TÉCNICO LISBOA

Master in Electrical and Computer Engineering

2021/2022 – P4

Estimação e Controlo Preditivo Distribuído

Distributed Predictive Control and Estimation

Problems for Self-study

Picture source: https://hortovanyi.wordpress.com/2017/06/06/model-predictive-controller-project/

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P1. Diference equations. By iterating the following difference equation, find the first 6 terms of its solution

$$y(k) = y(k-1) + y(k-2) \qquad k = 2,\ 3,\ \dots$$

starting from the initial conditions

$$y(0) = y(1) = 1$$

(These numbers are called Fibonacci numbers).

P2. Diference equations. Consider the difference equation

$$y(k+2) - 1{,}3y(k+1) + 0{,}4y(k) = 0 \qquad \text{(P2-1)}$$

for $k = 1, 2, 3, \dots$, with the initial conditions

$$y(0) = 1, \quad y(1) = 0.$$

Find $y(k)$ for any k.

Sugestion: assume that there are constants λ such that λ^k verifies the difference (P2-1). Since the equation is linear, a linear combination of these solutions is also a solution. The coefficients of the linear combination are obtained from the initial conditions. Your answer must be given in the form of a linear combination of powers k.

P3. Diference equations. The sequence of Fibonacci numbers is a sequence of integer numbers generated by iterating the difference equation

$$y(k+2) = y(k+1) + y(k) \qquad \text{(P4-1)}$$

for $k = 1, 2, 3, \dots$, with initial conditions $y(1) = y(2) = 1$.

Find $y(k)$ for any k.

Sugestão: assume that there are constants λ such that λ^k verifies the difference (P2-1). Since the equation is linear, a linear combination of these solutions is also a solution. The coefficients of the linear combination are obtained from the initial conditions. Your answer must be given in the form of a linear combination of powers k. **Do not approximate the square roots.**

Remark: It is interesting to observe that, although the general term of the solution of the difference equation involves irrational numbers, these numbers cancel out to produce a sequence of integer numbers.

P4. Diference equations. Consider the system with input u and output y described by the difference equation

$$y(k+2)+a_1y(k+1)+a_2y(k)=b_0u(k+1)+b_1u(k)$$

where $k=0,\ 1,\ 2,\ \dots$ is discrete time, the a_i, b_i are constant parameters and the initial cionditions are zero. By applying the Principle of Superposition, show that this system is linear.

P5. Diference equations. Consider the discrete system described by the difference equation:

$$y(k)-0.5y(k-1)+y(k-2)=2u(k-4)+u(k-5)$$

a) Find the transfer function;

b) Compute the zeros and poles (as well as their multiplicity).

c) Write a state modelo of the system.

P6. State model. Consider the discrete system described by the difference equation:

$$y(k+2)+a_1y(k+1)+a_2y(k)=b_0u(k+1)+b_1u(k)$$

a) Write the equation in which the most advanced variable is $y(k)$.

b) Write the transfer function.

c) Obtain an equivalent state-space model of the system.

P7. Diference equations. Consider the unit step responses shown on figure P7-1, that are identified by the letters A, B, C, D and E. Furthermore, consider also the discrete systems numbered from 1 to 5, that are described by the following transfer functions:

$$G_1(z)=\frac{1}{z-0.5} \qquad G_2(z)=\frac{0,2}{z-0,8} \qquad G_3(z)=\frac{0,35z+0,3}{z^2-z+0.7}$$

$$G_4(z)=\frac{1,5}{z+0,5} \qquad G_5(z)=\frac{0,5}{z-0,5}$$

a) State which number of the transfer functions correspond to each time response. Justify your answer.

b) Write the matrices of a linear state model that corresponds to $G_3(z)$.

