Distributed Predictive Control and Estimation

–File 2–

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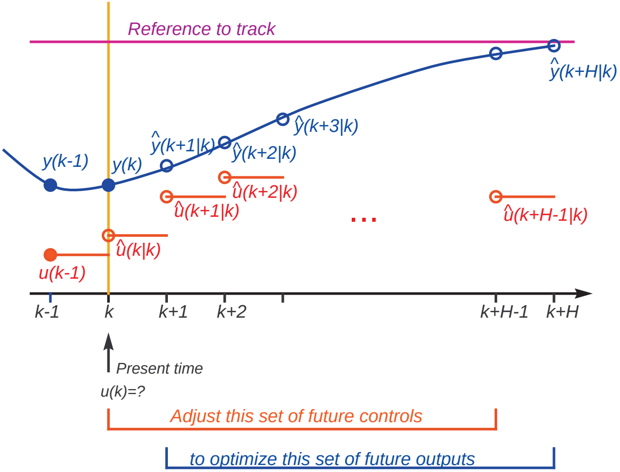
Unconstrained, linear receding horizon control

Basic MPC algorithm



According to the receding horizon (RH) strategy, apply to the plant only the first sample of the optimized sequence of future controls and repeat the process at time $k+1$

Linear-quadratic, unconstrained MPC problem

Consider a SISO plant to start.

Cost functional:

$$J_{RH}(u_k^{k+H-1};\, k) := \sum_{i=1}^{H} (y(k+i) - r(k+i))^2 + \rho u^2(k+i-1)$$

MPC problem

$$\min_{u_k^{k+H-1}} J_{RH}(u_k^{k+H-1};\, k)$$

$$\text{subject to} \quad \begin{array}{c} x(k+1) = Ax(k) + bu(k) \\ y(k) = Cx(k) \end{array}$$

Linear state prediction

Iterate the state model to get state predictors from present time k

$$x(k+1) = Ax(k) + Bu(k)$$

$$x(k+2) = Ax(k+1) + Bu(k+1) = A^2x(k) + Abu(k) + Bu(k+1)$$

$$x(k+3) = Ax(k+2)+Bu(k+2) = A^3x(k)+A^2Bu(k)+ABu(k+1)+Bu(k+2)$$

In general

$$x(k+i) = A^ix(k) + A^{i-1}Bu(k) + A^{i-2}Bu(k+1) + \ldots + Bu(k+i-1)$$

Output prediction

State predictors:

$$x(k+i) = A^i x(k) + A^{i-1} Bu(k) + A^{i-2} Bu(k+1) + \ldots + Bu(k+i-1)$$

Output predictors: multiply the state by C:

$$y(k+i) = CA^i x(k) + CA^{i-1} Bu(k) + CA^{i-2} Bu(k+1) + \ldots + CBu(k+i-1)$$

Pencil of predictive models

$$
\begin{bmatrix} y(k+1) \\ y(k+2) \\ y(k+3) \\ \vdots \\ y(k+H) \end{bmatrix} = \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2B & CAB & CB & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{H-1}B & CA^{H-2}B & \dots & CAB & CB \end{bmatrix} \begin{bmatrix} u(k) \\ u(k+1) \\ u(k+2) \\ \vdots \\ u(k+H-1) \end{bmatrix} +
$$

$$
+ \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^H \end{bmatrix} x(k)
$$

Compact notation

$$Y := \begin{bmatrix} y(k+1) \\ \vdots \\ y(k+H) \end{bmatrix}, \quad Y^* := \begin{bmatrix} r(k+1) \\ \vdots \\ r(k+H) \end{bmatrix}, \quad U := \begin{bmatrix} u(k) \\ \vdots \\ u(k+H-1) \end{bmatrix}$$

$$W := \begin{bmatrix} CB & 0 & 0 & \dots & 0 \\ CAB & CB & 0 & \dots & 0 \\ CA^2B & CAB & CB & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ CA^{H-1}B & CA^{H-2}B & \dots & CAB & CB \end{bmatrix} \quad \Pi := \begin{bmatrix} CA \\ CA^2 \\ CA^3 \\ \vdots \\ CA^H \end{bmatrix}$$

With this notation:

$$Y = WU + \Pi x(k), \quad J_{RH} = (Y - Y^*)^\top (Y - Y^*) + RU^\top U$$

The cost as a quadratic form of U

Use the model $Y = WU + \Pi x(k)$ in the cost to get

$$J_{RH} = (Y - Y^*)^\top (Y - Y^*) + \rho U^T U$$

$$J_{RH} = (U^\top W^\top + x^\top(k)\Pi^\top - Y^{*\top})(WU + \Pi x(k) - Y^*) + \rho U^T U$$

$$J_{RH} = U^\top MU + 2(x^\top(k)\Pi^\top - Y^{*\top})WU + (x^\top(k)\Pi^\top - Y^{*\top})(\Pi x(k) - Y^*)$$

$$M := U^\top U + \rho I$$

The cost is a quadratic form of U.

