ECPD - Solution to problems

Revised 6/July/2022

P1 $y(2) = y(1) + y(0) = 1 + 1 = 2$

$y(3) = y(2) + y(1) = 2 + 1 = 3$

$y(4) = y(3) + y(2) = 3 + 2 = 5$

$y(5) = y(4) + y(3) = 5 + 3 = 8$

$y(6) = y(5) + y(4) = 8 + 5 = 13$

$\boxtimes$

P2 $\lambda^{k+2} - 1,3\lambda^{k+1} + 0,4\lambda^{k} = 0$

since $\lambda \neq 0$, divide the equation by λ^k

$\lambda^2 - 1,3\lambda + 0,4 = 0$

$$\lambda_{1,2} = \frac{1,3 \pm \sqrt{1,3^2 - 1,6}}{2} = \begin{cases} \dfrac{1,3 + 0,3}{2} \\ \dfrac{1.3 - 0,3}{2} \end{cases}$$

$\lambda_1 = 0,8 \quad \lambda_2 = 0,5$

General solution

$y(k) = \alpha\, 0,8^k + \beta\, 0,5^k$

Coefficients α and β are found from the initial equations

$$\begin{cases} \alpha+\beta=1 & \text{(for } k=0) \\ 0{,}8\alpha+0{,}5\beta=0 & \text{(for } k=1) \end{cases}$$

$\beta=-1{,}6\alpha$

$-0.6\alpha=1 \longrightarrow \alpha=-1{,}67$

$\beta=1-\alpha=2{,}67$

solution

$y(k)=-1{,}67\times 0{,}8^k+2{,}67\times 0{,}5^k$ $\boxtimes$

P3 Assuming $y(k)=\lambda^k$, it follows that

$$\lambda^{k+2}-\lambda^{k+1}-\lambda^k=0$$

or, since $\lambda\neq 0$

$$\lambda^2-\lambda-1=0$$

This is the "characteristic equation". Its solutions are

$$\lambda_1=\frac{1+\sqrt{5}}{2}, \quad \lambda_2=\frac{1-\sqrt{5}}{2}$$

the general solution is

$$y(k)=\alpha\lambda_1^k+\beta\lambda_2^k$$

Using the initial conditions and writting the solution for $k=1$ and $k=2$, yields

$$\begin{cases} \alpha\lambda_1 + \beta\lambda_2 = 1 \\ \alpha\lambda_1^2 + \beta\lambda_2^2 = 1 \end{cases}$$

that has the solution

$$\alpha = \frac{1}{\sqrt{5}}, \quad \beta = -\frac{1}{\sqrt{5}}$$

therefore, the general term is

$$y(k) = \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]\frac{1}{\sqrt{5}}$$

It is interesting to remark that, although involving the irrational numbers, the values of $y(k)$ are always integer numbers.

$\boxtimes$

P4 Let $y^i(k)$ be the solution that corresponds to the input $u^i(k)$, for $i = 1, 2$.

Hence

$$y^1(k+2) + a_1 y^1(k+1) + a_2 y^1(k) =$$
$$= b_0 u^1(k+1) + b_1 u^1(k)$$
$$y^2(k+2) + a_1 y^2(k+1) + a_2 y(k) =$$
$$= b_0 u^2(k+1) + b_1 u^2(k)$$

By adding both these equations and regrouping

$$\left[y^1(k+2) + y^2(k+1)\right] + a_1\left[y^1(k+1) + y^2(k+1)\right] +$$
$$+ a_2\left[y^1(k) + y^2(k)\right] =$$
$$= b_0\left[u^1(k+1) + u^2(k+1)\right] + b_1\left[u^1(k) + u^2(k)\right]$$

that shows that $y^1(k) + y^2(k)$ is the solution that corresponds to $u^1(k) + u^2(k)$, i.e., the system verifies the principle of superposition and, hence, is linear.

$\square$

P5 a) Apply the z-transform with zero initial conditions

$$Y - 0{,}5 z^{-1} Y + z^{-2} Y = 2 z^{-4} U + z^{-5} U$$

Multiply by z^5

$$G(z) = \frac{Y(z)}{U(z)} = \frac{2z + 1}{z^5 - 0{,}5 z^4 + z^3}$$

b) the zeros are the roots of

$2z + 1 \longrightarrow z_1 = -0{,}5$

the poles are the roots of

$z^5 - 0{,}5 z^4 + z^3 = (z^2 - 0{,}5 z + 1) z^3$

there are 3 poles at 0 and 2 poles at the roots of $z^2 - 0{,}5 z + 1$

$$z_{1,2} = \frac{0{,}5 \pm \sqrt{0{,}25 - 4}}{2}$$

$z_1 = 0{,}25 + j\,0{,}97$

$z_2 = 0{,}25 - j\,0{,}97$

c)

$$U \longrightarrow \boxed{\dfrac{1}{z^5 - 0{,}5z^4 + z^3}} \xrightarrow{X_1} \boxed{2z+1} \xrightarrow{Y}$$

there are 5 state Variables (because the order of the system is 5).

