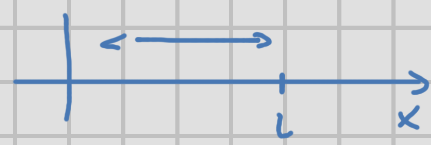
Aula 1 : 20 set. 2022

$\boxed{\text{Pr. } 1.1}$ $\quad m \longrightarrow$



Energia : $[E, E+\delta E]$

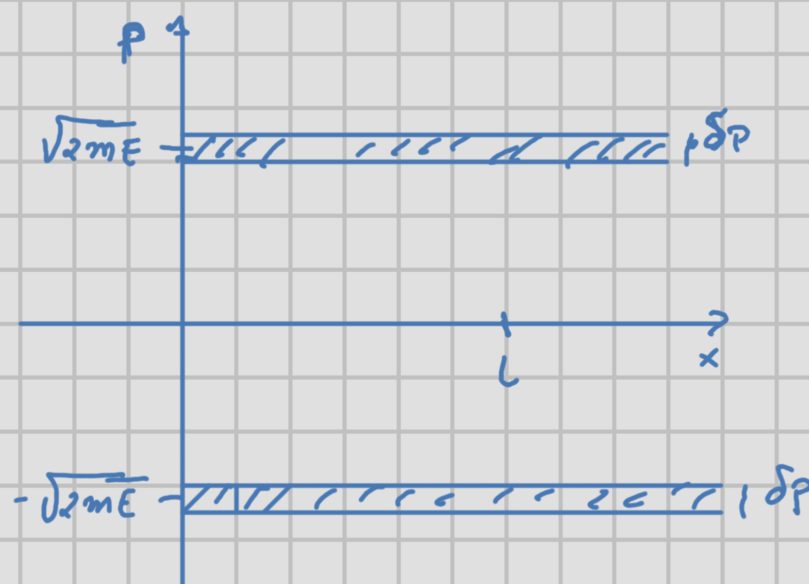
Espaço de fases x, p ?

$$p = mv \Rightarrow K = \frac{1}{2} m v^2 \quad \text{(energia cinética)}$$

$$K = \frac{1}{2} \frac{p^2}{m} = E \Rightarrow p = \sqrt{2mE}$$

$$\delta E \rightarrow dp = \frac{\partial p}{\partial E} dE = \frac{1}{\sqrt{2mE}} \cdot 2m \cdot dE = \sqrt{\frac{2m}{E}} \cdot dE$$

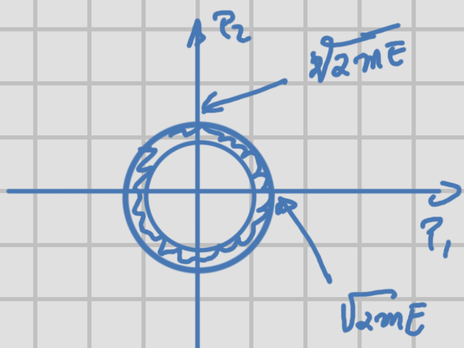
$$\delta p = \frac{\sqrt{2m}}{\sqrt{E}} \delta E \Rightarrow$$

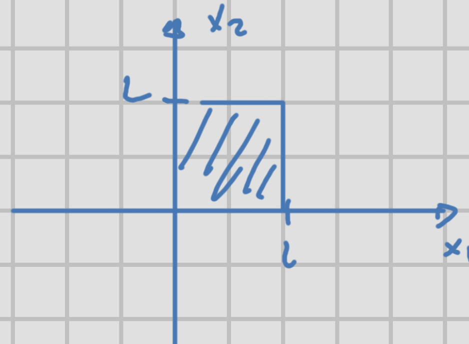


$\boxed{1.2}$ $x_1, x_2, p_1, p_2 \rightarrow$ partículas independentes

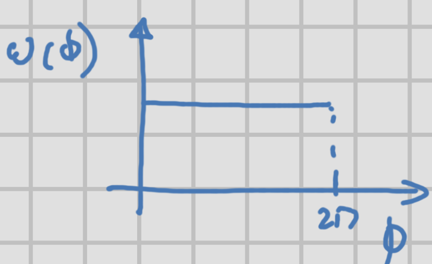
$$E_{TOT} = E_1 + E_2 = k_1 + k_2 = \frac{1}{2}\frac{p_1^2}{m} + \frac{1}{2}\frac{p_2^2}{m}$$

