

# Study of $f(R)$ theories with a non-minimal coupling and its applications in a cosmological context.

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## Abstract

In this work a dynamical system approach to nonminimal coupled  $f(R)$  theories is made. The solutions of three distinct models are obtained and their stability and physical interpretation are studied to ascertain their viability as candidates for dark energy. Comparison is drawn with previous works in the context of  $f(R)$  theories and nonminimally coupled models.

**Keywords:**  $f(R)$  theories, Nonminimal Coupling, Cosmology, Dynamical systems

## 1. Introduction

Despite its experimental success [1, 2], it is known that General Relativity (GR) does not exhibit the most general form to couple matter with curvature. In fact, these can be coupled in a nonminimal way [3] (for early proposals see Ref. [4, 5, 6, 7]) that has already been shown to be able to mimic dark matter [8, 9], dark energy [10, 11] and explain post-inflationary preheating [12, 13] and cosmological structure formation [14].

This nonminimal coupling (NMC) can give rise to several implications, from Solar System [15] and stellar dynamics [16, 17, 18, 19] to close like-time curves [20], wormholes [21] and modifications to the well-known energy conditions [22] (see Ref. [23] for a thorough review).

From a fundamental standpoint, a NMC can arise from one-loop vacuum-polarization effects in the formulation of Quantum Electrodynamics in a curved spacetime [24], as well as in the context of matter scalar fields [25, 26]. In the framework of Riemann-Cartan geometry, a NMC was considered in an earlier proposal [27] and another study showed that it clearly affects the features of the ground state [28]. Phenomenologically, it can be viewed as a natural continuation of so-called  $f(R)$  theories [29], where the standard Einstein-Hilbert action is replaced by a non-linear function  $f(R)$  of the scalar curvature — an extension of General Relativity that has garnered a strong interest in the past decade.

The purpose of this work is to make a dynamical system approach on NMC theories in a cosmological setting and derive the solutions for some models, presented in Ref. [30]; for a very recent and sim-

ilar study, albeit less general, see Ref. [31]. This work is organized as follows: the nonminimal gravitational model is discussed in section 2; the formulation of the equivalent dynamical system is presented in section 3; a confirmation of the dynamical system obtained in  $f(R)$  theories is shown in section 4; the discussions of the results obtained for two pure NMC models and for a power law correction model are presented in sections 5 and 6, respectively. Finally, the conclusions are presented in section 7.

## 2. The Model

Following the generalization of the Einstein-Hilbert action put forward in  $f(R)$  theories [29], a NMC model is embodied in the action [3],

$$S = \int d^4x \sqrt{-g} [\kappa f_1(R) + f_2(R)\mathcal{L}], \quad \kappa = \frac{c^4}{16\pi G}, \quad (1)$$

where  $f_i(R)$  are arbitrary functions of the scalar curvature  $R$ ,  $g$  is the metric determinant and  $\mathcal{L}$  is the matter Lagrangian density. The standard Einstein-Hilbert action is obtained by taking  $f_1(R) = R - 2\Lambda$  and  $f_2(R) = 1$ . The field equations are obtained by imposing a null variation of the action with respect to the metric,

$$FG_{\mu\nu} = \frac{1}{2}f_2T_{\mu\nu} + \Delta_{\mu\nu}F + \frac{1}{2}g_{\mu\nu}\kappa f_1 - \frac{1}{2}g_{\mu\nu}RF, \quad (2)$$

where  $F = \kappa f_1' + f_2'\mathcal{L}$ , the prime denotes derivation with respect to the scalar curvature (omitted),  $\Delta_{\mu\nu} \equiv \nabla_\mu \nabla_\nu - g_{\mu\nu}\square$ , and the matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}}. \quad (3)$$

The generalized Bianchi identities imply the non-covariant conservation law

$$\nabla^\mu T_{\mu\nu} = \frac{f'_2}{f_2} (g_{\mu\nu} \mathcal{L} - T_{\mu\nu}) \nabla^\mu R. \quad (4)$$

Since there is an equivalence between this model and a two-scalar field model, this non-conservation may be interpreted as an energy exchange between matter and those scalar fields [26].

To study the recent accelerated expansion of our universe, a flat universe is considered with the line element

$$ds^2 = -dt^2 + a^2(t)dV^2, \quad (5)$$

where  $a(t)$  is the scale factor and  $dV$  is the volume element with comoving coordinates, and matter is assumed to behave as a perfect fluid, with an energy-momentum tensor

$$T^{\mu\nu} = (\rho + P) u^\mu u^\nu + P g^{\mu\nu}, \quad (6)$$

derived from the Lagrangean density  $\mathcal{L} = -\rho$  (see Ref. [32] for a discussion), where  $\rho$  and  $P$  are the energy density and pressure of the perfect fluid, respectively, and  $u^\mu$  is its four-velocity.

One can see that the energy-momentum tensor is again conserved, just like in GR or  $f(R)$  theories, since Eq. (4) yields the continuity equation

$$\dot{\rho} + 3H(1+w)\rho = 0, \quad (7)$$

where  $H = \dot{a}/a$  is the Hubble parameter and  $w = P/\rho$  is the equation of state (EOS) parameter.