$x_2(k) = x_1(k+1)$ $\qquad X_2 = zX_1$

$x_3(k) = x_2(k+1)$ $\qquad X_3 = z^2X_1$

$x_4(k) = x_3(k+1)$ $\qquad X_4 = z^3X_1$

$x_5(k) = x_4(k+1)$ $\qquad X_5 = z^4X_1$

$X_1(z^5 - 0{,}5z^4 + z^3) = U$

$z(z^4X_1) = 0{,}5z^4X_1 - z^3X_1 + U$

$x_5(k+1) = 0{,}5x_5(k) - x_4(k) + u(x)$

the state model is

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = x_3(k) \\ x_3(k+1) = x_4(k) \\ x_4(k+1) = x_5(k) \\ x_5(k+1) = -x_4(k) + 0{,}5x_5(k) + u(k) \end{cases}$$

$Y = X_1 + 2zX_1$

$y(k) = x_1(k) + 2x_2(k)$

In matrix form

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \\ x_4(k+1) \\ x_5(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0{,}5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \\ x_4(k) \\ x_5(k) \end{bmatrix}$$

$\boxtimes$

P6 a)

$$y(k) + a_1 y(k-1) + a_2 y(k-2) = \\ = b_0 u(k-1) + b_1 (k-2)$$

b)

$$G(z) = \frac{Y}{U} = \frac{b_0 z + b_1}{z^2 + a_1 z + a_2}$$

c) $$\xrightarrow{u} \boxed{\frac{1}{z^2 + a_1 z + a_2}} \xrightarrow{X_1} \boxed{b_0 z + b_1} \xrightarrow{Y}$$

$$x_2(k) := x_1(k+1)$$

$$Z(ZX_1) = -a_2 X_1 - a_1 Z X_1 + U$$

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -a_2 x_1(k) - a_1 x_2(k) + u(k) \end{cases}$$

$$Y = b_1 X_1 + b_0 Z X_2$$

$$y(k) = b_1 x_1(k) + b_0 x_2(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

P7 A, C, D are responses of 1st order systems with poles between 0 and 1. Hence, they correspond to 1, 2 and 5

$G_1(z)$ has a static gain of $\frac{1}{1-0{,}5} = 2$, and hence it corresponds to C.

D is slower, and hence it corresponds to 2 (pole closer to 1), and A corresponds to 5.

G_4 has a negative pole at $-0,5$ and hence has a fast oscillatory response such as B.

E has an oscillatory response that corresponds to G_3

1 - C

2 - D

3 - E

4 - B

5 - A

$\boxtimes$

P8 a) Poles

$$z^2 - 1,5\,z + 0,56 = 0$$

$$z_{1,2} = \frac{1,5 \pm \sqrt{1,5^2 - 4 \times 0,56}}{2}$$

$z_1 = 0,8 \qquad z_2 = 0,7$

Since $|z_1| < 1$ and $|z_2| < 1$,

both poles are inside the unit circle and G_1 is asymptotically stable.

b) $y(k+2) - 1,5\, y(k+1) + 0,56\, y(k) = u(k+1) - 0,5\, u(k)$.

c) $U \longrightarrow \boxed{\dfrac{1}{z^2 - 1,5z + 0,56}} \xrightarrow{X_1} \boxed{z - 0,5} \xrightarrow{Y}$

$x_2(k) := x_1(k+1)$

$z(z X_1) = 1,5\, z X_1 - 0,56\, X_1 + U$

$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = -0,56\, x_1(k) + 1,5\, x_2(k) + u(k) \end{cases}$$

$y(k) = -0,5\, x_1(k) + x_2(k)$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -0,56 & 1,5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} -0,5 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

d) State realization of G_2

$$x_3(k+1) = 0{,}9\, x_3(k) + 2 u_2(k)$$

State realization of the series

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -0{,}56 & 1{,}5 & 0 \\ -1 & 2 & 0{,}9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(k)$$

Observe that

$$u_2(k) = y_1(k) = -0{,}5\, x_1(k) + x_2(k)$$

and hence

$$x_3(k+1) = -x_1(k) + 2 x_2(k) + 0{,}9\, x_3(k)$$

$\square$

<u>**P9**</u> a) $Y(z) = \dfrac{z}{z^2 - 0{,}5 z - 0{,}5} U(z)$

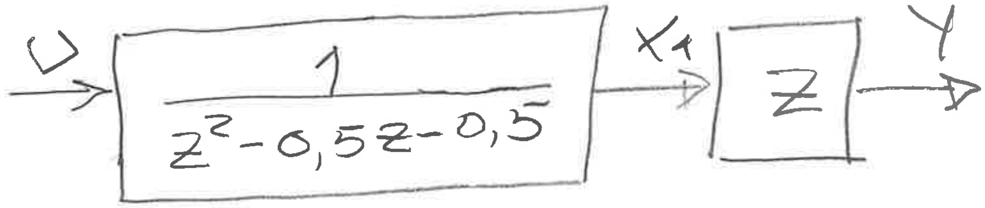
$$y(k+2) - 0{,}5\, y(k+1) - 0{,}5\, y(k) = u(k+1)$$

b) $y(2) = 0{,}5\, \underset{\overset{\shortparallel}{0}}{y(1)} + 0{,}5\, \underset{\overset{\shortparallel}{0}}{y(0)} + 1 = 1$

$y(3) = 0{,}5\, \underset{\overset{\shortparallel}{1}}{y(2)} + 0{,}5\, \underset{\overset{\shortparallel}{0}}{y(1)} + 1 = 1.5$

$y(4) = 0{,}5\, \underset{\overset{\shortparallel}{1{,}5}}{y(3)} + 0{,}5\, \underset{\overset{\shortparallel}{1}}{y(2)} + 1 = 2{,}25$

c)



$$x_2(k) := x_1(k+1)$$

$$z(z X_1) = 0,5 z X_1 + 0,5 X_1 + U$$

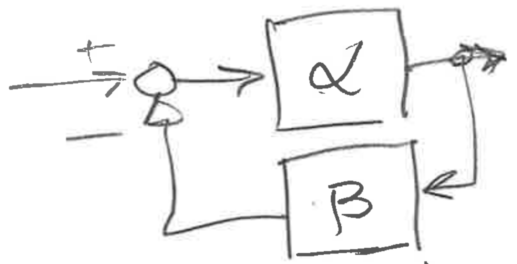
$$\begin{cases} x_1(k+1) = x_2(k) \\ x_2(k+1) = 0,5 x_1(k) + 0,5 x_2(k) + u(k) \end{cases}$$

$$Y = z X_1 \longrightarrow y(k) = x_2(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0,5 & 0,5 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

d) Use the rule



$$\Longleftrightarrow \quad \frac{\alpha}{1+\alpha\beta}$$

Closed-loop transfer function

$$G_{CL}(z) = \frac{\dfrac{Kz}{(z-1)(z+0,5)}}{1 + \dfrac{Kz}{(z-1)(z+0,5)}} =$$

$$= \frac{kz}{(z-1)(z+0,5)+kz}$$

e) the static gain is obtained for $z=1$

Static gain $= G_{CL}(1) = 1$

provided that the closed-loop is stable.