A detour: gradient rules

Gradient of a linear function:

$$x \in \mathbb{R}^n, \quad f(x) = b^\top x$$

$$\nabla_x f(x) = b^\top$$

Gradient of a quadratic function:

$$x \in \mathbb{R}^n, \quad f(x) = x^\top A x$$

$$\nabla_x f(x) = 2x^\top A$$

Optimized control sequence

$$J_{RH} = U^\top MU + 2(x^\top(k)\Pi^\top - Y^{*\top})WU + (x^\top(k)\Pi^\top - Y^{*\top})(\Pi x(k) - Y^*)$$

Taking the gradient with respect to U and transposing:

$$U^\top M = -((x^\top \Pi^T - Y^{*\top})W$$

$$U = -M^{-1}W^\top \Pi x(k) + M^{-1}W^\top Y^*$$

MPC control law for unconstrained LQ problems

$$U = -M^{-1}W^{\top}\Pi x(k) + M^{-1}W^{\top}Y^*$$

According to the receding horizon strategy, only the first control of this sequence is applied to the plant:

$$u(k) = -K_{RH}x(k) + K_{ff}y(k)$$

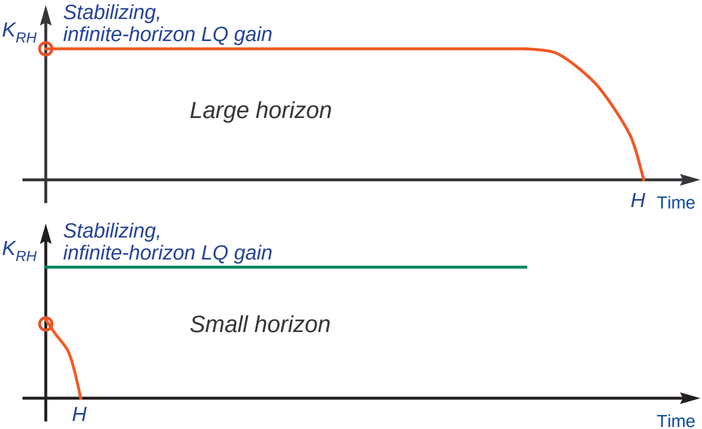
$$K_{RH} := e_1 M^{-1}W^{\top}\Pi \qquad K_{ff} := e_1 M^{-1}W^{\top}$$

where

$$e_1 = \begin{bmatrix} 1 & 0 & \dots & 0 \end{bmatrix}$$

Choice of the horizon: the Michelin star restaurant

Choice of the prediction horizon



The prediction horizon must be large enough so that the first gain is sufficiently close to the infinite horizon LQ gain that stabilizes the closed-loop.

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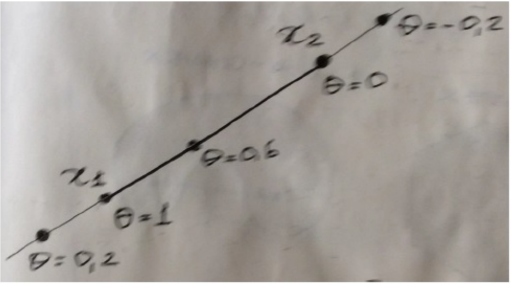
Optimization and constraints

Parametrization of a line segment between 2 points



Line passing through x_1 and x_2 described parametrically by

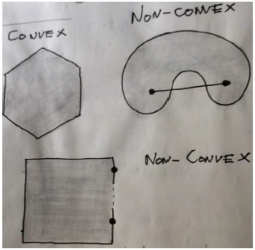
$$\theta x_1 + (1 - \theta) x_2, \ \theta \in \mathbb{R}$$

The segment between x_1 and x_2 is obtained for $\theta \in [0, 1]$.