Inserting the metric in the field equations (2), one obtains the modified field equations

$$H^2 = \frac{1}{3F} \left[ \frac{1}{2} f_2 \rho - 3HF' \dot{R} - 9H^2(1+w) f'_2 \rho - \left( -\frac{1}{2} \kappa f_1 + \frac{RF}{2} \right) \right], \quad (8)$$

$$2\dot{H} + 3H^2 = -\frac{1}{2F} \left[ f_2 w \rho + 2\ddot{F} + \kappa f_1 - RF + 4H\dot{F} \right]. \quad (9)$$

### 3. Dynamical System

One way to obtain the solutions of the field equations is via the study of the ensuing dynamical system, in which the field equations are replaced by an equivalent dynamical system of dimensionless variables

$$x = -\frac{F' \dot{R}}{FH}, \quad y = \frac{R}{6H^2}, \quad z = -\frac{\kappa f_1}{6FH^2}, \quad (10)$$

$$\Omega_1 = \frac{f_2 \rho}{6FH^2}, \quad \Omega_2 = -\frac{3(1+w)f'_2 \rho}{F}$$

with  $F' \equiv \kappa f'_1 - f'_2 \rho$  (note that the latter is the partial derivative of  $F$  with respect to the scalar

curvature  $R$ ). The modified Friedmann equation (8) becomes

$$1 = x + y + z + \Omega_1 + \Omega_2, \quad (11)$$

acting as a restriction to the phase space. Since  $\rho$  is indirectly dependent on  $R$ , one finds that for a constant  $F$ , (as studied in Ref. [33]),

$$-\frac{d \log F}{dN} = -\frac{F' \dot{R}}{FH} + \frac{f'_2 \dot{\rho}}{FH} = x + \Omega_2 = 0. \quad (12)$$

where  $N = \ln a$  is the number of e-folds.

The introduction of this coupling increases the number of variables of the problem — for  $f(R)$  theories, only four variables were required [34].

Calculating the variation of this variables with respect to  $N$ , one obtains the following autonomous system, equivalent to the field equations (8),

$$\begin{cases} \frac{dx}{dN} = x \left[ x - y + \Omega_2 \left( 1 + \frac{\alpha_2}{\alpha} \right) \right] - 1 - y - 3z \\ \quad + 3w\Omega_1 + \Omega_2 [3(1+w) - y] \\ \frac{dy}{dN} = y [-x/\alpha + 2(2-y)] \\ \frac{dz}{dN} = z \left[ x \left( 1 - \frac{\alpha_1}{\alpha} \right) + \Omega_2 + 2(2-y) \right] \\ \frac{d\Omega_1}{dN} = \frac{\Omega_2 x y}{3\alpha(1+w)} + \Omega_1 [1 - 3w + x + \Omega_2 - 2y] \\ \frac{d\Omega_2}{dN} = \Omega_2 \left[ x \left( 1 - \frac{\alpha_2}{\alpha} \right) - 3(1+w) + \Omega_2 \right] \end{cases} \quad (13)$$

with the dimensionless parameters,

$$\alpha = \frac{F'R}{F}, \quad \alpha_1 = \frac{f'_1 R}{f_1}, \quad \alpha_2 = \frac{f'_2 R}{f_2}. \quad (14)$$

These will depend on the choice of the functions  $f_1(R)$  and  $f_2(R)$  and must be computed as a function of the variables for each particular model (they are analogous to the  $\Upsilon$  parameter defined in Ref. [34]). It is possible to show that

$$\alpha_1 = \frac{y}{z} \left[ \frac{\Omega_2}{3(1+w)} - 1 \right], \quad (15)$$

from the variation of Eq. (11) with respect to  $N$ ,

$$0 = \frac{dx}{dN} + \frac{dy}{dN} + \frac{dz}{dN} + \frac{d\Omega_1}{dN} + \frac{d\Omega_2}{dN}. \quad (16)$$

Other useful relations are

$$\alpha = \frac{f'_1 R}{f_1} \left[ 1 - \frac{\Omega_2}{3(1+w)} \right] + \frac{\alpha_2 \Omega_2}{3(1+w)}, \quad (17)$$

and

$$f'_2 R = -\frac{\Omega_2 y}{3(1+w)\Omega_1} f_2. \quad (18)$$

The determination of the fixed points of any dynamical system analysis depends crucially on the choice of the variables. The number of dynamical variables for the pure NMC case is the same as in Ref. [31]. Since in Ref. [31],  $f_1(R) = R$  and  $f_2(R)$  remains unspecified, the fixed points appear as a function of  $y$  (in the present notation). Different values of  $y$  correspond to distinct cosmological eras, since this parameter is related to the decelerated parameter  $q = 1 - y$  (defined in the following section) and to  $w_{eff} = (2q - 1)/3$ . Furthermore, it also assumed  $w = 0$ , thus limiting its scope to a Universe filled with pressureless dust.

### 3.1. Physical Quantities

With the adopted metric (5), the Ricci scalar reads

$$R = 6(2H^2 + \dot{H}). \quad (19)$$

One important parameter used in cosmology is the deceleration parameter

$$q \equiv -\frac{\ddot{a}a}{\dot{a}^2} = 1 - y, \quad (20)$$

so that the scalar curvature may be written as

$$R = 6H^2(1 - q). \quad (21)$$

Since our universe appears to be expanding at an accelerated rate, one is searching for a model with  $q < 0 \rightarrow y > 1$ . In GR this parameter yields

$$q = \frac{1}{2}(1 + 3w). \quad (22)$$

which would require an exotic fluid with negative pressure,  $w < -1/3$ .

After determining the fixed points of the dynamical system for each particular choice of functions  $f_1(R)$  and  $f_2(R)$ , one may straightforwardly determine the scale factor for each fixed point. From Eq. (19), one can see that  $\dot{H} = (y - 2)H^2$ , and it is possible to determine the scale factor with this relation. Note that this quantity also depends on  $y$ , with the distinct evolutions,

$$a(t) = \begin{cases} \left(\frac{t}{t_0}\right)^{\frac{1}{2-y}}, & y \neq 2 \\ e^{H_0 t}, & y = 2 \end{cases} \quad (23)$$

For the first case, the scale factor evolves as a power of time, while in the second result the Hubble parameter will be constant and thus the scale factor will rise exponentially, *i.e.* a De Sitter phase.

Other important physical quantity is the energy density: one can determine its evolution for each fixed point from the continuity equation (7). The general solution for this equation is the familiar result

$$\rho(t) = \rho_0 a(t)^{-3(1+w)}. \quad (24)$$

Considering the definition of the variable  $\Omega_2$  from equation (10), one can see that

$$\rho = \frac{\kappa f'_1 \Omega_2}{f'_2 [\Omega_2 - 3(1+w)]}, \quad (25)$$

so, for a particular fixed point, it should be possible to determine the energy density from this relation. Note that for  $\Omega_2 = 3(1+w)$  there appears to be a divergence in the density: physically, a fixed point with this value of  $\Omega_2$  will correspond to a regime where  $f'_2 \rho \gg \kappa f'_1$ .

