Distributed Predictive Control and Estimation

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Stability constraints

Chapter objective

The objective of this chapter is to discuss conditions for stability of control loops with MPC.

We will:

- Provide a stability proof in the presence of a perfect model;
- Discuss extensions.

The technique is based Lyapunov's direct method for discrete systems.

As such, this method and notions associated to it are first introduced.

Historical note

The direct method for the study of stability of continuous-time dynamic systems was introduced by Alexander Lyapunov (1857-1918) in 1892, as part of his Ph. D. thesis in Saint Petersburgh (Russia).
The author of the extension to discrete-time is not clear. An early reference is due to W. Hahn (1958).
An elementary account is given in D. G. Luenberger (1979). *Introduction to Dynamic Systems*, Wiley, ch. 9.

A recent transcript of continuous-time theorems to discrete-time (greatly exceeds the objectives of this course) is given in

https://doi.org/10.48550/arXiv.1809.05289

Nonlinear discrete-time state model

Autonomous n-th order discrete-time state model

$$x_1(k+1) = f_1(x_1(k), x_2(k), \ldots, x_n(k))$$

$$x_2(k+1) = f_2(x_1(k), x_2(k), \ldots, x_n(k))$$

$$\vdots$$

$$x_n(k+1) = f_n(x_1(k), x_2(k), \ldots, x_n(k))$$

A set of n 1^{st} order difference equations.

Initial conditions given for: $x_1(0), x_2(0), \ldots, x_n(0)$

Nonlinear discrete-time state model

Matrix notation

$$x(k+1) = f(x(k)), \quad x(0) = x_0$$

Controlled state model

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0$$

Equilibrium points

A vector $\bar{x}$ is an equilibrium point if, once the state is made equal to $\bar{x}$, it remains equal to $\bar{x}$ for all future time.

Since

$$x(k+1) = f(x(k))$$

it follows that the equilibrium points satisfy

$$\bar{x} = f(\bar{x})$$

Equilibrium points: example

$$x_1(k+1) = \alpha x_1(k) + x_2(k)^2$$

$$x_2(k+1) = x_1(k) + \beta x_2(k)$$

An equilibrium point $\bar{x} = [\bar{x}_1 \ \bar{x}_2]^\top$ satisfies

$$\bar{x}_1 = \alpha \bar{x}_1 + \bar{x}_2^2$$

$$\bar{x}_2 = \bar{x}_1 + \beta \bar{x}_2$$

Solve the second equation to express $\bar{x}_1$ in terms of $\bar{x}_1$ and replace in the first to yield

$$(1-\alpha)(1-\beta)\bar{x}_2 = \bar{x}_2^2$$

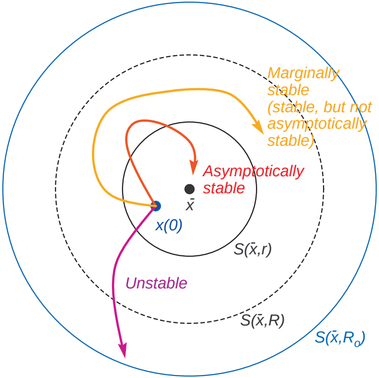
There are 2 equilibrium points

$$\bar{x}^A = (0, \, 0) \quad \text{and} \quad \bar{x}^B = ((1-\alpha)(1-\beta)^2, \, (1-\alpha)(1-\beta))$$

Stability

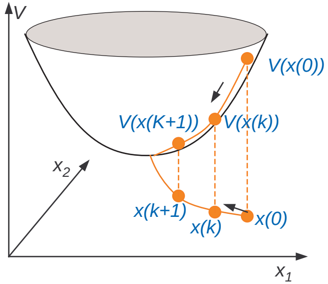
An equilibrium point $\bar{x}$ is

- Stable if there is an $R_0 > 0$ such that, for every $R < R_0$, there is an r, $0 < r < R$ such that, if $x(0)$is inside $S(\bar{x}, r)$, then $x(t)$ is inside $S(\bar{x}, R)$ for all $t > 0$.
- Asymptotically stable if it is stable and, in addition, $x(t) \to \bar{x}$ when $t \to \infty$.
- Marginally stable if it is stable, but not asymptotically stable.
- Unstable if it is not stable.