Convex sets

A set C is convex if the line segment between any two points in C lies in C, *i.e.*

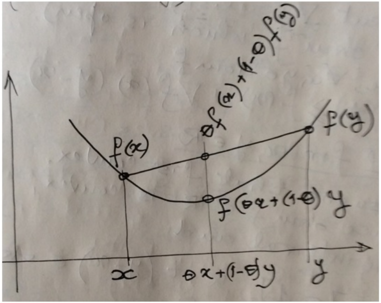
C is convex iff $\forall_{x_1, x_2 \in C}, \forall_{\theta \in [0,1]}$, then $\theta x_1 + (1-\theta)x_2 \in C$



Convex functions

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is a convex function iff its domain is a convex set and if for all $x, y \in dom\, f$, then

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta) f(y)$$



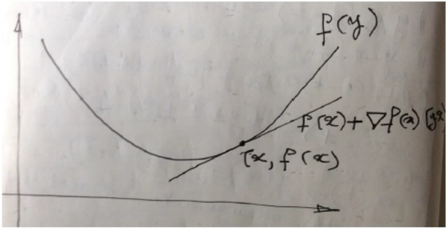
First order convexity condition

Assume that

1) $dom\ f$ is convex and open;

2) f is differentiable (∇f exists at each point in $dom\ f$, which is open;

Then, $\forall x, y \in dom\ f$, $f(y) \geq f(x) + \nabla f(x)(y - x)$



Uniqueness of the minimum

Uniqueness of the minimum of a C^1convex function

Let $f: D \subset \mathbb{R}^n \rightarrow \mathbb{R}$ be a C^1 convex function.

Consider a stationary point, i.e., a point x^* for which $\nabla f(x^*) = 0$.

By the first order convexity condition

$$f(y) \geq f(x^*) + \nabla f(x^*)(y - x^*) = f(x^*) + 0.(y - x^*) = f(x^*)$$

Hence

$$f(y) \geq f(x^*) \quad \forall y \in D,$$

meaning that x^* is the global minimum of f on D.

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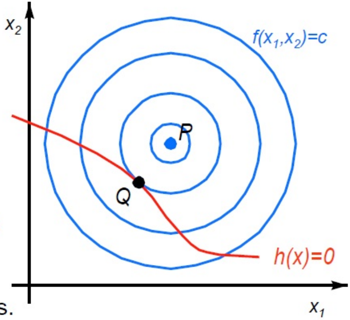
Equality Constraints

Equality constraints: An example in $\mathbb{R}^2$

The unconstrained minimum of f is P.

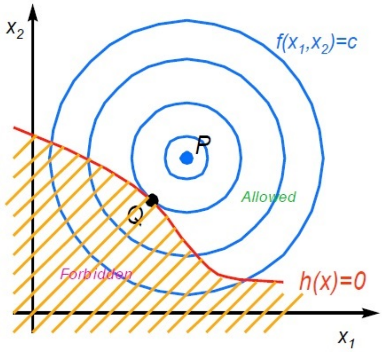
When x is constrained to be on the line defined by $h(x) = 0$, the constrained minimum is point Q.in

Point Q results from the intersection of the level curves of f with the line $h(x) = 0$, when this line is tangent to the level curves.

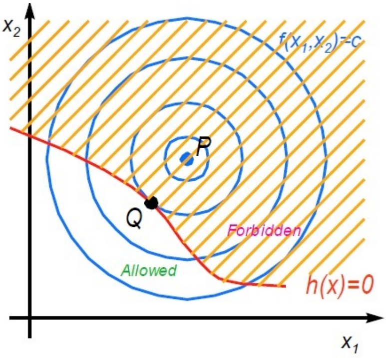


The MATLAB function *contour* plots the level curves of functions defined in $\mathbb{R}^2$

Inequality Constraints



(A)



(B)

Constrained Optimization

$D \subset \mathbb{R}^n$ a "surface" defined by the equality constraints

$$h_1(x) = h_2(x) = \cdots = h_m(x) = 0$$

$h_i: \mathbb{R}^n \to \mathbb{R}, \quad i = 1, \dots, m \quad C^1$ functions

f a C^1 function

Objective: Study the minima of f over D (constrained minima)

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Regular Points

$x^* \in D$ a local minimum

x^* is be assumed to a regular point. This means that $\nabla h_i, i = 1, \ldots, m$ are linearly independent at x^*.

This assumption rules out degenerate situations in which the necessary condition for minima to be presented may not hold.

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Dual optimization

Special case for equality constraints

Primal problem

Minimize $f(x)$ $\qquad f: \mathbb{R}^n \to \mathbb{R}$, $D = dom\, f$

Subject to $h_i(x), i = 1, \dots, m$

Lagrangian function: $\quad L: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \quad dom\, L = D \times \mathbb{R}^n$

$L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$

$\lambda = [\lambda_1 \quad \dots \quad \lambda_m]$ Lagrange multiplier vector or dual variable

KKT (Karush-Kuhn-Tucker) conditions

Lagrangian function $L(x, \lambda) = f(x) + \sum_{i=1}^{m} \lambda_i h_i(x)$

Special case for equality constraints [L2012] p.15

At (x^*, λ^*),

$$\begin{cases} \frac{\partial}{\partial x} L(x, \lambda) = 0 \\ \frac{\partial}{\partial \lambda} L(x, \lambda) = 0 \end{cases} \quad \text{or} \quad \begin{bmatrix} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x) \\ h(x^*) \end{bmatrix} = 0$$

Loosely speaking, adding Lagrange multipliers converts a constrained problem into an unconstrained one.