## 4. $f(R)$ Theories

In order to confirm the results obtained in Ref. [34], the  $f(R)$  theories case is considered

$$f_1(R) = f(R), \quad f_2(R) = 1 \rightarrow \alpha_1 = -y/z, \quad (26)$$

the variable  $\Omega_2$  vanishes trivially,  $\alpha$  will only depend on the derivatives of our arbitrary function  $f(R)$  and  $\alpha_2$  is not well determined, but does not appear in the equations. The dynamical system can be simplified to

$$\begin{cases} \frac{dx}{dN} = x(x - y) - y - 3z + 3w\Omega_1 - 1 \\ \frac{dy}{dN} = y \left[ 4 - \frac{x}{\alpha} - 2y \right] \\ \frac{dz}{dN} = z[x + 4 - 2y] + \frac{xy}{\alpha} \\ \frac{d\Omega_1}{dN} = \Omega_1(1 - 3w + x - 2y) \end{cases} \quad (27)$$

which is equivalent to the system presented in Ref. [34], as expected (for an extensive discussion of a dynamical system approach on  $f(R)$  theories see also Ref. [35]).

## 5. Pure Nonminimal Coupling Case

To study the influence of the NMC in cosmology, a simple case where  $f_1(R) = R$  and  $f_2(R) = f(R)$  is considered. From relation (15),  $\Omega_2$  can be written as

$$\Omega_2 = 3(1+w) \left( 1 + \frac{z}{y} \right). \quad (28)$$

Also, equation (17) implies that  $\alpha = \alpha_2(1 + z/y)$ , and thus the dynamical system can be written as

$$\begin{cases} \frac{dx}{dN} = \frac{3(1+w)[3(1+w)+x]z}{y} - y(4+3w+x) \\ + 8 + 9w^2 + 6x + x^2 - 6z + 3w(6+2x-z+\Omega_1) \\ \frac{dy}{dN} = y \left( 4 - \frac{xy}{(y+z)\alpha_2} - 2y \right) \\ \frac{dz}{dN} = z \left[ 4 + x - 2y + \frac{3(1+w)(y+z)}{y} \right. \\ \left. - \frac{xy}{(y+z)\alpha_2} \right] \\ \frac{d\Omega_1}{dN} = \frac{xy}{\alpha_2} + \Omega_1 \left[ 4 + x - 2y + \frac{3(1+w)z}{y} \right] \end{cases} \quad (29)$$

restricted by the modified Friedmann equation

$$1 = x + y + z + \Omega_1 + 3(1+w) \left( 1 + \frac{z}{y} \right). \quad (30)$$

### 5.1. Power law Nonminimal Coupling

The most simple case is to consider the function

$$f_2(R) = C + \left( \frac{R}{12M^2} \right)^n, \quad (31)$$

already studied in Ref. [10] with  $C = 1$ . The parameters  $C$  and  $M$  are both constant and the latter is related to the energy scale of the theory. For this model,  $\alpha_2 = n - 1$ , independently of  $C$ . For  $C = 0$ , the exponent  $n$  should be close to zero so as to introduce a small deviation from  $f_2(R) = 1$ ; conversely, for  $C = 1$ ,  $n$  may take any value. The fixed points obtained for both cases are the same, but the evolution of the physical quantities will differ.

The fixed points of this system are obtained imposing a null variation of the dynamic variables. Their values associated with the solutions are shown in Table 1.

Comparing these fixed points with the solutions obtained in the article Ref. [10], one can verify that point  $\mathcal{B}$  (with  $w = 0$ ) corresponds to the  $f_2\rho \gg \kappa$  regime; point  $\mathcal{C}$  has some similar features to the  $f_2\rho \ll \kappa$  regime, although there is not an exact equivalence, as discussed in the corresponding paragraph. Also there appears to be an extra solution not mentioned in Ref. [10], corresponding to a De Sitter phase of exponential expansion of the Universe.

To have a point that can replicate the effects of dark energy, one requires it to be stable with  $q < 0$ . Also, since the model is a power law of  $R$ , one requires that  $n < 0$ , so that it dominates only for late times, when the scalar curvature is sufficiently small.

In the regime  $f_2\rho \gg \kappa$ , one can see that

$$\Omega_1 \sim -\frac{f_2}{6f_2'H^2} = -\frac{y}{n} \left[ 1 + C \left( \frac{12M^2}{R} \right)^n \right], \quad (32)$$

This relation will be used to further explore the physical significance of relevant fixed points.

#### 5.1.1 Point $\mathcal{A}$

This point corresponds to a De Sitter solution in the regime  $f_2\rho \gg \kappa$ , since  $\Omega_2 = 3(1+w)$  and considering Eq. (25). For  $C = 0$ , the above yields  $y = -n\Omega_1$ , which leads to the restriction  $n = 2/(4+3w)$ . For  $C = 1$ , the relation (32) yields

$$H_0 = M \left[ n \left( 2 + \frac{3w}{2} \right) - 1 \right]^{-1/(2n)}, \quad (33)$$

for  $n > 2/(4+3w)$ . The stability of the point is shown in Fig. 1.

Notice that the NMC exponent is positive,  $n > 0$ , so that its effect should be dominant at early times, when the curvature is high,  $R \gg M^2$ . Furthermore, Fig. 1 shows that the fixed point is never an attractor for any pair  $(w, n)$ , but unstable or a saddle point. Thus, it is not a viable candidate for dark energy, but could have some bearing on inflation.