f) the poles of the closed-loop system are the roots of

$$(z-1)(z+0,5)+0,5z=0$$

$$z^2-0,5=0$$

$$(z+0,707)(z-0,707)=0$$

$$\lambda_1=-0,707$$

$$\lambda_2=0,707$$

Since $|\lambda_1|<1$ and $|\lambda_2|<1$, all the closed-loop poles are inside the unit circle and therefore the closed-loop system is asymptotically stable.

$\boxed{/}$

P 10 a) The system has the same number of poles and zeros and hence the matrix $D \neq 0$.

$$\begin{array}{l} z^2 - 0{,}64 \\ z^2 - 0{,}25 \\ \hline -0{,}39 \end{array} \qquad \begin{array}{|l} z^2 - 0{,}25 \\ \hline 1 \end{array}$$

$$\frac{(z-0{,}8)(z+0{,}8)}{(z-0{,}5)(z+0{,}5)} = 1 + \frac{-0{,}39}{z^2 - 0{,}25} \quad (*)$$

$$\begin{cases} X_1 = \dfrac{-0{,}39}{z^2 - 0{,}25} U \\ X_2 = z X_1 \end{cases}$$

$$z(zX_1) = 0{,}25 X_1 - 0{,}39 U$$

$x_1(k+1) = x_2(k)$

$x_2(k+1) = 0{,}25 x_1(k) - 0{,}39 u$

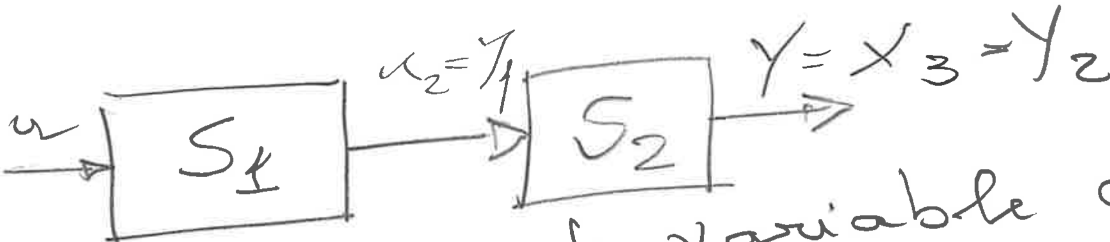
From (*) above

$y(k) = x_1(k) + u(k)$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0,25 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -0,39 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + u(k)$$

b)



x_3 is the state variable of S_2

$$X_3 = Y_2 = \frac{3}{z-0,9} U_2$$

$$x_3(k+1) = 0,9\, x_3(k) + 3\, u_2(k)$$

$$u_2(k) = x_1(k) + u(k)$$

$$x_3(k+1) = 3x_1(k) + 0,9\, x_3(k) + 3\, u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0,25 & 0 & 0 \\ 3 & 0 & 0,9 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + \begin{bmatrix} 0 \\ -0,39 \\ 3 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix}$$

P11 $$x(1) = A\underbrace{x(0)}_{=0} + b\underbrace{u(0)}_{=1}$$

$$x(1) = b$$
$$y(1) = c\,x(1) = cb$$

$$x(2) = A\underbrace{x(1)}_{=b} + b\underbrace{u(1)}_{=0}$$

$$x(2) = Ab \qquad y = cAb$$

From this time on $u(k) = 0$ and hence $x(k+1) = Ax(k)$,

Hence
$$x(3) = Ax(2) = A^2 b$$
$$y(3) = c\,x(3) = cA^2 b$$

In general
$$y(0) = 0$$
$$y(1) = cb$$
$$y(2) = cAb$$
$$y(3) = cA^2 b$$
$$\cdots$$
$$y(k) = cA^{k-1} b$$

$\square$

P12

$$\min_{x_1,x_2} x_1^2 + x_2^2$$

$$\text{s.t. } (x_1-1)^2 + (x_2-1)^2 = 1$$

$$\mathcal{L} = x_1^2 + x_2^2 + \lambda\left((x_1-1)^2 + (x_2-1)^2 - 1\right)$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = \cancel{2}x_1 + \cancel{2}\lambda(x_1-1) = 0$$

$$x_1(1+\lambda) = \lambda \rightarrow x_1 = \frac{\lambda}{1+\lambda}$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = \cancel{2}x_2 + \cancel{2}\lambda(x_2-1) = 0$$

$$x_2 = \frac{\lambda}{1+\lambda}$$

From the constraint

$$\frac{\lambda}{1+\lambda} - 1 = \frac{\lambda-(1+\lambda)}{1+\lambda} = \frac{-1}{1+\lambda}$$