$$2mE = p_1^2 + p_2^2 \rightarrow \text{círculo de raio } \sqrt{2mE}$$





$\boxed{1.3}$ $x = A\cos(\omega t + \phi)$



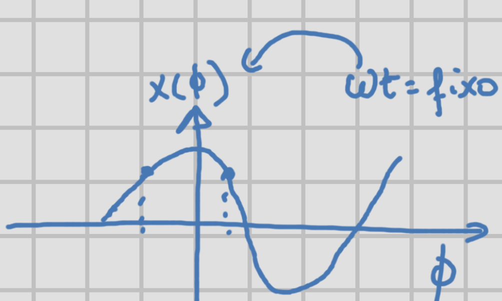
$$\omega(\phi) = \frac{1}{2\pi} \Rightarrow \omega(\phi)\,d\phi = \frac{1}{2\pi}d\phi \quad : \text{Probabilidade para } \phi$$

Por ϕ em função de x

$$\left\{ \begin{array}{l} \cos(\omega t + \phi) = \frac{x}{A} \\ \omega t + \phi = \arccos\frac{x}{A} \Rightarrow \phi = \arccos\frac{x}{A} - \omega t \end{array} \right.$$

$\omega(\phi)\,d\phi \rightarrow$ mudança de variável

$$\omega(\phi(x))\,d\phi(x) = \omega(\phi(x))\left|\frac{d\phi}{dx}\right| \cdot dx \cdot 2 = \frac{1}{\cancel{2}\pi}\left|\frac{d\phi}{dx}\right| dx \cdot \cancel{2} \Rightarrow$$



Para cada valor de x, dois valores de ϕ!

$$\frac{d\phi}{dx} = \frac{d}{dx}\left[\arccos\frac{x}{A} - \omega t\right] = -\frac{1}{\sqrt{1-(x/A)^2}}\cdot\frac{1}{A} =$$

$$= -\frac{1}{\sqrt{A^2 - x^2}}$$

$$\boxed{P(x)\,dx = \frac{1}{\pi}\,\frac{1}{\sqrt{A^2 - x^2}}\,dx}$$

b) $[E, E+\delta E] \to$ energia do oscilador

$$E = \frac{p^2}{2m} + \frac{1}{2}\,\omega^2 m x^2$$

Elipse:

Área $= \pi a b$

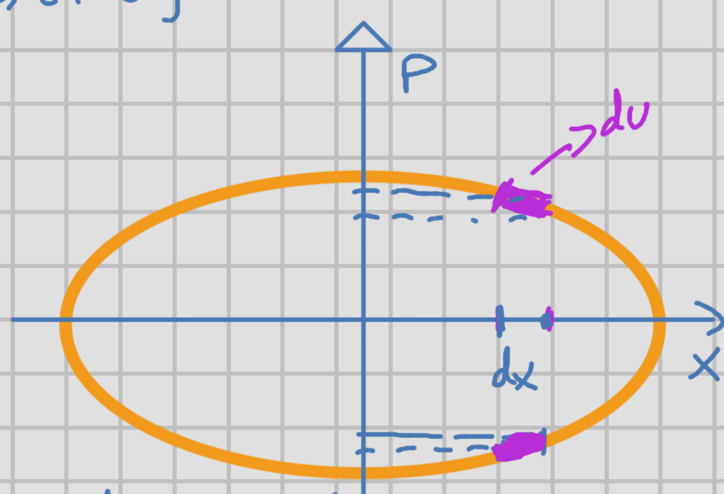
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Vamos a calcular $P(x)dx$ como

$$P(x)\,dx = \frac{\delta V}{V}$$

$\delta V \to$ Volume no espaço de fases com energia entre $[E, E+\delta E]$ e x entre $[x, x+dx]$

$V \to$ Volume no espaço de fases com energia entre $[E, E+\delta E]$



$$1 = \frac{1}{2mE}\,p^2 + \frac{\omega^2 m}{2E}\,x^2$$

$$\text{Área} = \pi\cdot\sqrt{2\cancel{m}E}\cdot\sqrt{\frac{2E}{\omega^2\cancel{m}}} = \frac{2\pi E}{\omega}$$

largura infinitesimal $\delta E \Rightarrow$ área infinitesimal

$$\delta A = \frac{\partial A}{\partial E}\cdot\delta E = \frac{2\pi}{\omega}\,\delta E = V$$

