Derivadas de funções compostas

Teorema: *Sejam $A \subset \mathbb{R}^n$ e $B \subset \mathbb{R}^m$ conjuntos abertos e*

$$\begin{aligned} \boldsymbol{f} &: A \subset \mathbb{R}^n \to \mathbb{R}^m \\ \boldsymbol{g} &: B \subset \mathbb{R}^m \to \mathbb{R}^p \end{aligned}$$

funções tais que $\boldsymbol{f}(A) \subset B$, diferenciáveis em $\boldsymbol{a} \in A$ e em $\boldsymbol{f}(\boldsymbol{a})$, respectivamente. Então $\boldsymbol{g} \circ \boldsymbol{f}$ é diferenciável em $\boldsymbol{a}$ e a sua derivada é

$$D(\boldsymbol{g} \circ \boldsymbol{f})(\boldsymbol{a}) = D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))D\boldsymbol{f}(\boldsymbol{a}) \ .$$

Demonstração: Seja

$$\boldsymbol{X}(\boldsymbol{h}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h} \ .$$

Então, pela definição de derivada, $\boldsymbol{g} \circ \boldsymbol{f}$ é diferenciável em $\boldsymbol{a}$ com derivada $D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))D\boldsymbol{f}(\boldsymbol{a})$ se e só se

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\boldsymbol{X}(\boldsymbol{h})}{||\boldsymbol{h}||} = \boldsymbol{0} \ .$$

Para provar a diferenciabilidade seja então $\boldsymbol{k} = \boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{a})$. Temos $\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{k}) = \boldsymbol{f}(\boldsymbol{a}) + \boldsymbol{k}$ e assim

$$\boldsymbol{X}(\boldsymbol{h}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h} \ .$$

Somando e subtraindo $D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k}$ vem

$$\begin{aligned} \boldsymbol{X}(\boldsymbol{h}) &= \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k} \\ &\quad + D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k} - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h} \\ &= \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k} \\ &\quad + D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))(\boldsymbol{k} - D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}) \\ &= \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k}) - \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})) - D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k} \\ &\quad + D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))(\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{a}) - D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}) \ , \end{aligned}$$

pelo que, uma vez que $D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))$ é linear,

$$\begin{aligned}\frac{\boldsymbol{X}}{||\boldsymbol{h}||} &= \frac{\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))-D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k}}{||\boldsymbol{h}||} \\ &\quad + D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\left(\frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})-D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||}\right) .\end{aligned}$$

Como $\boldsymbol{f}$ tem derivada $D\boldsymbol{f}(\boldsymbol{a})$ em $\boldsymbol{a}$ e $D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))$ é uma função contínua, o limite da segunda parcela é $\mathbf{0}$, pelo que resta demonstrar que

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{h}||} = \mathbf{0} ,$$

onde $\boldsymbol{Y}(\boldsymbol{k}) = \boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))-D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k}$. Tem-se

$$\frac{||\boldsymbol{Y}(\boldsymbol{k})||}{||\boldsymbol{h}||} = \frac{||\boldsymbol{Y}(\boldsymbol{k})||}{||\boldsymbol{k}||}\frac{||\boldsymbol{k}||}{||\boldsymbol{h}||} .$$

Notando que temos os seguintes limites devido à diferenciabilidade de $\boldsymbol{g}$ em $\boldsymbol{f}(\boldsymbol{a})$ e à continuidade de $\boldsymbol{f}$ em $\boldsymbol{a}$,

$$\begin{aligned}\lim_{\boldsymbol{k}\to\mathbf{0}} \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||} &= \lim_{\boldsymbol{k}\to\mathbf{0}} \frac{\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a})+\boldsymbol{k})-\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))-D\boldsymbol{g}(\boldsymbol{f}(\boldsymbol{a}))\boldsymbol{k}}{||\boldsymbol{k}||} \\ &= \mathbf{0} , \\ \lim_{\boldsymbol{h}\to\mathbf{0}} \boldsymbol{k} &= \lim_{\boldsymbol{h}\to\mathbf{0}} \boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a}) \\ &= \mathbf{0} ,\end{aligned}$$

conclui-se que

$$\lim_{\boldsymbol{h}\to\mathbf{0}} \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||} = \mathbf{0} .$$

Por outro lado,

$$\begin{aligned}\frac{\boldsymbol{k}}{||\boldsymbol{h}||} &= \frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})}{||\boldsymbol{h}||} \\ &= \frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})-D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}+D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||} \\ &= \frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})-D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||} + D\boldsymbol{f}(\boldsymbol{a})\frac{\boldsymbol{h}}{||\boldsymbol{h}||} ,\end{aligned}$$

pelo que

$$\begin{aligned}\frac{||\boldsymbol{k}||}{||\boldsymbol{h}||} &\leq \left\|\frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})-D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||}\right\| + \left\|D\boldsymbol{f}(\boldsymbol{a})\frac{\boldsymbol{h}}{||\boldsymbol{h}||}\right\| \\ &\leq \left\|\frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h})-\boldsymbol{f}(\boldsymbol{a})-D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||}\right\| + ||D\boldsymbol{f}(\boldsymbol{a})|| ,\end{aligned}$$

onde $||D\boldsymbol{f}(\boldsymbol{a})||$ é a *norma* da transformação linear:

$$||D\boldsymbol{f}(\boldsymbol{a})|| = \sup\{||D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{u}|| : ||\boldsymbol{u}|| = 1\} \ .$$

Conclui-se assim, usando a diferenciabilidade de $\boldsymbol{f}$ em $\boldsymbol{a}$, que

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||}\frac{||\boldsymbol{k}||}{||\boldsymbol{h}||} = \boldsymbol{0} \ ,$$

pois tem-se

$$\begin{aligned} \left\| \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||}\frac{||\boldsymbol{k}||}{||\boldsymbol{h}||} \right\| \ &\leq \ \left\| \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||} \right\| \left\| \frac{\boldsymbol{f}(\boldsymbol{a}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{a}) - D\boldsymbol{f}(\boldsymbol{a})\boldsymbol{h}}{||\boldsymbol{h}||} \right\| \\ &\quad + \left\| \frac{\boldsymbol{Y}(\boldsymbol{k})}{||\boldsymbol{k}||} \right\| ||D\boldsymbol{f}(\boldsymbol{a})|| \end{aligned}$$

e ambas as parcelas da direita têm $\lim_{\boldsymbol{h}\to\boldsymbol{0}}$ nulo.