#### 5.1.2 Point $\mathcal{B}$

Since  $\Omega_2 = 3(1+w)$ , this point is in the regime  $f_2\rho \gg \kappa$  again due to Eq. (25). If  $n = 2/(4+3w)$ , it is equal to  $\mathcal{A}$ , so a De Sitter phase is attained. The stability of the point is shown in Fig. 2.

Notice that one has  $y = -\Omega_1/n$ : from Eq. (32), this is only physical when  $C = 0$  or, if  $C = 1$ , when  $(R/12M^2)^n \gg 1$  — so that the NMC should be dominant in the latter case. This is in accordance with the corresponding regime  $f_2\rho \gg \kappa$  studied in Ref. [10], and confirmed by the value for the deceleration parameter when  $w = 0$ ,  $q = (1+n)/(1-n)$ , and requires a negative value for the exponent  $n$  for the NMC to dominate at late times.

In GR, for the era of matter dominance,  $w = 0 \rightarrow q = 1/2$  and for the radiation era,  $w = 1/3 \rightarrow q = 1$ . In table 2 it is shown that it is possible to have quite different values of  $q$  with respect to the latter: for example, it is possible to have an evolution characteristic of the radiation era in GR, i.e.  $q = 1$ , even when  $w = 0$ .

#### 5.1.3 Point $\mathcal{C}$

Notice that this point has  $x = -\Omega_2$  so, from Eq. (12),  $F$  is constant and its value can be determined

$$z = -\frac{\kappa R}{6FH^2} \Rightarrow F = -\frac{\kappa y}{z} = \kappa \frac{1-4n-3w}{1-2n-3w}. \quad (34)$$

	$q$	$w_{GR}$	$w$	$n$	$w$	$n$
$\mathcal{B}$	-1	-1	0	$\infty$	1/3	2/5
	1/2	0	0	-1/3	1/3	-1/2, 1/3
	1	1/3	0	0	1/3	0, 1/4
$\mathcal{C}$	1/2	0	0	0	1/3	-1/3
	1	1/3	0	1/4	1/3	0

Table 2: Values of  $n$  needed to obtain the usual deceleration values for different  $w$ .

For this case, one has

$$\rho(t) = \frac{24\kappa M^2}{1-2n-3w} \left[ \frac{3^2(1+w)^2 M^2}{(1-n)(1-4n-3w)} t^2 \right]^{n-1}, \quad (35)$$

independent of  $C$ . For  $w = 0$ , this reads

$$\rho_0 = \frac{8}{3} \left(\frac{3}{4}\right)^n (1-5n+4n^2)^{-n} \frac{(1-n)(1-4n)}{(1-2n)} \left(\frac{t_0}{t_2}\right)^{2n} \frac{\kappa}{t_0^2}, \quad (36)$$

with  $t_2 \equiv 1/(2\sqrt{3}M)$ .

One can see that the result obtained here is different from the one attained in the  $f'_2\rho \ll \rho$  regime studied in Ref. [10]: in the latter,  $f'_2\rho = 0$  was effectively assumed, and thus  $F = \kappa$ . Conversely, Eq. (34) with  $w = 0$  (a universe filled with pressure-less dust) reads

$$F = \kappa \frac{1-4n}{1-2n} \neq \kappa \rightarrow \frac{f'_2\rho}{\kappa} = \frac{2n}{1-2n}. \quad (37)$$

and indeed one finds that  $f'_2\rho/\kappa$  can be of the order unity or larger.

Nevertheless, both point  $\mathcal{C}$  here obtained (for  $w = 0$ ) and the regime  $f'_2\rho \ll \kappa$  studied in Ref. [10] predict the same evolution for the scale factor, typified by a deceleration factor  $q = -1 + 3/[2(1-n)]$ .

Also, one can see that

$$\Omega_1 = \frac{1}{1-n} \left[ 1 + C \left( \frac{12M^2}{R} \right)^n \right] \rightarrow C \left( \frac{12M^2}{R} \right)^n = 0, \quad (38)$$

and a consistent solution is obtained when  $C = 0$ , or alternatively if  $C = 1$  and  $(R/12M^2)^n \gg 1$ : the latter implies that the NMC must be dominant, again requiring a negative value for the exponent  $n$  in order to replicate late time dark energy.

The stability of the point is shown in Fig. 3.  $\mathcal{C}$  is not a viable candidate for dark energy, since the stable region corresponds to  $q > 0$ . In table 2 the values of  $n$  are shown for typical values of  $q$  and  $w$ . As expected, when  $n = 0$ , these coincides exactly with the results of GR. Again, this fixed point allows one type of matter to mimic another (*e.g.* NMC dust leads to a behaviour typical of radiation in GR), as depicted on Table 2.

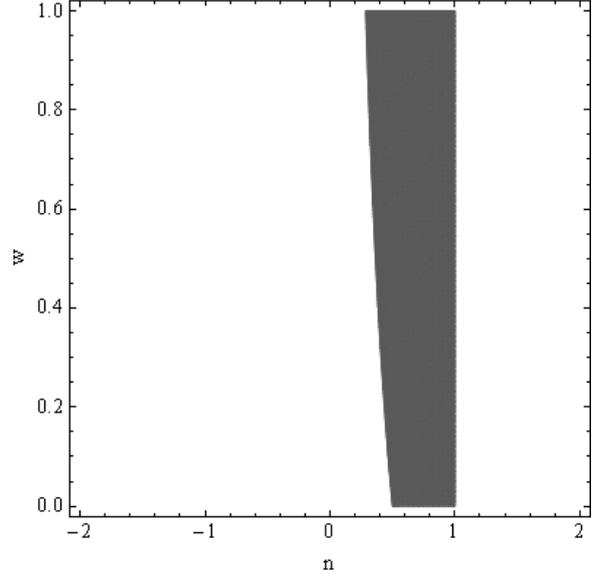


Figure 1: The dark grey region corresponds to the unstable region of point  $\mathcal{A}$ . There is no stable region and the remaining phase space corresponds to a saddle point.

