For the case of a (2+1)-dimensional shell in an asymptotically AdS rotating spacetime we have obtained the shell's rest mass, the pressure and the angular velocity by using the junction conditions (see Chapter 4), and the weak and dominant energy conditions. First we analyzed the problem for the slowly rotating limit and afterward we solved the exact problem. Because of rotation, the shell's surface energy-stress tensor was not in the form of a perfect fluid. We overcame this difficulty by attempting to write the tensor in a perfect fluid form like in [49–51].

Futhermore in Chapter 5 we generalized the Martinez [114] work on the thermodynamics of a self-gravitating (3+1)-dimensional thin matter shell by adding a negative cosmological constant. Using the first law of thermodynamics, with the results from Chapter 3 we arrived to a differential for the entropy. By choosing a suitable ansatz for the thermal equation of state we analyzed the thermodynamic stability of the shell. Taking the black hole limit, i.e., pushing the shell to its gravitacional radius, we found that the entropy was equal to the Bekenstein-Hawking entropy of a Schwarzschild AdS black hole.

The thermodynamics of a self-gravitating (d+1)-dimensional thin matter shell in a asymptotically AdS spacetime for $d > 3$ is an open problem.

In Chapter 6 we have generalized the Lemos and Quinta [53] work on the thermodynamics of a self-gravitating (2+1)-dimensional thin matter shell by considering rotation, yet slowly rotation. We arrived to a differential for the entropy, by using the first law of thermodynamics with the pressure, rest mass and angular velocity from Chapter 4. Since the entropy must be exact we obtained integrability equations which gave us the shell's inverse temperature and an equation for the thermodynamic's angular velocity. This angular velocity is precisely the angular velocity obtained by dynamics blueshifted, since this last one is defined at infinity and we need to pull it to the shell. Moreover, we analyzed thermodynamic stability of the shell. Finally we took the shell to its own gravitacional radius, which by quantum arguments fixed the inverse temperature equal to the inverse of the Hawking temperature and we found that the entropy was equal to the Bekenstein-Hawking entropy of a BTZ black hole. However, we were only able to solve the thermodynamic problem for the slowly rotating limit. The general case is an open problem to solve. The difficulty relies on interpretating the pressure terms that appear in the first law of thermodynamics.

A

Thin shell formalism

A.1 Generics

Consider a d dimensional spacetime, divided in two regions V^+ and V^- by a $d-1$ dimensional hypersurface Σ. The hypersurface is either timelike or spacelike. In the region V^+ the metric is $g^+_{\alpha\beta}$ expressed in coordinates x^α_+, whereas in the region V^- the metric is $g^-_{\alpha\beta}$ expressed in coordinates x^α_- [49].

One can ask for the conditions that the metric must satisfy to ensure both regions are joined smoothly at Σ. First we assume that the hypersurface has coordinates y^a on both sides of the hypersurface. The latin indices run from 0 to $d-1$. We can also have coordinates x^α in an open region that intersects Σ and its both sides. This coordinate system overlaps with $x^\alpha_\pm$ in a open region of $V^\pm$ that contains Σ and we are going to make all calculations in these coordinates x^α. In this coordinates the hypersurface is described by the following parametric equations

$$x^\alpha = x^\alpha(y^a)\,, \tag{A.1}$$

which implies that the vectors e^α_a, tangent to the lines of constant y^a on Σ are given by

$$e^\alpha_a = \frac{\partial x^\alpha}{\partial y^a}\,. \tag{A.2}$$

We imagine Σ to be orthogonally perforated by a congruence of geodesics, which are parametrized by their proper distance l, adjusted such that $l = 0$ at Σ, $l > 0$ in V^+ and $l < 0$ in V^-. Then we can define the normal vector, perpendicular to the $d-1$ tangent vectors e^α_a, as

$$n_\alpha = \epsilon \frac{\partial l}{\partial x^\alpha}\,, \tag{A.3}$$

where $\epsilon = n^\alpha n_\alpha$ and takes values $+1$ or -1 if the hypersurface is timelike or spacelike, respectively.