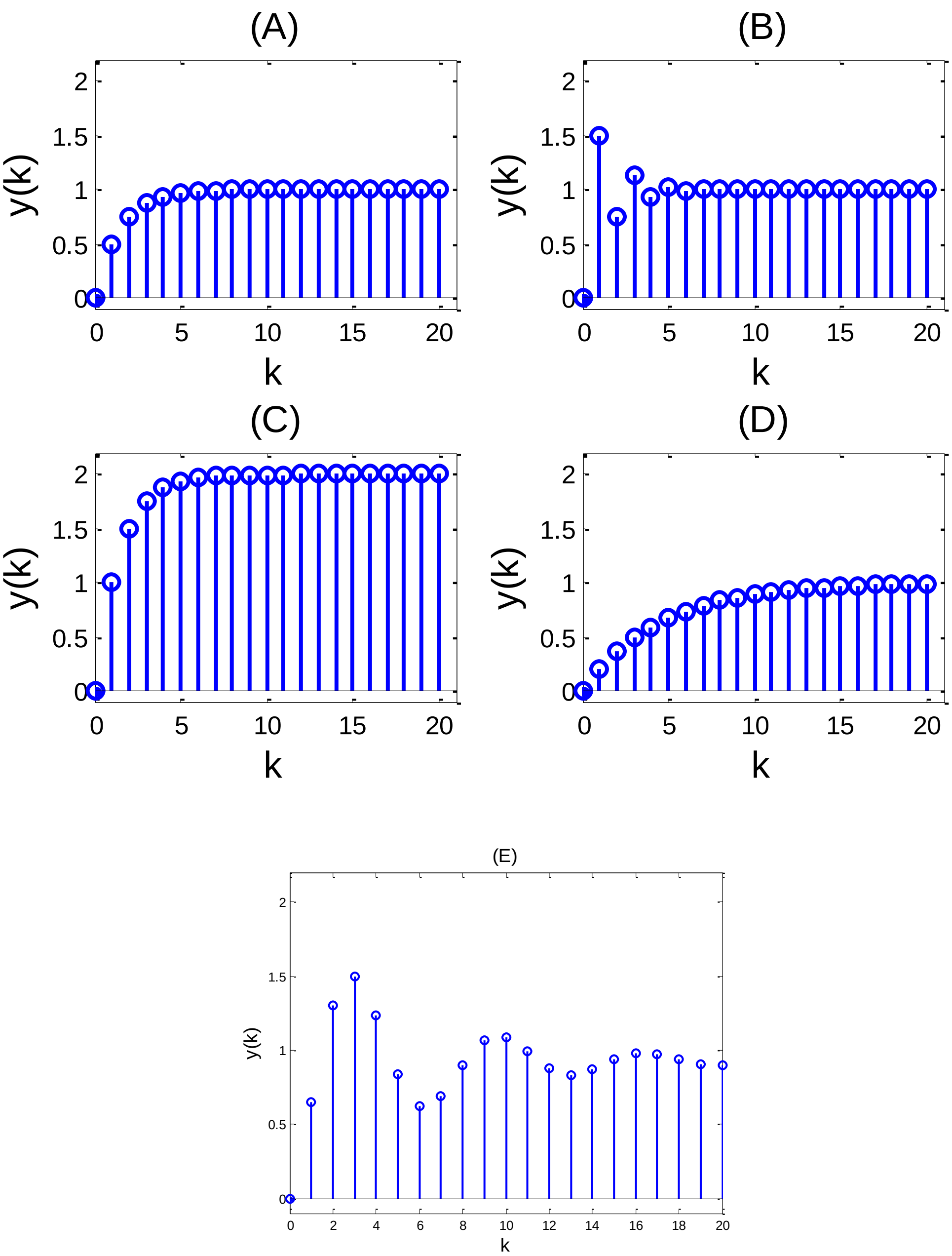

Figure P7-1. Problem P7. Time responses to the unit step input of 5 different linear discrete systems.

P8. State model. Consider the linear time invariant system, in discrete time, described by the digital transfer function

$$G_1(z) = \frac{z - 0{,}5}{z^2 - 1{,}5z + 0{,}56}$$

a) Say, and justify, if the system is, or is not, asymptotically stable.

b) Write the difference equation that is equivalent to the above transfer function.

c) Obtain a state realization of this system. Give the answer in matrix form.

d) Consider the system described by the transfer function

$$G_2(z) = \frac{2}{z - 0{,}9}$$

Obtain a state model for the association shown in figure P8-1.

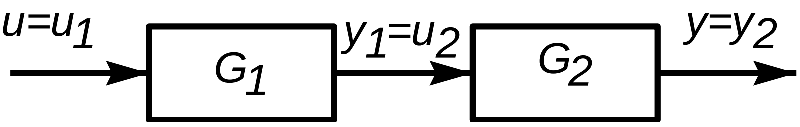

Figure P8-1 Series association of systems G_1 and G_2.