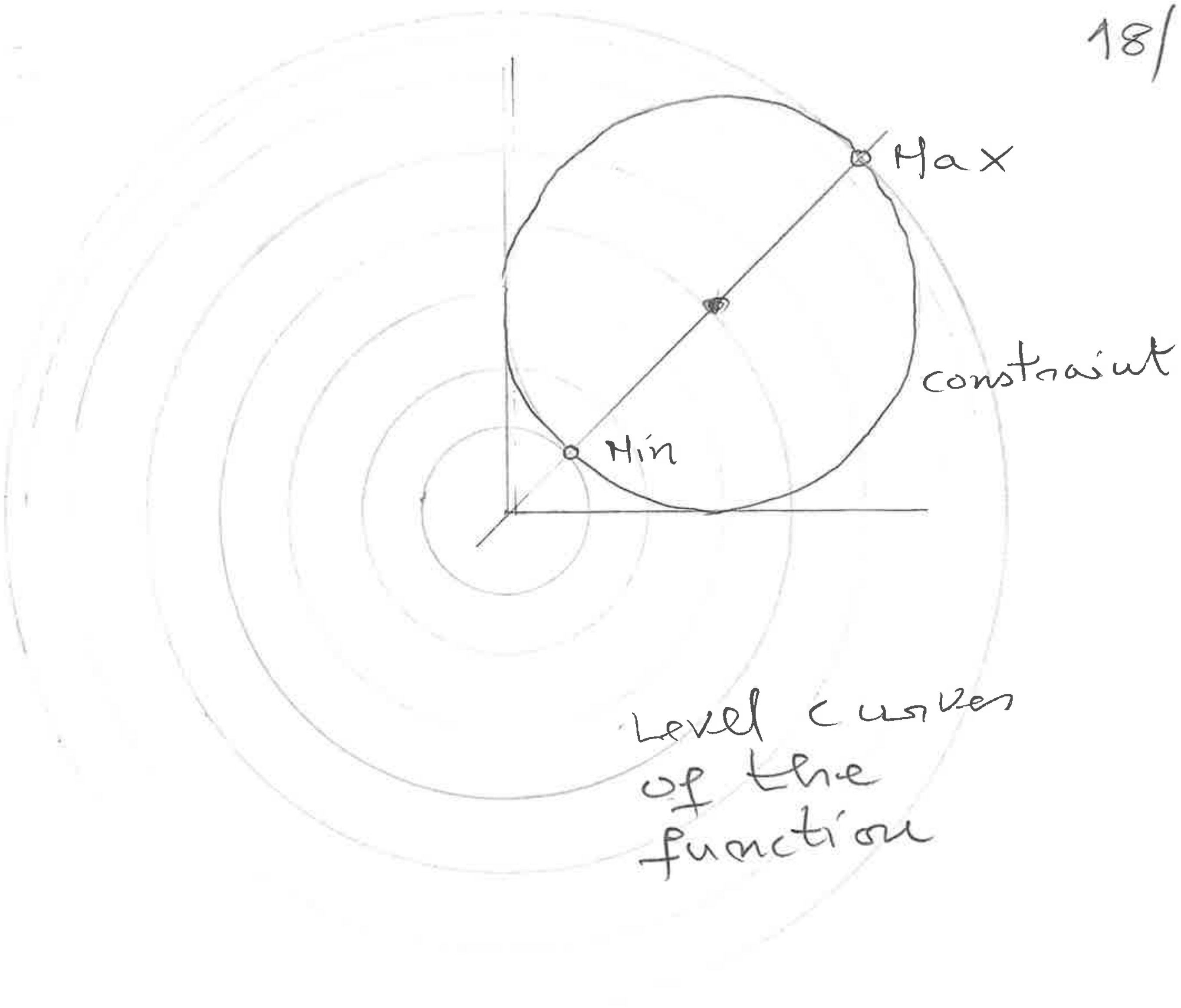
$$\frac{1}{(1+\lambda)^2} + \frac{1}{(1+\lambda)^2} = 1 \rightarrow \frac{2}{(1+\lambda)^2} = 1$$

$$(1+\lambda)^2 = 2 \rightarrow 1 + 2\lambda + \lambda^2 = 2$$

$$\lambda = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}$$

$$x_1 = x_2 = \frac{-1+\sqrt{2}}{1-1+\sqrt{2}} = \frac{\sqrt{2}-1}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}} \simeq 0{,}293$$

$$x_1 = x_2 = \frac{-1-\sqrt{2}}{\cancel{1-}1-\sqrt{2}} = \frac{1+\sqrt{2}}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} \simeq 1{,}71$$


Max
constraint
Min
Level curves
of the
function

P 13

$$\mathcal{L} = (Ax-b)^T(Ax-b) + \lambda^T(Gx-h)$$
$$= x^T A^T A x - 2b^T A x + b^T b + \lambda^T G x - \lambda^T h$$
$$= x^T A^T A x + (\lambda^T G - 2b^T A)x + b^T b - \lambda^T h$$

$$\nabla_x \mathcal{L} = 2x^T A^T A + \lambda^T G - 2b^T A = 0$$

Transposing and dividing by 2:

$$A^T A x + \tfrac{1}{2} G^T \lambda - A^T b = 0$$

$$x = -(A^T A)^{-1}\left[\tfrac{1}{2} G^T \lambda - A^T b\right] \qquad (*)$$

From the constraint $Gx - h = 0$

$$G(A^T A)^{-1}\left[\tfrac{1}{2} G^T \lambda - A^T b\right] + h = 0$$

$$\lambda = 2\left[G(A^T A)^{-1} G^T\right]^{-1}\left[-G(A^T A)^{-1} A^T b + h\right]$$

Insert this expression for λ in $(*)$

$$x = -(A^T A)^{-1}\left\{ G^T \left[G(A^T A)^{-1} G^T\right]^{-1}\left[-G(A^T A)^{-1} A^T b + h\right] - A^T b\right\}$$

$\square$

P14 a) Let k denote the present time. The value of the control to apply to the plant at this time is obtained by solving the optimization problem

$$\min_{\hat{u}_k^{k+H-1}} \sum_{i=1}^{H} \left[\hat{x}^T(k+i|k)\, Q\, \hat{x}(k+i|k) + \rho\, \hat{u}^2(k+i-1|k) \right]$$

s.t.

$$\hat{x}(k+i|k) = f\left(\hat{x}(k+i-1|k), \hat{u}(k+i-1|k)\right)$$

$$i = 1, \dots, H$$

$$\hat{x}(k|k) = x(k)$$

Of the resulting optimal sequence $\hat{u}_k^{k+H-1}$ only apply to the plant the first entry, i.e. make $u(k) = \hat{u}(k|k)$ and repeat the whole procedure at $k+1$.

b) MPC is a closed-loop law 2 1/ since, at each discrete time k the control action depends on the measured state x_k.