We will also use the following notation

$$[A] \equiv A(V^+)\big|_\Sigma - A(V^-)\big|_\Sigma\,, \tag{A.4}$$

for the jump of quantity A across Σ.

Note that

$$[n^\alpha] = [e^\alpha_a] = 0\,, \tag{A.5}$$

since x^α, y^a and l are continuous across the hypersurface.

It is also essential to introduce the concept of induced metric, h_{ab}, which arises when one wants to know the metric on an hypersurface alone instead of on the whole spacetime. It is defined such that

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta = h_{ab} dx^a dx^b\,, \tag{A.6}$$

and is explicitly given by

$$h_{ab} = g_{\alpha\beta} e^\alpha_a e^\beta_b\,. \tag{A.7}$$

A.2 First junction condition

We will write the metric $g_{\alpha\beta}$ using the language of distributions

$$g_{\alpha\beta} = \Theta(l)g^{+}_{\alpha\beta} + \Theta(-l)g^{-}_{\alpha\beta}\,, \tag{A.8}$$

where all metric tensors are expressed in the coordinates x^{α}. $\Theta(l)$ is the Heaviside distribution equal to $+1$ if $l > 0$, indeterminate if $l = 0$ and -1 if $l < 0$. We will use the following properties

$$\Theta^2(l) = \Theta(l)\,, \tag{A.9}$$

$$\Theta(l)\Theta(-l) = 0\,, \tag{A.10}$$

$$\frac{\mathrm{d}}{\mathrm{d}l}\Theta(l) = \delta(l)\,, \tag{A.11}$$

where $\delta(l)$ is the Dirac distribution. Our initial question for the conditions that the metric must satisfy in order to both regions join smoothly is then the same asking if the metric of Eq. (A.8) is a valid distributional solution of the Einstein equations

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi G_d T_{\alpha\beta}\,, \tag{A.12}$$

where $R_{\alpha\beta}$ is the Ricci tensor, $R = g_{\alpha\beta}R^{\alpha\beta}$ is the Ricci scalar curvature, G_d is Newton's gravitational constant in d dimensions, $T_{\alpha\beta}$ is the energy-stress tensor and natural units $c = 1$ were used. The left hand side of Eq. (A.12) is defined as the Einstein's tensor $G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R$. The Ricci tensor is a contraction of the Riemann tensor $R_{\alpha\beta} = R^{\mu}_{\ \alpha\mu\beta}$, while the explicit form of the Riemann tensor is

$$R^{\alpha}_{\ \beta\gamma\delta} = \Gamma^{\alpha}_{\delta\beta,\gamma} - \Gamma^{\alpha}_{\gamma\beta,\delta} + \Gamma^{\alpha}_{\gamma\lambda}\Gamma^{\lambda}_{\delta\beta} - \Gamma^{\alpha}_{\delta\lambda}\Gamma^{\lambda}_{\gamma\beta}\,, \tag{A.13}$$

where we used the notation $f_{,\alpha} = \dfrac{\partial f}{\partial x^{\alpha}}$ and the Γ are the Christoffel symbols defined as

$$\Gamma^{\alpha}_{\beta\delta} = \frac{1}{2}g^{\alpha\lambda}\left(g_{\lambda\beta,\delta} + g_{\lambda\delta,\beta} - g_{\beta\delta,\lambda}\right)\,. \tag{A.14}$$

Thus to answer the initial question we have to verify that geometrical quantities obtained from the metric which Eq. (A.12) is made of, like the Ricci tensor, are properly defnied as distribuitions. The first quantity to analyze is the derivative of the metric which is

$$g_{\alpha\beta,\gamma} = \Theta(l)g^{+}_{\alpha\beta,\gamma} + \Theta(-l)g^{-}_{\alpha\beta,\gamma} + \epsilon\delta(l)[g_{\alpha\beta}]n_{\gamma}\,, \tag{A.15}$$

where Eq. (A.3) was used. Whereas the first two terms are well behaved, the last one will create terms of the form $\Theta(l)\delta(l)$ in the Christoffel symbols and that terms are not defined as distributions. To eliminate this therm we impose $[g_{\alpha\beta}] = 0$, which means the continuity of the metric across the hypersurface Σ. Still this condition only holds in coordinates x^{α}, but we can turn this into a coordinate-invariant condition $[g_{\alpha\beta}]e^{\alpha}_{a}e^{\beta}_{b} = [g_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b}] = [h_{ab}]$, so that

$$[h_{ab}] = 0\,. \tag{A.16}$$

This is our first junction condition and it states that the induce metric must be the same on both sides of the hypersurface. It is a requirement to the hypersurface have a well-defined geometry.