Lyapunov functions

A Lyapunov function measures the distance with respect to the equilibrium point $\bar{x}$ the stability of which is to be studied.



A Lyapunov function V is a function that maps the state in a region Ω around $\bar{x}$ in $\mathbb{R}_0^+$, such that

- V is continuous;
- $V(x) > 0$ for all $x \in \Omega$ and $V(\bar{x}) = 0$;
- Along any trajectory of the system contained in Ω, the value of V never increases, $V(x(k+1)) - V(x(k)) \leq 0$.

$$V(f(x(k))) - V(x(k)) \leq 0$$

Lyapunov's direct method

Let $\bar{x}$ be an equilibrium point of the dynamical system

$$x(k+1) = f(x(k)).$$

If there exists a Lyapunov function $V(x)$ in a spherical region $S(\bar{x}, R_0)$, with center in $\bar{x}$ then, the equilibrium point is stable (at least).

If $V(f(x)) - V(x) < 0$ for any x and $f(x)$ in $S(\bar{x}, R_0)$, then the stability is asymptotic.

Stability proof using Lyapunov's direct method: example

Consider the 2nd order system

$$x_1(k+1) = \frac{x_2(k)}{1 + x_2(k)^2}$$

$$x_2(k+1) = \frac{x_1(k)}{1 + x_2(k)^2}$$

The point $x = (0,\ 0)$ is an equilibrium point.

Define the candidate Lyapunov function $V(x_1, x_2) = x_1^2 + x_2^2$.

This function is continuous and has a unique minimum at the equilibrium point.

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Stability proof using Lyapunov's direct method: example (cont.)

Furthermore

$$Vx(k+1)) = x_1(k+1)^2 + x_2(k+1)^2$$

$$= \frac{x_2(k)^2}{[1+x_2(k)^2]^2} + \frac{x_1(k)^2}{[1+x_2(k)^2]^2}$$

$$= \frac{x_1(k)^2 + x_2(k)^2}{[1+x_2(k)^2]^2}$$

$$= \frac{V(x(k))}{[1+x_2(k)^2]^2} \leq V(x(k)).$$

Hence, $x = (0,\, 0)$ is stable.

Terminal constraints ensure stability

Plant model $x(k+1) = f(x(k),\ u(k))$, such that $0 = f(0,0)$.

Apply the control according to a receding horizon strategy, by minimizing

$$V(k) = \sum_{i=1}^{H} L(\hat{x}(k+i|k), \hat{u}(k+i-1|k))$$

where $L(x,u) \geq 0$, and $L(x,u) = 0$ iff $x = 0$ and $u = 0$, and satisfies the decrescend property:

$$\|[x_1^\top \ u_1^\top]\| > \|[x_2^\top \ u_2^\top]\| \Rightarrow L(x_1,u_1) > L(x_2,u_2),$$

subject to the terminal constraint

$$\hat{x}(k+H|k) = 0$$

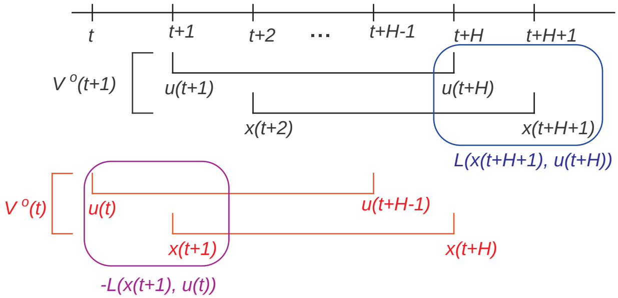
The control and state satisfy the operational constraints $\hat{u}(k+i|k) \in \mathcal{U}$ and $\hat{x}(k+i|k) \in \mathcal{X}$, with $\mathcal{U}$ and $\mathcal{X}$ prescribed sets.

Then, the equilibrium point $x = 0$, $u = 0$ is stable, provided that the optimization problem is feasible and is solved at each step.