P9. State model. In an electrical DC motor the applied tension u and the shaft angular position y are related by the following transfer function, in which $Y(z)$ and $U(z)$ are the Z transforms of y and u

$$Y(z) = \frac{z}{(z-1)(z+0{,}5)} U(z)$$

a) Write a difference equation that relates the samples u with the ones of y.

b) Taking as initial conditions $y(0) = 0$, $y(1) = 0$, use the difference equation to compute $y(k)$ for $k = 1, \dots, 4$, when $u(k) = 1$, $k \geq 0$.

c) Define a state vector and obtain the state equations in matrix form.

d) In order to control the process so that the angular position goes to a value close to the reference r, the motor is connected to a proportional controller, as shown in figure P1-1. Find the closed-loop transfer function, $y(z)/R(z)$, for a generical K.

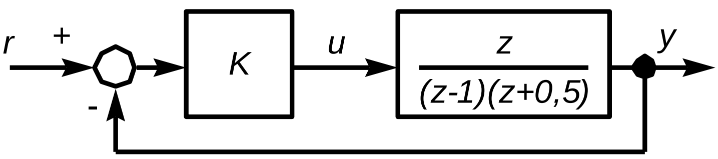

Fig. P9-1. Problem P9.

e) Compute the static gain of the controlled system.

f) Say whether the controlled system is stable for $K = 0{,}5$. Justify your answer.

P10. State model. Consider the system S_1, with input u_1 and output y_1, defined by the transfer function

$$Y_1(z) = \frac{(z-0{,}8)(z+0{,}8)}{(z-0{,}5)(z+0{,}5)} U_1(z)$$

a) Obtain a minimum state realization for this system Define the state vector. Write the model in matrix form.

b) Consider now system S_2, with input u_2 and output y_2, defined by the transfer function

$$Y_2(z) = \frac{3}{z - 0{,}9} U_2(z)$$

Write in matrix form the state realization of the system that is obtained by connecting S_1 and S_2 in series, such that the input of S_2 is the output of S_1 and the state of the overall system is the concatenation of the states of S_1 and S_2. The input of the global system is $u = u_1$ and its output is $y = y_2$.

P11. State model. Consider the discrete time system described by the state model

$$x(k+1) = Ax(k) + bu(k) \qquad x(0) = 0$$

$$y(k) = Cx(k),$$

Where both the input u and the output y are scalar. Compute the value of the samples of the time response to the impulse, defined by

$$u(k) = \begin{cases} 1, & for\ k = 0 \\ 0, & for\ k > 0 \end{cases}$$

Remark:These samples are also known as *Markov parameters*. They play an important role in the theory of linear dynamic systems and linear MPC.

P12. Constrained optimization. Using the method of Lagrange multipliers, solve the constrained optimization problem

$$minimize\ x_1^2 + x_2^2$$

$$subject\ to \quad (x_1 - 1)^2 + (x_2 - 1)^2 = 1$$

Provide a graphical interpretation.

P13. Constrained optimization. Equality constrained least-squares. Solve the equality constrained least-squares problem

$$minimize\ \|Ax - b\|^2$$

$$subject\ to\ Gx = h$$

P14. MPC. Consider a plant described by the nonlinear state model

$$x(k+1) = f(x(k), u(k))$$

in which $u(k)$ is scalar. Associated to it, consider the quadratic cost, defined from present time k by

$$J = \sum_{i=1}^{H} \hat{x}^T(k+i|k)Q\hat{x}(k+i|k) + \rho\hat{u}^2(k+i-1|k).$$

a) Explain the MPC algorithm associated to the above elements, including the receding horizon strategy. Formulate mathematically the MPC problem as a constrained optimization problem.
b) Explain whether the MPC algorithm is an open-loop or a closed-loop controller.

P15. MPC. Consider the plant described by the linear state model in discrete time

$$x(k+1) = Ax(k) + bu(k)$$

$$y(k) = Cx(k)$$

In order to simplify, it is assumed that, in addition to the output y, we have also access to the state x at each instant of discrete time k ($k = 0, 1, 2, 3, \ldots$). Both u and y are scalar.