A.3 Second junction condition

So far, using the first junction condition, Eq. (A.16), we have the Christoffel symbols without problematic terms

$$\Gamma^{\alpha}_{\beta\gamma} = \Theta(l)\Gamma^{+\alpha}_{\beta\gamma} - \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma} \,. \tag{A.17}$$

Now to find the second junction condition we have to compute the distribution-valued Riemann tensor for which we need to differentiate the Christoffel symbols

$$\Gamma^{\alpha}_{\beta\gamma,\delta} = \Theta(l)\Gamma^{+\alpha}_{\beta\gamma,\delta} - \Theta(-l)\Gamma^{-\alpha}_{\beta\gamma,\delta} + \epsilon\delta(l)[\Gamma^{\alpha}_{\beta\gamma,\delta}]n_{\delta} \,. \tag{A.18}$$

Thus the Riemann tensor is

$$R^{\alpha}_{\beta\gamma\delta} = \Theta(l)R^{+\alpha}_{\beta\gamma\delta} - \Theta(-l)R^{-\alpha}_{\beta\gamma\delta} + \delta(l)A^{\alpha}_{\beta\gamma\delta} \,, \tag{A.19}$$

where

$$A^{\alpha}_{\beta\gamma\delta} = \epsilon\left([\Gamma^{\alpha}_{\beta\delta}]n_{\gamma} - [\Gamma^{\alpha}_{\beta\gamma}]n_{\delta}\right) \,. \tag{A.20}$$

This last quantity is a tensor since it is the differente between two Christoffel symbols. From Eq. (A.19) the Riemann tensor is well defined as a distribution, however the δ term gives a curvature singularity at Σ. We will investigate further this term, which will lead us to the second junction condition. First note that the metric is continuous across the hypersurface implies that its tangential derivatives are also continuous. Thus it can only have a discontinuity directed along the normal vector n^{α}, such that

$$[g_{\alpha\beta,\gamma}] = \kappa_{\alpha\beta}n_{\delta} \,, \tag{A.21}$$

which constrains the tensor $\kappa_{\alpha\beta}$ to be given by

$$\kappa_{\alpha\beta} = \epsilon[g_{\alpha\beta,\gamma}]n^{\delta} \,. \tag{A.22}$$

Therefore we can compute the jump of the Christoffel symbols

$$[\Gamma^{\alpha}_{\beta\delta}] = \frac{1}{2}\left(\kappa^{\alpha}_{\beta}n_{\gamma} + \kappa^{\alpha}_{\gamma}n_{\beta} - \kappa_{\beta\gamma}n^{\alpha}\right) \,. \tag{A.23}$$

Inserting this in Eq. (A.20) yields

$$A^{\alpha}_{\beta\gamma\delta} = \frac{\epsilon}{2}\left(\kappa^{\alpha}_{\delta}n_{\beta}n_{\gamma} - \kappa^{\alpha}_{\gamma}n_{\beta}n_{\delta} - \kappa_{\beta\delta}n^{\alpha}n_{\gamma} + \kappa_{\beta\gamma}n^{\alpha}n_{\delta}\right) \,, \tag{A.24}$$

which corresponds to the δ-part of the Riemann tensor, Eq. (A.19).