$\boxed{/}$

P13 a)

$$x(k+1) = Ax(k) + b\,u(k)$$

$$x(k+2) = A^2 x(k) + Ab\,u(k) + b\,u(k+1)$$

$$x(k+i) = A^i x(k) + A^{i-1} b\,u(k) + \dots + b\,u(k+i-1)$$

$$y(k+i) = CA^i x(k) + CA^{i-1} b\,u(k) + \dots + cb\,u(k+i-1)$$

b)

$$\underbrace{\begin{bmatrix} y(k+1) \\ y(k+2) \\ \vdots \\ y(k+H) \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} cb & 0 & \cdots & 0 \\ CAb & cb & \ddots & \vdots \\ \vdots & & \ddots & 0 \\ CA^{H-1}b & \cdots & & cb \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+H-1) \end{bmatrix}}_{U} + \underbrace{\begin{bmatrix} CA \\ CA^2 \\ \vdots \\ CA^H \end{bmatrix}}_{\Pi} x(k)$$

$$Y = WU + \Pi x(k)$$

c) $J = (Y - Y^*)^T (Y - Y^*) + \rho U^T U$

$= (WU + \Pi x_k - Y^*)^T (WU + \Pi x_k - Y^*) +$

$+ \rho U^T U =$

$= U^T (W^T W + \rho I) U + 2 ((\Pi x_k - Y^*)^T W U)$

$+ (\Pi x_k - Y^*)^T (\Pi x_k - Y^*)$

$M := W^T W + \rho I \qquad M = M^T$

$\nabla_U J = 2U^T M + 2 (\Pi x_k - Y^*)^T W = 0$

Transpose, and solve with respect to U

$U = - M^{-1} W^T \Pi x_k + M^{-1} W^T Y^*$

d) $e_1^T = [1 \ 0 \ \ldots \ 0]$

$F = \quad e_1^T M^{-1} W^T \Pi$

$u_{ff}(k) = e_1^T M^{-1} W^T Y^*$

$\boxed{/}$

P16 a) the problem is formulated as

$$\min_U J = \sum_{i=1}^{H} \hat{y}^2(k+i) + \rho \hat{u}^2(k+i-1)$$

s. t. $\hat{x}(k+i) = A\hat{x}(k+i-1) + b\hat{u}(k+i-1)$

$i = 1, \dots, H$

$\hat{x}(k) = x(k)$

$\hat{x}(k+H|k) = 0$

$$U := \begin{bmatrix} u(k) \\ u(k+1) \\ \vdots \\ u(k+H-1) \end{bmatrix}$$

the H-steps ahead state predictor is

$$\hat{x}(k+H) = A^H x(k) + A^{H-1}b\,u(k) + A^{H-2}u(k+1) + \dots + b\,u(H-1)$$

or

$$\hat{x}(k+H) = \underbrace{\left[A^{H-1}b \mid A^{H-2}b \mid \dots \mid b\right]}_{:= G^T} U + A^H x(k)$$

Define

$$Y := \begin{bmatrix} \hat{y}(k+1) \\ \vdots \\ \hat{y}(k+H) \end{bmatrix}$$

Predictive model

$$Y = WU + \Pi x(k)$$

Cost

$$J = Y^T Y + \rho U^T U$$

The problem can be formulated as

$\min_U J$

s.t. $Y = WU + \Pi x(k)$

$G^T U + A^H x(k) = 0$

this problem is solved using Lagrange multipliers.

Lagrangian

$\mathcal{L} = Y^T Y + \rho U^T U + \lambda^T (G^T U + A^H x(k)) =$

$= (WU + \Pi x(k))^T (WU + \Pi x(k)) + \rho U^T U +$

$+ \lambda^T (G^T U + A^H x(k)) =$

$= U^T H U + 2 x^T(k) \Pi^T W U + x^T(k) \Pi^T \Pi x(k) +$

$+ \lambda^T (G^T U + A^H x(k))$

where $H := W^T W + \rho I$.

Gradient of the Lagrangian

$\nabla_U \mathcal{L} = 2U^T H + 2x^T(k) \Pi^T W + \lambda^T G^T$

Equate the Lagrangean to zero and transpose:

$HU + W^T \Pi x(k) + \frac{1}{2} \lambda^T G^T = 0$

$U = -H^{-1} \left[W^T \Pi x(k) + \frac{1}{2} G \lambda \right]$ (*)

Use the constraint to obtain an equation for λ

$G^T U + A^H x(k) = 0$

$G^T H^{-1} W^T \Pi x(k) + \frac{1}{2} G^T H^{-1} G \lambda = A^H x(k)$

Solve with respect to λ

$$\lambda = -2\left(G^T H^{-1} G\right)^{-1}\left[G^T H^{-1} W^T \Pi - A^H\right] x(k)$$

Insert in (*) to express the control in $x(k)$:

$$U = -H^{-1}\left\{W^T \Pi\, x(k) - G\left(G^T H^{-1} G\right)^{-1}\left[\ldots\right.\right.$$
$$\left.\left.\ldots G^T H^{-1} W^T \Pi - A^H\right] x(k)\right\}$$

or

$$U = -H^{-1}\left\{W^T \Pi - G\left(G^T H^{-1} G\right)^{-1}\left[G^T H^{-1} W^T \Pi - A^H\right]\right\} x(k)$$

According to a receding horizon strategy

$$u(k) = -F\, x(k)$$

with

$$F := e_1 H^{-1}\left\{W^T \Pi - G\left(G^T H^{-1} G\right)^{-1}\left[G^T H^{-1} W^T \Pi - A^H\right]\right\}$$

$$e_1 = \left[1\ 0 \ldots 0\right]$$

the stability constraint causes a correction of the gain.

b) For the equation

$$G^T U + A^H x(k) = 0$$

to have a solution with respect to U for any $x(k)$, the columns of G must span the whole space, meaning that rank G = dim $x = n$

According to the Cailey-Hamilton theorem it is enough to consider $N = n$ columns since all columns $n+1, n+2, \ldots$ are linearly dependent on the first n ones.