Assume that you are at instant k, and that you know $x(k)$. We want to compute a sequence of manipulated variable values $u(k)$, $u(k+1)$, ..., $u(k+H-1)$, where H is an integer number selected by the designer (called the "control horizon"). The computation of these control samples is made by minimizing

$$J(u(k), \ldots, u(k+H-1)) = \sum_{i=1}^{H} (y(k+i) - r(k+i))^2 + Ru^2(k+i-1)$$

where r is a reference to track. Define $Y^* = [r(k+1) \quad \ldots \quad r(k+H)]^T$.

a) In order to express J as a function of only $x(k)$ and $u(k), \ldots, u(k+H-1)$, write $y(k+i)$ as a function of $x(k)$ and $u(k), \ldots, u(k+H-1)$. Suggestion: use the state model to express $x(k+i)$ as a function of $x(k)$ and $u(k), \ldots, u(k+H-1)$.
b) Define $Y = [y(k+1) \quad \ldots \quad y(k+H)]^T$, $U = [u(k) \quad \ldots \quad u(k+H-1)]^T$. Using the result of a), show that there are matrices W and Π such that $Y = WU + \Pi x(k)$. Write the elements of these matrices.
c) Find, as a function of $x(k)$, Y^*, R, W, and Π the value of U that minimizes J.
d) In MPC, of all the sequence U only the first element, $u(k)$, is applied to the process, the whole process being repeated at discrete $k+1$, a procedure called "receding horizon strategy". Show that, according to the receding horizon strategy, the optimal control law has the form $u(k) = -Fx(k) + u_{ff}(k)$, and give expressions for the feedback gain F and the feedforward term $u_{ff}(k)$.

P16. MPC. Consider the linear plant

$$x(k+1) = Ax(k) + bu(k),$$

in which u is scalar. Associated to this plant, consider the problem of minimizing in a receding horizon sense the quadratic cost

$$V(k) = \sum_{i=1}^{H} \hat{y}^2(k+i|k) + \rho \hat{u}^2(k+i-1|k)$$

subject to the stability constraint

$$\hat{x}(k+H|k) = 0.$$

It is assumed that (A, b) is controllable and that the horizon verifies $H \geq \dim x$.

a) Using Lagrange multipliers, find the optimal control law in a receding horizon sense.
b) Explain why it is required to assume that (A, b) is controllable and that the horizon verifies $H \geq \dim x$.

P17. MPC. Consider the linear, time invariante, system described by the state model in discrete time

$$x(k+1) = \Phi x(k) + \Gamma u(k),$$

$$y(k) = Cx(k),$$

where

$$\Phi = \begin{bmatrix} 0 & 1 \\ 0{,}5 & 0{,}4 \end{bmatrix}, \quad \Gamma = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

The initial condition is

$$x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Compute the sequence of control actions $u(0), u(1), u(2)$ that transfers the system output to the reference value $r = 5$, at time $k = 3$, that is to say, that is such that $y(3) = r = 5$, and that minimizes the control energy, given by

$$J(u) = u(0)^2 + u(1)^2 + u(2)^2.$$

Start by obtaining general expressions for $u(0), u(1), u(2)$ as a function of Φ, Γ, C and r,and only aftarwards obtain numerical values.

P18. Lyapunov's direct method. Let $f(x): \mathbb{R}^n \to \mathbb{R}$ be a function to be minimized with respect to the n-dimensional vector x. For this sake, the steepest-descent method is a numerical procedure to find the minimum, $\bar{x}$. This method starts from an initial estimate x_0 of $\bar{x}$ and recursively updates the estimate according to

$$x_{k+1} = x_k - \alpha_k g(x_k),$$

where $g(x_k)$ is the transpose of the gradient of f, with respect to x, at x_k. The scalar step α_k is chosen, at each step k, such as to minimize $f(x_k - \alpha g(x_k))$ with respect to α.

It is assumed that the function f satisfies the following properties:

1. The function f has a unique minimum at the point $\bar{x}$.
2. The function f has continuous partial derivatives, and the gradient of f vanishes only at $\bar{x}$.
3. $f(x) \to \infty$ when the absolute value of any component of x goes to infinity.

Using Lyapunov's direct method, show that the above iterative procedure converges to the minimum $\bar{x}$ from any starting point.

Suggestion: Use as Lyapunov function $f(x_k)$.

P19. Lyapunov's direct method. Consider the discrete-time autonomous state model

$$x(k+1) = Ax(k).$$

Find conditions, in the form of the positive semidefiniteness of a symmetric matrix, for $V(x) = x^T P x$, with $P = P^T$ a positive matrix, to be a Lyapunov function.