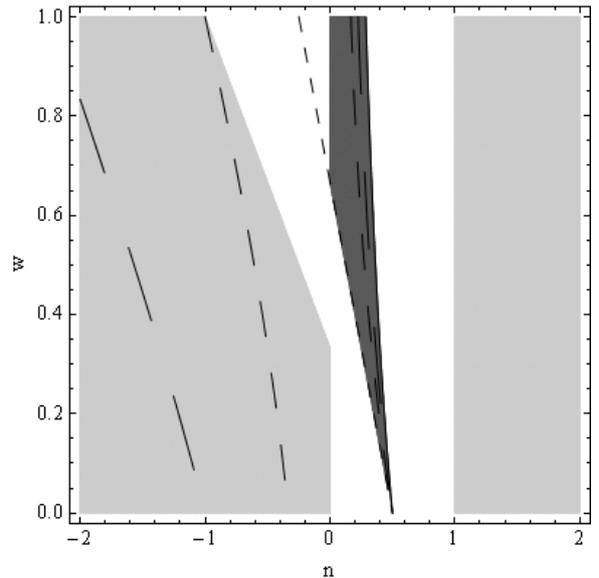


Figure 2: Stability region of point  $\mathcal{B}$ . The light grey region corresponds to the a stable fixed point, the dark grey region to an unstable fixed point and the remaining to a saddle point. Large, medium and short dash indicate  $q = 0$ ,  $q = 1/2$  and  $q = 1$ , respectively. The continuous line corresponds to  $q = -1$ .

## 5.2. Exponential Nonminimal Coupling

The study of an exponential model,

$$f_2(R) = \exp\left(\frac{R}{R_0}\right), \quad (39)$$

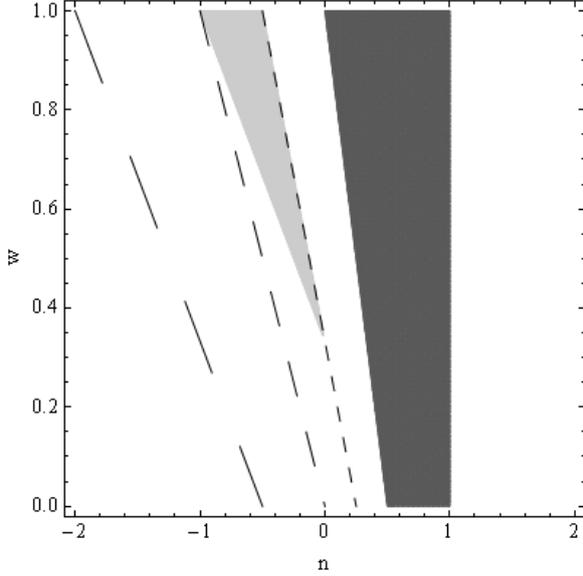


Figure 3: Stability region of point  $\mathcal{C}$ . The light grey region corresponds to a stable fixed point, the dark grey region to an unstable fixed point and the remaining to a saddle point. Large, medium and short dash indicate  $q = 0$ ,  $q = 1/2$  and  $q = 1$ , respectively.

might be of interest because when the scalar curvature tends to zero, the NMC vanishes asymptotically. One can see that the effects of an exponential function in  $f(R)$  theories, Ref. [36], is richer than in NMC theories. For this model, one can determine  $\alpha_2$  using equation (18), since for this particular case  $\alpha_2 = f_2''R/f_2' = f_2''R/f_2'$ . Also, the relation  $\alpha = \alpha_2(1 + z/y)$  is still valid, and thus the dynamical system is well determined. The fixed points obtained are shown in Table 3.

	Coordinates $(x, y, z, \Omega_1, \Omega_2)$
$\mathcal{A}$	$\left(-\frac{3}{2}(1+w), 2, -1, 0, \frac{3}{2}(1+w)\right)$
$\mathcal{B}$	$(0, 2, 0, -4 - 3w, 3(1+w))$
$\mathcal{C}$	$(-4 - 3w, 2, 0, 0, 3(1+w))$

Table 3: Fixed points of the model, Eq. (39).

### 5.2.1 Point $\mathcal{A}$

This is a saddle point with no physical meaning. First, it presents the unusual case where  $y = 2$ , so that the scalar curvature is constant, but  $x \neq 0$ . One can see that  $x = -\Omega_2$ , which implies that  $F$  is constant, from Eq. (12). From Eq. (25), one can see that  $\rho = -\kappa/f_2'$ . Also, considering the definitions presented in Eq. (10), in order to have  $x \neq 0$  with a constant curvature, one needs  $H = 0$

or  $F = 0$ . Note that,

$$F = \kappa - f_2'\rho = 0 \Rightarrow \rho = \kappa/f_2', \quad (40)$$

which disagrees with the previous result unless  $R_0 \rightarrow \infty$  and GR is recovered. For  $H = 0$ ,

$$\Omega_1 = \frac{f_2\rho}{6FH^2} \rightarrow \infty \neq 0, \quad (41)$$

unless  $\rho = 0$ , but it will also disagree with the previous result — thus, proving the inconsistency of this point.

### 5.2.2 Point $\mathcal{B}$

This is a saddle point in the regime  $f_2'\rho \gg \kappa$ , since  $\Omega_2 = 3(1+w)$ , with

$$H_0^2 = \frac{R_0}{6(4+3w)}. \quad (42)$$

This is the only consistent fixed point for this model. For  $y = 2$ ,  $R = 12H_0^2$  is constant, implying  $x = 0$ . Also,

$$z \sim \frac{\kappa R}{6f_2'\rho H_0^2} \sim \frac{\kappa}{f_2'\rho} \rightarrow 0. \quad (43)$$

### 5.2.3 Point $\mathcal{C}$

This is a stable point in the regime  $f_2'\rho \gg \kappa$ , since  $\Omega_2 = 3(1+w)$ . This appears to be another case where  $y = 2$ ,  $R$  constant with  $x \neq 0$  and it is inconsistent. The definition of  $x$  from Eq. (10) and  $f_2'\rho \gg \kappa$  implies that

$$x \sim -\frac{\dot{R}}{R_0 H}. \quad (44)$$

Since  $\dot{R} = 0$ , to obtain  $x \neq 0$ , it is necessary that  $H = 0$ , to induce an indetermination. When considering the definition of  $\Omega_1$ ,

$$\Omega_1 \sim -\frac{R_0}{6H^2} \rightarrow \infty \neq 0, \quad (45)$$

which makes this an inconsistent point. Note that this was the only point of this model with a stable region: although all the fixed points correspond to a De Sitter phase, none of them can be used to describe dark energy.