Hence, it is a necessary condition for $G^{+}U + A^{H} x(k) = 0$ to have a solution in all cases that $N \geq n$.

$\boxtimes$

P17 $x(k+1) = \Phi x(k) + \Gamma u(k)$

$y(k) = C x(k)$

$y(3) = C\Phi^2\Gamma u(0) + C\Phi\Gamma u(1) + C\Gamma u(2)$

Find $u(0), u(1), u(2)$ such that $y(3) = r$, with r specified, and

$J(u) = u^2(0) + u^2(1) + u^2(3)$

is minimum.

Use the method of Lagrange multipliers.

$$L(u,\lambda) = u_0^2 + u_1^2 + u_2^2 + \lambda\left(y(3) - r\right)$$

$$L = u_0^2 + u_1^2 + u_2^2 + \lambda\left(C\Phi^2\Gamma u_0 + C\Phi\Gamma u_1 + C\Gamma u_2 - r\right)$$

Extremum conditions:

$$\frac{\partial L}{\partial u_0} = 2u_0 + \lambda C\Phi^2\Gamma = 0$$

$$\frac{\partial L}{\partial u_1} = 2u_1 + \lambda C\Phi\Gamma = 0$$

$$\frac{\partial L}{\partial u_2} = 2u_2 + \lambda C\Gamma = 0$$

$$\frac{\partial L}{\partial \lambda} = C\Phi^2\Gamma u_0 + C\Phi\Gamma u_1 + C\Gamma u_2 - r) = 0$$

$$\mu_0 = -\frac{1}{2}\lambda C \underline{\Phi}^2 T'$$

$$\mu_1 = -\frac{1}{2}\lambda C \underline{\Phi} T' \qquad (*)$$

$$\mu_2 = -\frac{1}{2}\lambda C T'$$

Use these in the constraint to eliminate μ_0, μ_1 and μ_2 and get an equation for λ:

$$-\frac{1}{2}\lambda\left[(C\underline{\Phi}^2T')^2 + (C\underline{\Phi}T')^2 + (CT')^2\right] - r = 0$$

$$\lambda = -\frac{2r}{(C\underline{\Phi}^2T')^2 + (C\underline{\Phi}T')^2 + (CT')^2}$$

Eliminate now λ in $(*)$ to get the final result

$$\mu(0) = \frac{C\underline{\Phi}^2T' r}{(C\underline{\Phi}^2T')^2 + (C\underline{\Phi}T')^2 + (CT')^2}$$

$$\mu(1) = \frac{C\underline{\Phi}T' r}{(C\underline{\Phi}^2T')^2 + (C\underline{\Phi}T')^2 + (CT')^2}$$

$$\mu(2) = \frac{CT' r}{(C\underline{\Phi}^2T')^2 + (C\underline{\Phi}T')^2 + (CT')^2}$$

$$\Phi = \begin{bmatrix} 0 & 1 \\ 0{,}5 & 0{,}4 \end{bmatrix} \quad \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$r = 5$

$\mu_0 = 1{,}2979$

$\mu_1 = 1{,}7142$

$\mu_2 = 1{,}2244$

$\square$

P18 Take as Lyapunov function f.

$$f(x_{k+1}) = f(x_k - \alpha_k g(x_k))$$

Since α_k is selected at each step to minimize $f(x_k - \alpha_k g(x_k))$, for any other

$$f(x_k - \alpha_k g(x_k)) < f(x_k - \alpha g(x_k)) \quad \forall \alpha \neq \alpha_k$$

In particular, for $\alpha = 0$,

$$f(x_k - \alpha_k g(x_k)) < f(x_k)$$

and thus

$$f(x_{k+1}) < f(x_k),$$

and f is a Lyapunov function.

$\square$

P19 From the model

$$V(x_{k+1}) = x_{k+1}^T P x_{k+1} =$$

$$= x_k^T A^T P A x_k$$

For V to be a Lyapunov function

$$V(x_{k+1}) - V(x_k) \leq 0$$

Using the definition of V in this condition

$$x_k^T A^T P A x_k - x^T P x \leq 0$$

or

$$x^T (A^T P A - P) x_k \leq 0$$

therefore, for V to be a Lyapunov function there must exist a positive semidefinite matrix Q such that

$$-Q = A^T P A - P \quad (*)$$

Remark: Eq. (*) is known as the Lyapunov equation. The converse proposition is also true (but it is much harder to prove): if all the eigenvalues of A are inside the unit circle, for any $Q > 0$, (*) has a solution $P > 0$.

$\square$

P20

$$V(x_{k+1}) - V(x_k) =$$

$$= x_{k+1}^T S x_{k+1} - x_k^T S x_k$$

$$= \left((A-bF)x_k\right)^T S\left((A-bF)x_k\right) - x_k^T S x_k$$

$$= x_k^T (A-bF)^T S (A-bF) x_k - x_k^T S x_k$$

$$= x_k^T \left[(A-bF)^T S (A-bF) - S\right] x_k$$

$$= -x_k^T \left(Q + \rho F^T F\right) x_k < 0$$

$\boxtimes$

P21. Take as candidate Lyapunov function the optimal cost $V^0(k)$

$$V^0(k) = \min_U \sum_{i=1}^{H} \left[\hat{x}^T(k+i)\, Q\, \hat{x}(k+i) + \rho\, \hat{u}^2(k+i-1)\right]$$

$$U := \left[\hat{u}(k) \quad \hat{u}(k+1) \ \ldots \ \hat{u}(k+H-1)\right]^T$$

Relate $V^0(k+1)$ with $V^0(k)$.