$\delta V = 2\,dx\,dp$

$dp \to$ por em função de δE

$$p(E,x) = \sqrt{2mE - \omega^2 m^2 x^2}$$

$$\delta p = \frac{\partial p}{\partial E}\,\delta E = \frac{1}{\cancel{2}\sqrt{2mE - \omega^2 m^2 x^2}} \cdot \cancel{2}m \cdot \delta E$$

$$\delta V = \frac{2m}{\sqrt{2mE - \omega^2 m^2 x^2}}\,\delta E\,dx$$

$$P(x)\,dx = \frac{\delta V}{V} = \frac{\cancel{2}m}{\sqrt{2mE - m^2\omega^2 x^2}}\,\cancel{\delta E} \cdot dx \cdot \frac{1}{\frac{\cancel{2}\pi}{\omega}\cancel{\delta E}} =$$

$$= \frac{m\omega}{\pi\sqrt{2mE - m^2\omega^2 x^2}}\,dx =$$

$$= \frac{1}{\pi}\,\frac{1}{\sqrt{\frac{2E}{m\omega^2} - x^2}}\,dx$$

$x = x_{max} \Rightarrow p = 0 \Rightarrow E = \frac{1}{2}\omega^2 m A^2 \Rightarrow A^2 = \frac{2E}{\omega^2 m}$

Pelo tanto: $P(x)\,dx = \frac{1}{\pi}\,\frac{1}{\sqrt{A^2 - x^2}}\,dx$ // c.q.d.

$\boxed{1.4}$ $N_1, N_2 \rightarrow E = -(n_1 - n_2)\mu H$

$\uparrow \mu$

spin +	spin −
$\uparrow$	$\downarrow$
$E = -\mu H$	$E = +\mu H$
$\downarrow$	$\downarrow$
n_1	n_2

a) $\Omega(E) = ?$

$N = n_1 + n_2 \Rightarrow n_2 = N - n_1$

$E = -[n_1 - (N - n_1)] \cdot \mu H = 2n_1 - N)\mu H = (N - 2n_1)\mu H$

n_1 em função de E: $n_1 = \frac{1}{2}\left(N - \frac{E}{\mu H}\right)$

$\Omega(E) \rightarrow$ contar o nº de possíveis microestados com n_1 partículas com spin + e $N - n_1$ com spin − .

↳ Nº de possíveis formas de escolher n_1 partículas em N sítios $\rightarrow n_1!$

$\longrightarrow$

— — — — . . . . — $\to N$ átomos.

↳ Primeiro spin +: há N lugares a escolher

— — $\overset{\uparrow}{—}$ — . . . . —

↳ Segundo spin +: $(N-1)$ lugares

⋮

Spin + nº n_1 : $N-(n_1-1)$ lugares

Total de combinações

$N(N-1)\cdots(N-(n_1-1))$

$\downarrow$

$$\frac{N!}{(N-n_1)!}$$

Mas... a localização dos átomos da rede = fixa $\Rightarrow$ temos de descontar as permutações dos n_1 elementos (dão exactamente a mesma configuração $\to$ equivalente a partículas idênticas) $\to n_1!$

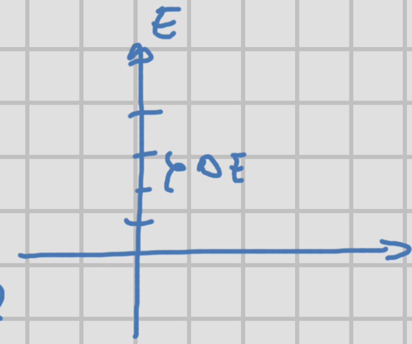
Total de combinações: $\dfrac{N!}{(N-n_1)!\,n_1!} = \Omega$

Pôr n_1 em função de E: $N-n_1 = N - \frac{1}{2}\left(N - \frac{E}{\mu H}\right) =$

$$= \frac{N}{2} + \frac{E}{2\mu H} = \frac{1}{2}\left(N + \frac{E}{2\mu H}\right)$$

$$\Omega(E) = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu H}\right)!\left(\frac{N}{2} + \frac{E}{2\mu H}\right)!}$$

$\delta E \gg \mu H$



Separação entre níveis de energia

↳ $\Delta E = 2\mu H$ (i.e. variação da energia total do sistema quando um spin muda a sua orientação)

Nº de níveis de energia correspondentes a δE : $\frac{\delta E}{\Delta E} = \frac{\delta E}{2\mu H}$

$$\boxed{\Omega(E)\,dE = \frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu H}\right)!\left(\frac{N}{2} + \frac{E}{2\mu H}\right)!}\,\frac{\delta E}{2\mu H}}$$

b) $\ln \Omega(E)$?