## 6. Power law NMC and curvature term

A power law correction to GR model is considered with the functions

$$f_1(R) = R + 12M_1^2 \left(\frac{R}{12M_1^2}\right)^{n_1}, \quad (46)$$

$$f_2(R) = 1 + \left(\frac{R}{12M_2^2}\right)^{n_2},$$

where  $M_i$  are characteristic energy scales. One can see that  $\alpha_2 = n_2 - 1$  and relation (15) is still valid; to determine  $\alpha$ , one resorts to equation (17) and writes

$$\frac{f_1'' R}{f_1'} = n_1 \left( 1 - \frac{1}{\alpha_1} \right) \quad (47)$$

The dynamical system is obtained by replacing all this parameters in the initial system (13). The fixed points obtained from this dynamic system are shown in Table 6 and the corresponding solutions in Table 7. For a detailed view of the stability of the points see the figures in the original thesis.

Note that for  $n_1 = 1 \rightarrow f_1(R) = 2R$ , the fixed points coincide with the ones presented in the subsection 5.1, considering the restriction for  $\Omega_2$ , Eq. (28). For the GR case ( $n_1 = n_2 = 0$ ), the fixed points obtained will collapse to only two: an unstable matter dominance and a stable cosmological constant dominance.

### 6.0.1 Points $\mathcal{A}$ , $\mathcal{B}$ and $\mathcal{C}$

These correspond exactly to the points presented in section 5.1 of the power law pure NMC case. The stability of point  $\mathcal{A}$  is shown in Fig. 1. Despite this correspondence between the points of both models, the stability of  $\mathcal{B}$  and  $\mathcal{C}$  is altered by the NMC.

### 6.0.2 Points $\mathcal{D}$ , $\mathcal{E}$ and $\mathcal{F}$

These are points of little interest with an evolution similar to the radiation era. Since  $\Omega_2 = 0$ , one might expect these points to be related to pure  $f(R)$  solutions. In fact, they appear in Ref. [34], that studies the  $f(R) = R + R_0(R/R_0)^n$  model.

### 6.0.3 Point $\mathcal{G}$

In this point the energy density is null so the NMC is neglected,  $f_2' \rho \ll \kappa f_1'$ . This corresponds to a point based on a pure  $f(R)$  theory that was studied in Ref. [34].

It is easy to see that a De Sitter phase is obtained when  $n_1 = 2$ , which corresponds to  $x = 0$ ,  $y = 2$  and  $z = -1$  — the same as the previous fixed point, but their origin is completely different. The Starobinsky inflation model, Ref. [13], corresponds to  $n_1 = 2$  and  $n_2 = 0$ . Since the NMC is neglected, this point corresponds to that solution.

The scale factor and deceleration parameter are independent of  $w$ , but the stability has some dependency. One can see that when

$$n_1 = (7 \pm \sqrt{73})/12 \rightarrow q = 1/2, \quad (48)$$

and

$$n_1 = 0, 5/4 \rightarrow q = 1. \quad (49)$$

### 6.0.4 Point $\mathcal{H}$

The scale factor only depends on  $n_1$ , reflecting a stronger influence of  $\kappa f_1'$ . This is also visible, since  $\Omega_1 \neq 0$  and  $\Omega_2 = 0$ . The regime  $f_2' \rho \ll \kappa f_1'$  is verified, but with non-null energy density. This fixed point also appears in Ref. [34], and is based on pure  $f(R)$  theory.

A De Sitter solution is only obtained when  $w = -1$ , which is similar to the use of a cosmological constant. Table 4 shows the values of  $n_1$  needed to obtain the usual values of  $q$ , when  $w = 0, 1/3$ . For  $n_1 = 1$ , the result  $q = (1 + 3w)/2$ , typical of GR, is recovered; interestingly, this does not depend on  $n_2$ .

$q$	$w_{GR}$	$w$	$n_1$	$w$	$n_1$
1/2	0	0	1	1/3	4/3
1	1/3	0	3/4	1/3	1

Table 4: Values of  $n$  needed to obtain the usual deceleration values for different  $w$  for point  $\mathcal{H}$ .

### 6.0.5 Point $\mathcal{I}$

This is stable when  $1 < n_1 < 2$  and a saddle point in rest of the region. It corresponds to a De Sitter phase, with no matter. The Hubble parameter has to satisfy the condition

$$H_0^2 = M_1^2 \left( \frac{1}{n_1 - 2} \right)^{1/(n_1 - 1)}, \quad (50)$$

for  $n_1 \neq 1$  and  $n_1 \neq 2$ . This point also appears in Ref. [34].

### 6.0.6 Point $\mathcal{J}$

This point appears to be a generalization of  $\mathcal{C}$  and  $\mathcal{H}$ , has no dominant regime and it is the only point that depends explicitly on both functions. This point is divergent when  $n_1 = n_2$ .

A De Sitter solution is only obtained when  $w = -1$ , which is similar to the use of a cosmological constant. Table 5 shows the relation between  $n_1$  and  $n_2$  needed to obtain the usual values of  $q$ , when  $w = 0, 1/3$ . The normal results for GR are obtained when  $n_1 - n_2 = 1$ .

$q$	$w_{GR}$	$w$	$n_1 - n_2 = 1$	$w$	$n_1 - n_2 = 4/3$
1/2	0	0	$n_1 - n_2 = 1$	1/3	$n_1 - n_2 = 4/3$
1	1/3	0	$n_1 - n_2 = 3/4$	1/3	$n_1 - n_2 = 1$

Table 5: Values of  $n$  needed to obtain the usual deceleration values for different  $w$  for point  $\mathcal{J}$ .