For that sake, observe that

$$V^{o}(k+1) = \min_{U} \sum_{i=1}^{H} \left[\hat{x}^{T}(k+1+i)\, Q\, \hat{x}(k+1+i) + \rho\, \hat{u}^{2}(k+i) \right]$$

$$= \min_{U} \left\{ \sum_{i=1}^{H} \left[\hat{x}^{T}(k+i)\, Q\, \hat{x}(k+i) + \rho\, u^{2}(k+i-1) + \right. \right.$$

$$+ \hat{x}^{T}(k+1+H)\, Q\, \hat{x}(k+1+H) +$$

$$+ \rho\, \hat{u}^{2}(k+H) -$$

$$\left. - \hat{x}^{T}(k+1)\, Q\, \hat{x}(k+1) - \rho\, \hat{u}^{2}(k) \right\} \leq$$

$$\leq V^{o}(k) + \min_{U} \left[\hat{x}^{T}(k+1+H)\, Q\, \hat{x}(k+1+H) + \rho\, u^{2}(k+H) \right]$$

But $\hat{x}(k+1+H) = 0$ because of the stability constraint.

Furthermore, the optimum of $u^{2}(k+H)$ is zero. Hence

$$V^{o}(k+1) \leq V^{o}(k)$$

that allows to conclude that the closed-loop is stable. $\square$

P22 a)

$\det(zI - A + LC) =$

$$= \det\left\{\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}\begin{bmatrix} 1 & 0 \end{bmatrix}\right\} =$$

$$= \begin{vmatrix} z - 1 + L_1 & -h \\ L_2 & z - 1 \end{vmatrix} =$$

$$= (z - 1 + L_1)(z - 1) + h L_2 =$$

$$= z^2 - z(2 - L_1) + 1 - L_1 + h L_2$$

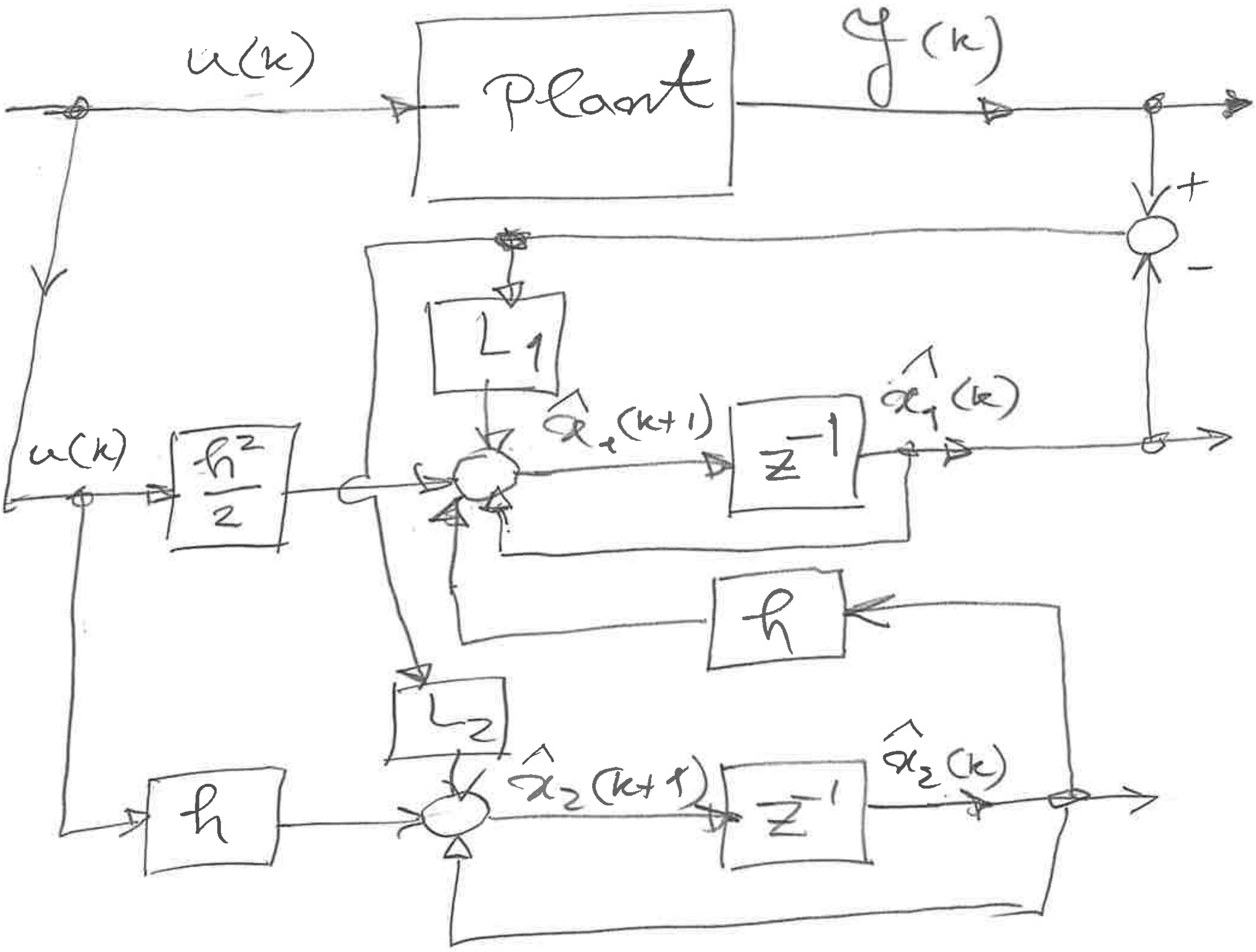
Desired polynomial : z^2. Hence

$$L_1 = 2, \quad L_2 = \frac{L_1 - 1}{h} = \frac{1}{h}$$

b) $x_1^+ = x_1 + h x_2 + \frac{h^2}{2} u$

$x_2^+ = x_2 + h u$

$$\hat{x}(k+1) = A\hat{x}(k) + bu(k) + L(y(k) - C\hat{x}(k))$$


u(k)
Plant
y(k)
+
-
L1
x̂1(k+1)
z^-1
x̂1(k)
u(k)
h^2/2
h
L2
h
x̂2(k+1)
z^-1
x̂2(k)