Proof (stability with terminal constraints)

$$V^o(t+1) = \min_u \sum_{i=1}^{H} L(x(t+1+i), u(t+i))$$

$$= \min_u \left\{ \sum_{i=1}^{H} L(x_{t+i}, u_{t+i-1}) - L(x_{t+1}, u_t) + L(x_{t+H+1}, u_{t+H}) \right\}$$



Proof (stability with terminal constraints – cont.)

$$V^o(t+1) = \min_u \sum_{i=1}^{H} L(x(t+1+i), u(t+i))$$

$$= \min_u \left\{ \sum_{i=1}^{H} L(x_{t+i}, u_{t+i-1}) - L(x_{t+1}, u_t) + L(x_{t+H+1}, u_{t+H}) \right\}$$

$$\leq V^o(t) - L(x_{t+1}, u_t^o) + \min_u \{L(x_{t+H+1}, u_{t+H})\}$$

Proof (stability with terminal constraints – cont.)

$$V^o(t+1) \leq V^o(t) - L(x_{t+1}, u_t^o) + \min_u \{L(x_{t+H+1}, u_{t+H})\}$$

But $x(t+H) = 0$.

By the plant dynamics
$x(t+H+1) = f(x(t+H), u(t+H)) = f(0, u(t+H))$.

The best choice for $u(t+H)$ is $u(t+H) = 0$, that leads to
$x(t+H+1) = 0$ and

$$\min_u \{L(x(t+H+1), u(t+H))\} = 0$$

Hence $V^o(t+1) \leq V^o(t) - L(x_{t+1}, u_t^o) \leq V^o(t)$

Inequality stability terminal constraints

The key for the above stability proof is to impose the terminal equality constraint

$$\hat{x}(k+H|k) = 0$$

In practical numerical terms, it might be very difficult to solve the optimization problem with this constraint.

Another possibility to build a stability proof is to impose the inequality terminal constraint

$$\|\hat{x}(k+H|k)\| \leq \varepsilon$$

For ε sufficiently small, it is possible to prove stability of the MPC law.

The key issue is to ensure that the state is sufficiently contracted at the end of the prediction horizon. Otherwise, the state might grow.

A detour: controllability and observability

Before presenting the MPC control law in the linear case with no operational constraints, we need to present important system theory concepts on

- controllability,
- observability,
- reconstructibility.

The MPC control law in the linear case with no operational constraints is a stabilizing constant state feedback.

Since it can be replace by a LQ controller gain, that is simpler, it has not much practical interest. However, it is interesting to see what MPC amounts to in a situation with no operational constraints.

Controllability (definition)

The state realization of order n

$$x(k+1) = Ax(k) + bu(k)$$

Is said to be completely controllable (or reachable) iff for an initial condition $x(0) = 0$ and any x_f there exists N finite and a sequence of control inputs

$$u(0), u(1), \ldots, u(N-1)$$

such that

$$x(N) = x_f$$

Controllability criterum

The discrete system

$$x(k+1) = Ax(k) + bu(k)$$

Is completely controllable iif the matrix

$$C(A,b) = \left[b \mid Ab \mid A^2 b \mid \cdots \mid A^{n-1} b \right]$$

Called controllability matrix, has rank $n = \dim x$.

This result is proved in the following slides.

Cailey-Hamilton theorem

Given a square matrix A with characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = \det(sI - A)$$

The matrix verifies the equation

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

With an abuse of language we say that a matrix verifies its characteristic equation.

Reference: Strang (1980). *Linear Algebra and its Applications*. Academic Press.

Lemma

$\forall_{N \geq n}$ we have

$$rank\left[b \mid Ab \mid \cdots \mid A^{n-1}b \mid \cdots \mid A^{N-1}b\right] = rank\left[b \mid Ab \mid \cdots \mid A^{n-1}b\right]$$

The proof of this lemma is a consequence of the Cailey-Hamilton theorem (try to do it before looking at the next slide).

Proof of the lemma

Let $a(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n$ be the characteristic polynomial of A.

From Cailey-Hamilton theorem

$$A^n + a_1 A^{n-1} + \ldots + a_n I = 0$$

Multiply in the right by b

$$A^n b + a_1 A^{n-1} b + \ldots + a_n b = 0$$

This means that $A^n b$ is a linear combination of $A^{n-1}b, \ldots, b$. The proof that $A^{n+i}b \quad i \geq n$ is also a linear combination of the same vector is made by induction.