### 6.1. Modified Friedmann equation

In this section we attempt a comparison between the results obtained above and those of Ref. [33], which is based upon the phenomenological study of modifications of the Friedmann equation, of the form

$$H^2 \sim \rho^{1+\beta}. \quad (51)$$

As shown in that study, the above relation can be obtained in the regime  $F = \text{const.}$  when  $f_1(R) \sim R$  and  $f_2(R) \sim R^{\beta/(1+\beta)}$ . From Eq. (24), the scale factor is given by

$$a(t) \sim t^{\frac{2}{3(1+w)(1+\beta)}}, \quad (52)$$

and the deceleration parameter is

$$q = -1 + \frac{3(1+\beta)(1+w)}{2}. \quad (53)$$

From the above, a comparison between this solutions and the solutions obtained from the fixed points is possible.

Considering the exponents of the functions defined in Eq. (46) as  $n_1 = 1$  and  $n_2 = \beta/(1+\beta)$ , one can see that  $\mathcal{C}$ ,  $\mathcal{H}$ ,  $\mathcal{I}$  and  $\mathcal{J}$  correspond to a constant  $F$ .  $\mathcal{I}$  corresponds to a De Sitter phase and  $q$  is only equal to Eq. (53) when  $\beta = -1$ , which corresponds to  $n_2 \rightarrow \infty$ .  $\mathcal{C}$  and  $\mathcal{J}$  have a deceleration parameter exactly like Eq. (53).  $\mathcal{B}$  has  $F = \text{const.}$  and satisfies Eq. (53) when  $\beta = (1-3w)/(1+3w)$ .  $\mathcal{H}$  also satisfies Eq. 53 when  $\beta = 0$ , which corresponds to GR case. Thus, one concludes that the modifications to Friedmann equation due to a NMC are indeed obtainable from a dynamical system's approach, as correctly argued in Ref. [33].

### 6.2. Linear NMC

Note that, for  $n_2 = 1$  divergences appear in some points and a more detail study by direct substitution is required, which is done in Ref. [31]. A direct comparison between the points obtained above and the ones attained in Ref. [31] is not done due to the different choice of the variables: in particular, note that that study resorts to a variable proportional to  $\rho^2$ .

Nevertheless, one can compare the deceleration parameter obtained in both works ( $n_1 = n$ ), which clearly marks the physical significance of the underlying fixed points: one finds that  $\mathcal{J}$  presented here has the same deceleration parameter,  $q = (5-2n)/(2(n-1))$ , of the fourth point of the mentioned article. Also, there is a fixed point in Ref. [31] with  $q = -1$ , that can be related to the fixed points obtained here with the same value. These comparisons are only valid when the power law term dominates over  $R$ , since our choice of model was Eq. (46).

## 7. Conclusion

In this work, a dynamical system approach was made on NMC theories. The dynamical system for the most general case with two arbitrary functions was obtained. Also, the solutions and their stability for three different models were obtained and compared with previous works.

As expected, the NMC dynamical system can be particularized to a pure  $f(R)$  theory when  $f_2(R) = 1$ , yielding the same results obtained in Ref. [34]. One can see that the variable  $\Omega_2$  introduced by the NMC, is the key to determine whether  $F$  is constant or if the NMC dominates over the usual  $f(R)$  theory.

In the pure NMC case described by a power law, the solutions obtained are in agreement with the ones presented in Ref. [10]. In addition, the obtained result for the energy density for the point  $\mathcal{C}$  is different from the one in Ref. [10], due to the assumption of the latter that  $\rho = 0 \rightarrow F = \kappa$ , which differs from the result here obtained, Eq. (34).

Furthermore, the pure NMC exponential case appears to have less diversity of solutions than the usual exponential  $f(R)$  model, as seen in Ref. [36].

The last model considered of power law corrections to GR yields the solutions for the pure  $f(R)$  case and the pure NMC case, as if it considered the regimes for which function dominates over the other, and also a solution that depends simultaneously on both models. Furthermore, it was determined which fixed points correspond to the general solution of  $H^2 \sim \rho^{1+\beta}$ , presented in Ref. [33].

This method is a good way to determine the solutions of a particular model, since it does not assume solutions a priori. Note that, in the pure NMC power law case, there is a solution obtained by this method not considered in Ref. [10].

However, and despite its success in determining a variety of solutions, this method depends on the chosen variables. In addition, the existence of fixed points with desired stability does not imply that there is a trajectory in the phase space (i.e. a history for the Universe) that connects these points. Even if that can be guaranteed, only a more profound mathematical study of the field equations (including their numerical solution) can ensure that the universe may evolve from a matter dominated to an accelerated expansion era, in a way compatible with the experimental observations and with reasonable physical parameters,  $n$  and  $M$ .

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	$(x, y, z, \Omega_1, \Omega_2)$	$a(t)$	$\rho(t)$	$q$
$\mathcal{A}$	$(0, 2, 0, -4 - 3w, 3(1 + w))$	$e^{H_0 t}$	$f_2' \rho \rightarrow \infty$	$-1$
$\mathcal{B}$	$\left( \frac{4 - 2n(4 + 3w)}{2n - 1}, \frac{n(-2 + 4n + 3w)}{1 - 3n + 2n^2}, 0, \frac{2 - 4n - 3w}{1 - 3n + 2n^2}, 3(1 + w) \right)$	$\left( \frac{t}{t_0} \right)^{\frac{1 - 3n + 2n^2}{2 - n(4 + 3w)}}$	$f_2' \rho \rightarrow \infty$	$-1 + \frac{2 - n(4 + 3w)}{1 - 3n + 2n^2}$
$\mathcal{C}$	$\left( \frac{6n(1 + w)}{1 - 4n - 3w}, -\frac{1 - 4n - 3w}{2(n - 1)}, \frac{1 - 2n - 3w}{2(n - 1)}, \frac{1}{1 - n}, -\frac{6n(1 + w)}{1 - 4n - 3w} \right)$	$\left( \frac{t}{t_0} \right)^{\frac{2(1 - n)}{3(1 + w)}}$	$\left( \frac{t}{t_0} \right)^{2n - 2}$	$-1 + \frac{3(1 + w)}{2(1 - n)}$

Table 1: Fixed points and respective solutions of the model, Eq. (31).