P23 a) $x_2(k+1) = 0{,}8\,x_2(k) + u(k)$

$$x_1(k+1) = x_1(k) + x_2(k+1)$$

$$x_1(k+1) = x_1(k) + 0{,}8\,x_2(k) + u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0{,}8 \\ 0 & 0{,}8 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix}$$

b) Observability

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0{,}8 \end{bmatrix}$$

rank $\mathcal{O} = 2 \Rightarrow$ observable

$\det(zI - A + LC) =$

$$= \left| \begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1 & 0{,}8 \\ 0 & 0{,}8 \end{bmatrix} + \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \right| =$$

$$= \begin{vmatrix} z + L_1 - 1 & -0{,}8 \\ L_2 & z - 0{,}8 \end{vmatrix} =$$

$$= (z + L_1 - 1)(z - 0{,}8) + 0{,}8 L_2$$

$$= z^2 + (L_1 - 1{,}8) z + (L_1 - 1)\, 0{,}8 + 0{,}8 L_2 \quad (*)$$

Desired characteristic polynomial

$$(z-0{,}5)(z-0{,}6)=$$

$$= z^2 - 1{,}1z + 0{,}3 \qquad (**)$$

Comparing the coefficients of $(*)$ and $(**)$:

$$L_1 - 1{,}8 = -1{,}1 \longrightarrow L_1 = 0{,}7$$

$$0{,}8(L_2 - L_1 - 1) = 0{,}3$$

$$L_2 - 1{,}7 = \frac{3}{8}$$

$$L_2 = 2{,}08$$

P24

$$x(k+1) = x(k) + u(k) \qquad (*)$$

$$y(k) = x(k) + \eta(k)$$

Observer

$$\hat{x}(k+1) = \hat{x}(k) + u(k) + L\left(y(k) - \hat{x}(k)\right) \qquad (**)$$

Estimation error

$$\tilde{x}(k) := x(k) - \hat{x}(k)$$

Subtract $(*)$ and $(**)$ to obtain an equation for the estimation error.

$$\tilde{x}(k+1) = \tilde{x}(k) - L\big(\underbrace{x(k)+\eta(k)}_{=y(k)} - \hat{x}(k)\big)$$

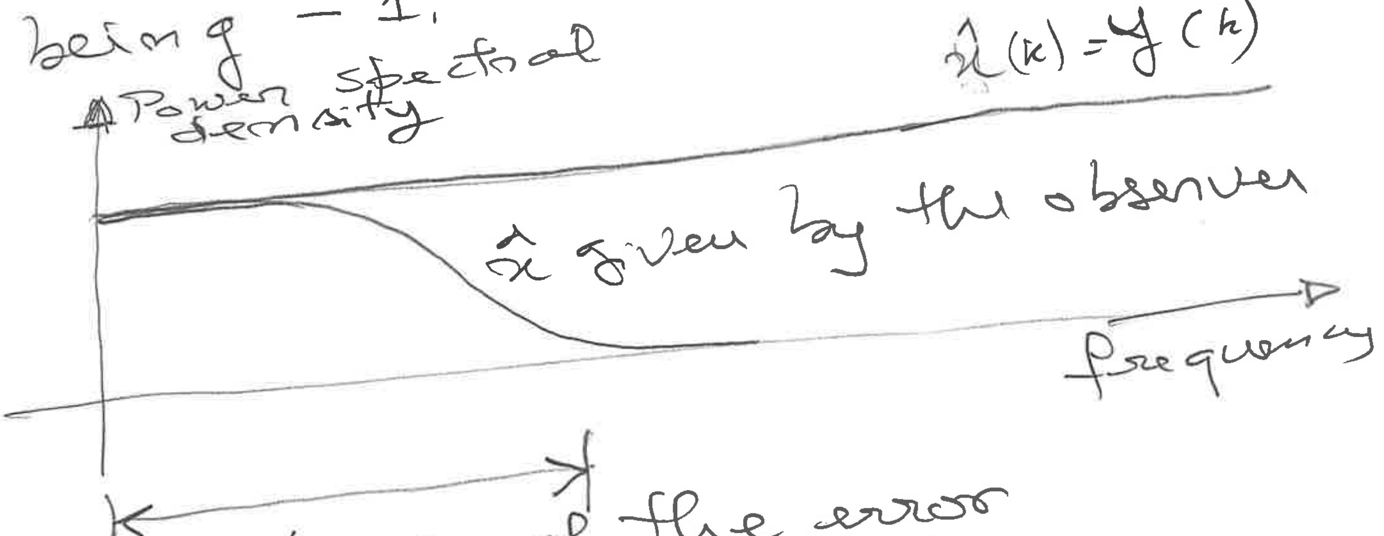
$$\tilde{x}(k+1) = (1-L)\,\tilde{x}(k) - L\,\eta(k)$$

Take the Z-transform and let

$$\Xi := \mathcal{Z}(\eta(k))$$

$$\tilde{X}(z) = \frac{-L}{z-(1-L)}\,\Xi(z)$$

the static gain of this filter is obtained by making $z=1$, being -1.



the power of the error with an observer is related to this frequency range

Instead the power of the error with $\hat{x}(k) = y(k)$ is <u>much bigger</u>, because it receives, in addition, high frequency contributions