Contracting the first and third indexes gives the δ-part of the Ricci tensor

$$A_{\alpha\beta} \equiv A^{\mu}_{\ \alpha\mu\beta} = \frac{\epsilon}{2}\left(\kappa_{\mu\alpha}n^{\mu}n_{\beta} + \kappa_{\mu\beta}n^{\mu}n_{\alpha} - \kappa n_{\alpha}n_{\beta} - \epsilon\kappa_{\alpha\beta}\right) \,, \tag{A.25}$$

where $\kappa \equiv \kappa^{\alpha}_{\alpha}$. Then, contracting the two indexes of $A_{\alpha\beta}$ results in the δ-part of the Ricci scalar

$$A \equiv A^{\alpha}_{\alpha} = \epsilon\left(\kappa_{\mu\nu}n^{\mu}n^{\nu} - \epsilon\kappa\right) \,. \tag{A.26}$$

Finally we can form the δ-part of the Enstein tensor

$$A_{\alpha\beta} - \frac{1}{2}Ag_{\alpha\beta} \equiv 8\pi S_{\alpha\beta} \,. \tag{A.27}$$

In fact, using the Einstein field equations, Eq. (A.12), we see that $S_{\alpha\beta}$ is related to the stress-energy tensor by

$$T_{\alpha\beta} = \Theta(l)T^{+}_{\alpha\beta} - \Theta(-l)T^{-}_{\alpha\beta} + \delta(l)S_{\alpha\beta}\,. \tag{A.28}$$

This gives an interpretation to $S_{\alpha\beta}$. While the first two terms are the stress-energy tensors of regions V^+ and V^-, respectively, the last term is associated with the distribuition of matter of the hypersurface Σ, the so-called thin matter shell. We call $S_{\alpha\beta}$ the surface energy-tensor, which the explict form is obtained by combining Eqs. (A.25)-(A.27)

$$16\pi\epsilon S_{\alpha\beta} = \kappa_{\mu\alpha}n^{\mu}n_{\beta} + \kappa_{\mu\beta}n^{\mu}n_{\alpha} - \kappa n_{\alpha}n_{\beta} - \epsilon\kappa_{\alpha\beta} - \left(\kappa_{\mu\nu}n^{\mu}n^{\nu} - \epsilon\kappa\right)g_{\alpha\beta}\,. \tag{A.29}$$

Since $S_{\alpha\beta}n^{\beta} = 0$ the surface stress-energy tensor is tangent to the hypersurface and therefore can be written in the basis of the tangent vectors to Σ, so that

$$S^{\alpha\beta} = S^{ab}e^{\alpha}_{a}e^{\beta}_{b}\,. \tag{A.30}$$

This implies

$$16\pi\epsilon S_{ab} = -\kappa_{\alpha\beta}e^{\alpha}_{a}e^{\beta}_{b} + h^{mn}\kappa_{\mu\nu}e^{\mu}_{m}e^{\nu}_{n}h_{ab}\,, \tag{A.31}$$

using Eq. (A.29). It is useful to introduce a new geometric quantity that characterize the hypersurface: the extrinsic curvature defined by

$$K_{ab} = n_{\alpha;\beta}e^{\alpha}_{a}e^{\beta}_{b}\,, \tag{A.32}$$

where ; denotes the covariant derivative. After some calculations one can arrives to

$$S_{ab} = -\frac{\epsilon}{8\pi G_d}\left([K_{ab}] + [K]h_{ab}\right)\,. \tag{A.33}$$

Recalling that in order to have a smooth transition across Σ we cannot have any δ-part in the Einstein equations, which happens only if $S_{ab} = 0$. From Eq. (A.33) this implies

$$[K_{ab}] = 0\,, \tag{A.34}$$

and this is the second junction condition, a condition that is independent of the x^{α} coordinates. Whereas the first junction condition, Eq. (A.16), must also be satisfied in order for both regions join smoothly, the same does not happen with the second junction condition. If this condition is violated we have the spacetime singular at Σ, but this singularity can be interpretated as a thin matter shell with a surface energy-tensor S_{ab}, which is located at the hypersurface. In conclusion the most simplest way to obtain the surface energy-tensor is the following form

$$S^{a}_{b} = -\frac{\epsilon}{8\pi G_d}\left([K^{a}_{b}] + [K]h^{a}_{b}\right)\,. \tag{A.35}$$

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