Proof of the controllability criterium

From the formula of variation of constants with zero initial conditions, the sate at time $N \geq n$ is given by

$$x(N) = A^{N-1}bu(0) + A^{N-2}bu(1) + \ldots + bu(N-1)$$

The points of the state space that can be reached are thus the linear combination of

$$b,\ Ab, \ldots,\ A^{N-2}b,\ A^{N-1}b$$

By the Lemma, the subspace generated by these vectors is the same as the subspace generated by

$$b,\ Ab, \ldots,\ A^{n-2}b,\ A^{n-1}b$$

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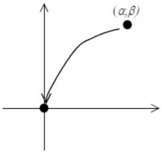
Problem

Consider the discrete system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

a) Show that the system is not controllable, i. e. that it is not always possible to transfer the origin to an arbitrarily specified state;

b) Show that starting from a generic state $x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, with α, β given, there exists a control law that brings the state to the origin in 1 step, *i.e.*, exists $u(0)$ function of α, β such that $x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



Problem (solution a))

a) The state realization is not controllable. Indeed:

$$C(A,b) = [b \quad Ab] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad rank\, C(A,b) = 1 < \dim x = 2$$

According to the criteria of controllability the system is not controllable and there are states that cannot be achieved staring from the origin..

Problem (solution b))

b) Since

$$x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u(0)$ must verify the system of equations

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0)$$

That has the solution

$$u(0) = -\alpha - \beta$$

Controllability and reachability

As the previous problem shows, in discrete systems, the concepts of controllability to the origin (being able to attain the origin from any state) and controllability (being able to attain any state from the origin, also called reachability) are not equivalent.
In continuous systems both concepts are equivalent.

The criteria refers to controllability (reachability) and not to controllability to the origin.

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Stabilizability (definition)

A state relization

$$x(k+1) = Ax(k) + bu(k)$$

is said to be stabilizable if there is a state feedback $u(k) = -Fx(k)$ such that the closed-loop is asymptotically stable.

Equivalently, there is a vector of gains F such that the eigenvalues of $A - bF$ are all inside the unit circle.

All the non-controllable modes of a stabilizable system must be asymptotically stable.

Observability (definition)

The discrete-time state realization

$$x(k+1) = Ax(k), \quad y(k) = Cx(k)$$

is said to be completely observable if exists finite k_1, $0 < k_1 < \infty$ such that the knowledge of the output $y(k)$ for $0 \leq k \leq k_1$ is sufficient to compute the inital state $x(0)$.

Observability criterium

The discrete-time state realization

$$x(k+1) = Ax(k), \quad y(k) = Cx(k)$$

is completely observable iff the observability matrix

$$\mathcal{O}(A, C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank $n = dim\, x$.

Reconstructibility (definition)

The discrete-time state realization

$$x(k+1) = Ax(k), \quad y(k) = Cx(k)$$

is said to be reconstructible if exists finite k_1, $0 < k_1 < \infty$ such that the knowledge of the output $y(\tau)$ for $k - k_1 \leq \tau \leq k$ is sufficient to compute the current state $x(k)$.

Example

$$[x(k) = [y(k)\; y(k-1)\; \ldots y(k-n)\; u(k-1)\; u(k-2)\; \ldots u(k-n)]^\top$$

yields a controllable and reconstructible state realization (but not observable).

Scalar LQ MPC with stability constraint

$$\min_{\hat{u}_{k|k}^{k+H-1|k}} V(k) = \sum_{i=1}^{H} \hat{x}^\top(k+i|k) Q \hat{x}(k+i|k) + \rho \hat{u}^2(k+i|k)$$

$$\text{subject to} \quad x(k+1) = Ax(k) + bu(k)$$

$$\hat{x}(k+H|k) = 0$$

No operational constraints

$$(A, b) \text{ controllable}, \quad (A, \sqrt{Q}) \text{ observable}, \quad Q = \sqrt{Q}^\top \sqrt{Q} \quad \rho > 0$$

Then, the solution is given by $u^o(k) = -Fx(k)$, and stabilizes the closed-loop ($A - bF$ has all its eigenvalues inside the unit circle).

The vector of gains F is computed using Lagrange multipliers and depends on A, b, Q, ρ.

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