P20. Lyapunov's direct method. Asymptotic stability of the LQ regulator. Consider the linear plant

$$x(k+1) = Ax(k) + bu(k),$$

in which u is scalar. The control that results from minimizing the infinite horizon cost

$$J_\infty = \sum_{k=0}^{\infty} x^T(k) Q x(k) + \rho u^2(k),$$

with $Q = Q^T \succ 0$ and $\rho > 0$, is given by the state feedback control law

$$u(k) = -Fx(k).$$

The vector gain F is given by

$$F = (\rho + b^T S b)^{-1} b^T S A,$$

where $S = S^T$ is the positive definite solution of the discrete-time algebraic Riccati equation (ARE)

$$S = (A - bF)^T S (A - bF) + Q + \rho F^T F.$$

Using as candidate Lyapunov function

$$V(x) = x^T S x,$$

prove that the closed-loop that results from the above control law is asymptotically stable.

P21. MPC. Stability. Consider the linear plant

$$x(k+1) = Ax(k) + bu(k),$$

in which u is scalar. Associated to this plant, consider the problem of minimizing in a receding horizon sense the quadratic cost

$$V(k) = \sum_{i=1}^{H} \hat{x}^T(k+i|k) Q \hat{x}(k+i|k) + \rho \hat{u}^2(k+i-1|k)$$

subject to the stability constraint

$$\hat{x}(k+H|k) = 0.$$

It is assumed that (A, b) is controllable and that the horizon verifies $H \geq \dim x$.

Using Lyapunov's direct method, show that the equilibrium defined by $u = 0$, $x = 0$ is stable.

Suggestion: Relate $V(k+1)$ with $V(k)$; then, use the stability constraint to show that $V(k+1) - V(k) \leq 0$.

P22. State estimation. Consider the discrete time system of 2nd order, that corresponds to the sampling of a double integrator, and is described by the state model

$$x(kh+h) = \begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix} x(kh) + \begin{bmatrix} \frac{h^2}{2} \\ h \end{bmatrix} u(kh), \qquad y(k) = x_1(k)$$

where h is the sampling period (continuous time between two consecutive samples), $k = 0, 1,$ 2, ... is discrete time, u is the control variable and $x(kh) = \begin{bmatrix} x_1(kh) \\ x_2(kh) \end{bmatrix}$ is the state vector at time kh.

a) Compute the gains F_1 and F_2 of a predictive observer such that the poles of the error estimation Dynamics are placed at the roots of the polynomial

$$\alpha_o(z) = z^2.$$

Express your result as a function of the sampling period h.

b) Draw the block diagram of the predictive observer. Use only scalar blocks.

P23. State estimation. Figure P23-1 represents the simplified block diagram of a robot arm joint. The manipulated variable is the tension applied to the motor that actuates the joint, x_2 is the joint angular velocity and x_1 is the joint angular position. There is only one sensor that measures the x_1, which means that this variable is the process output, y, and x_1 and x_2 are state variables.

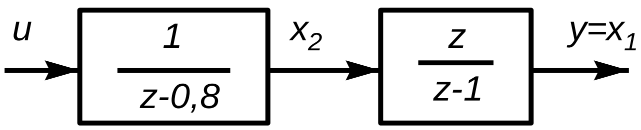

Figure P23-1. The system considered in problem P24.

a) For the input, output and state variables described above, write the state equations in matrix form.

b) Assume now that you don't have access to the state. Design a state observer of predictive type such that the error estimation poles are $0{,}5$ and $0{,}6$.

P24. State estimation. Consider the linear, first order, state model

$$x(k+1) = x(k) + u(k)$$
$$y(k) = x(k) + \eta(k)$$

where η is a sequence of zero mean, independent, identically distributed white noise (the spectral power density is constant in frequency).

Using an argument based on the frequency representation of the state estimation error, discuss whether there is any advantage of using a predictive state-observer with respect to just estimating the state by doing $\hat{x}(k) = y(k)$.

Suggestion: Write the equation of the state observer and, from this one, write an equation for the state estimation error $\tilde{x}(k) = x(k) - \hat{x}(k)$. Compute the transfer function that relates the Z-transform of η with the Z-transform of $\tilde{x}$.

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