$$\ln \Omega(E) = \ln\left(\frac{N!}{\left(\frac{N}{2} - \frac{E}{2\mu H}\right)!\left(\frac{N}{2} + \frac{E}{2\mu H}\right)!}\right) =$$

$$= \ln N! - \ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right)! - \ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right)! =$$

$$\simeq N\ln N - \cancel{N} - \left[\left(\frac{N}{2} - \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) - \cancel{\frac{N}{2}} + \cancel{\frac{E}{2\mu H}}\right] +$$

$$- \left[\left(\frac{N}{2} + \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right) - \cancel{\frac{N}{2}} - \cancel{\frac{E}{2\mu H}}\right] =$$

$$= N\ln N - \left(\frac{N}{2} - \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} - \frac{E}{2\mu H}\right) - \left(\frac{N}{2} + \frac{E}{2\mu H}\right)\ln\left(\frac{N}{2} + \frac{E}{2\mu H}\right)$$

c) Perceber a forma de $\ln \Omega(E)$ → casos específicos:

$E=0$ → tantos spins + como −

$$\ln \Omega(E=0) = N\ln N - \frac{N}{2}\ln\frac{N}{2} - \frac{N}{2}\ln\frac{N}{2} = N\left(\ln N - \ln\frac{N}{2}\right) =$$
$$= N\ln\left(\frac{N}{N/2}\right) = N\ln 2 \iff \Omega = 2^N$$

$E = +N\mu H$ → todos os spins alinhados → 1 possível combinação

$E = -N\mu H$ → 1 possível combinação

↳ Valores mínimos de $\Omega(E)$

Para E próximo ao máximo → expandir em série de Taylor:

$$\ln\Omega(E) = \ln\Omega(E_0) + \left.\frac{\partial \ln\Omega(E_0)}{\partial E}\right|_{E=0} + \frac{1}{2}\left.\frac{\partial^2 \ln\Omega}{\partial E^2}\right|_{E=E_0}(E-E_0)^2 + \cdots$$

$$\frac{\partial \ln\Omega(E)}{\partial E} = -\frac{1}{2\mu H}\ln\left(\frac{N}{2}+\frac{E}{2\mu H}\right) - \cancel{\left(\frac{N}{2}+\frac{E}{2\mu H}\right)\cdot\frac{1}{\frac{N}{2}+\frac{E}{2\mu H}}}\cdot\frac{1}{2\mu H} +$$
$$+\frac{1}{2\mu H}\ln\left(\frac{N}{2}-\frac{E}{2\mu H}\right) + \cancel{\left(\frac{N}{2}-\frac{E}{2\mu H}\right)\frac{1}{\frac{N}{2}-\frac{E}{2\mu H}}}\cdot\frac{1}{2\mu H} \Longrightarrow$$

$$\frac{\partial \ln \Omega}{\partial E} = \frac{1}{2\mu H} \underbrace{\ln\left[\frac{N - \frac{E}{\mu H}}{N + E/\mu H}\right]}_{\to\; = 0 \text{ quando } E=0 \text{ (i.e. } \ln 1)}$$

$$\frac{\partial^2 \ln \Omega}{\partial E^2} = \frac{1}{2\mu H}\left[\frac{1}{N - \frac{E}{\mu H}}\cdot\left(-\frac{1}{\mu H}\right) - \frac{1}{N + \frac{E}{\mu H}}\left(\frac{1}{\mu H}\right)\right] =$$

$$= \frac{-1}{2\mu H}\left[\frac{1}{N\mu H - E} + \frac{1}{N\mu H + E}\right] = -\frac{1}{\cancel{2}\cancel{\mu H}}\left[\frac{\cancel{2} N \cancel{\mu H}}{(N\mu H)^2 - E^2}\right]$$

$$\frac{\partial^2 \ln \Omega}{\partial E^2} = -\frac{N}{(N\mu H)^2 - E^2}$$

$$\ln \Omega(E) \simeq N \ln 2 - \frac{1}{2}\frac{\cancel{N}}{N^{\cancel{2}}\mu^2 H^2}\cdot E^2 = N \ln 2 - \frac{1}{2N}\left(\frac{E}{\mu H}\right)^2$$

Expansão entorno de $E=0$

$$\Omega(E) \simeq e^{N \ln 2} \cdot e^{-\frac{1}{2N}\left(\frac{E}{\mu H}\right)^2} = 2^N \cdot e^{-\frac{1}{2N}\left(\frac{E}{\mu H}\right)^2}$$

$$\boxed{\Omega(E) = 2^N \cdot e^{-\frac{1}{2N}\left(\frac{E}{\mu H}\right)^2}}$$