	Coordinates $(x, y, z, \Omega_1, \Omega_2)$
$\mathcal{A}$	$(0, 2, 0, -4 - 3w, 3(1 + w))$
$\mathcal{B}$	$\left( \frac{4 - 2n_2(4 + 3w)}{-1 + 2n_2}, \frac{n_2(-2 + 4n_2 + 3w)}{1 - 3n_2 + 2n_2^2}, 0, \frac{2 - 4n_2 - 3w}{1 - 3n_2 + 2n_2^2}, 3(1 + w) \right)$
$\mathcal{C}$	$\left( -\frac{6n_2(1 + w)}{-1 + 4n_2 + 3w}, \frac{1 - 4n_2 - 3w}{2 - 2n_2}, \frac{1 - 2n_2 - 3w}{2(-1 + n_2)}, \frac{1}{1 - n_2}, \frac{6n_2(1 + w)}{-1 + 4n_2 + 3w} \right)$
$\mathcal{D}$	$(-4, 0, 5, 0, 0)$
$\mathcal{E}$	$(1, 0, 0, 0, 0)$
$\mathcal{F}$	$(-1 + 3w, 0, 0, 2 - 3w, 0)$
$\mathcal{G}$	$\left( -\frac{2(n_1 - 2)}{2n_1 - 1}, \frac{n_1(-5 + 4n_1)}{1 - 3n_1 + 2n_1^2}, \frac{5 - 4n_1}{1 - 3n_1 + 2n_1^2}, 0, 0 \right)$
$\mathcal{H}$	$\left( \frac{3(-1 + n_1)(1 + w)}{n_1}, -\frac{3 - 4n_1 + 3w}{2n_1}, \frac{3 - 4n_1 + 3w}{2n_1^2}, \frac{-3(1 + w) - 2n_1^2(4 + 3w) + n_1(13 + 9w)}{2n_1^2}, 0 \right)$
$\mathcal{I}$	$(0, 2, -1, 0, 0)$
$\mathcal{J}$	$\left( -\frac{3(1 + w)[(-1 + n_2)(3 + 4n_2 + 3w) + n_1[7 - 2n_2^2 + 3w - 9n_2(1 + w)] + n_1^2[-4 + n_2(8 + 6w)]}{(n_1 - n_2)[4n_1 - 4n_2 - 3(1 + w)]}, \frac{3 - 4n_1 + 4n_2 + 3w}{2n_1 - 2n_2}, \frac{3 - 5n_2 - 2n_2^2 + 3w - 9n_2w + n_1[-4 + n_2(8 + 6w)]}{2(n_1 - n_2)^2}, \frac{-4n_2 - 3(1 + w) - 2n_1^2(4 + 3w) + n_1(13 + 2n_2 + 9w)}{2(n_1 - n_2)^2}, \frac{3n_2(1 + w)[3 + 4n_2 + 3w + n_1^2(8 + 6w) - n_1(13 + 2n_2 + 9w)]}{(n_1 - n_2)[4n_1 - 4n_2 - 3(1 + w)]} \right)$

Table 6: Fixed points of the model, Eq. (46).

	$a(t)$	$\rho(t)$	$q$
$\mathcal{A}$	$e^{H_0 t}$	$e^{-3(1+w)H_0 t}$	$-1$
$\mathcal{B}$	$\left( \frac{t}{t_0} \right)^{\frac{1 - 3n_2 + 2n_2^2}{2 - 4n_2 - 3n_2 w}}, n_2 \neq \frac{2}{4 + 3w}$	$\left( \frac{t}{t_0} \right)^{\frac{3(n-1)(2n-1)(1+w)}{n(4+3w)-2}}$	$-1 + \frac{2 - n_2(4 + 3w)}{1 - 3n_2 + 2n_2^2}$
$\mathcal{C}$	$\left( \frac{t}{t_0} \right)^{\frac{2(1 - n_2)}{3(1 + w)}}$	$\left( \frac{t}{t_0} \right)^{2(n_2 - 1)}$	$-1 + \frac{3(1 + w)}{2(1 - n_2)}$
$\mathcal{D}$	$\sqrt{t/t_0}$	$0$	$1$
$\mathcal{E}$	$\sqrt{t/t_0}$	$0$	$1$
$\mathcal{F}$	$\sqrt{t/t_0}$	$\left( \frac{t}{t_0} \right)^{-3(1+w)/2}$	$1$
$\mathcal{G}$	$\begin{cases} e^{H_0 t}, & n_1 = 2 \\ \left( \frac{t}{t_0} \right)^{\frac{-1 + 3n_1 - 2n_1^2}{n_1 - 2}}, & n_1 \neq 2 \end{cases}$	$0$	$-1 + \frac{2 - n_1}{1 - 3n_1 + 2n_1^2}$
$\mathcal{H}$	$\left( \frac{t}{t_0} \right)^{3(1 + w)}$	$\left( \frac{t}{t_0} \right)^{-2n_1}$	$-1 + \frac{3(1 + w)}{2n_1}$
$\mathcal{I}$	$e^{H_0 t}$	$0$	$-1$
$\mathcal{J}$	$\left( \frac{t}{t_0} \right)^{\frac{2(n_1 - n_2)}{3(1 + w)}}$	$\left( \frac{t}{t_0} \right)^{2(n_2 - n_1)}$	$-1 + \frac{3(1 + w)}{2(n_1 - n_2)}$

Table 7: Solutions associated with the fixed points of the model, Eq. (46).