Warning: There are cases in which the stationary point of the Lagrangian function is not the constrained minimum.

How to solve the KKT conditions

1) In $\frac{\partial L}{\partial x} = 0$ express x in terms of λ;

2)Insert the expression of x in terms of λ in $\frac{\partial L}{\partial \lambda} = h(x) = 0$;

3)Solve the equation on λ that results from step 2) to get the stationary points for λ (λ^*);

4)Go back to the expressions of x in terms of λ obtained in step 1), and use the results of step 3) co cancel λ and obtain x^*.

Exercise on KKT

Consider the problem

$$\underset{x_1,x_2}{minimize}\ f(x_1,x_2) = x_1 + 1$$

Subject to $h(x_1,x_2) = x_1^2 + x_2^2 - 1 = 0$

a) Write the Lagrangian function

b) Write the KKT conditions

c)Find the stationary points of the Lagrangian function

d)Make a sketch to provide a geometrical interpretation

Exercise on KKT (cont.)

Solution

a) $L(x_1, x_2, \lambda) = x_1 + 1 + \lambda(x_1^2 + x_2^2 - 1)$

b) $\frac{\partial L}{\partial x_1} = 1 + 2\lambda x_1 = 0 \qquad \frac{\partial L}{\partial x_2} = 2\lambda x_2 = 0 \qquad \frac{\partial L}{\partial \lambda} = x_1^2 + x_2^2 - 1 = 0$

c) From the first two equations: $x_1 = -\frac{1}{2\lambda} \qquad x_2 = 0$

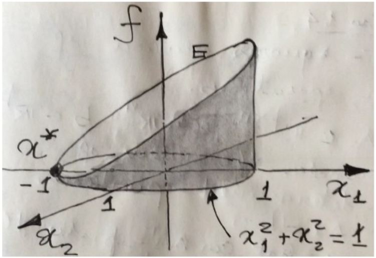
Insert these values for x_1 and x_2 in the 3rd equation and solve with respect to λ

$$\frac{1}{4\lambda^2} - 1 = 0 \;\rightarrow\; \lambda = \frac{1}{2}$$

Hence, $x_1 = -1$

Exercise on KKT (cont.)

e)The constraint $x_1^2 + x_2^2 - 1 = 0$ defines a circumference in the (x_1, x_2) plane. The function $f(x_1, x_2) = x_1 + 1$ defines a plane in the space (x_1, x_2, f). The intersection of this plane with the cylinder having as a basis that circumference yields



the elipse E, made by points with ordinate given by the value of f at the feasible points (the points of the circumference that satisfy the constraint). The minimum value for the constrained problem is attained at $(-1,0)$, in accordance with the stationary point of the KKT conditions.

Exercise: Least-squares solution of linear equations

Let $A \subset \mathbb{R}^{p \times n}$ a matrix, and $x \in \mathbb{R}^n$, $b \in \mathbb{R}^p$ vectors.

Consider the system of under-determined equations

$$Ax = b$$

Assume that $rank\ (AA^T) = p$ so that the inverse of AA^T exists.

Since there are infinite values of x that satisfy the equation, one possibility is to look for the minimum norm solution. As such, use the KKT conditions to solve the following minimization problem with equality constraints

$$\text{Minimize } x^T x$$

$$\text{Subject to } Ax - b = 0$$

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Solution

$$L(x,\lambda) = x^T x + \lambda^T (Ax - b)$$

$$\nabla_x L = 2x^T + \lambda^T A = 0 \quad \text{or, transposing} \quad 2x + A^T \lambda = 0$$

From which $x = -\frac{1}{2} A^T \lambda$

$$\nabla_\lambda L = x^T A^T - b^T = 0 \rightarrow Ax - b = 0$$

Insert now $x = -\frac{1}{2} A^T \lambda$ in this expression:

$$-\frac{1}{2} A A^T \lambda - b = 0 \quad \rightarrow \quad \lambda = -2(AA^T)^{-1} b$$

$$x = -\frac{1}{2} A^T (-2(AA^T)^{-1} b) \quad \rightarrow \quad x = A^T (AA^T)^{-1} b$$

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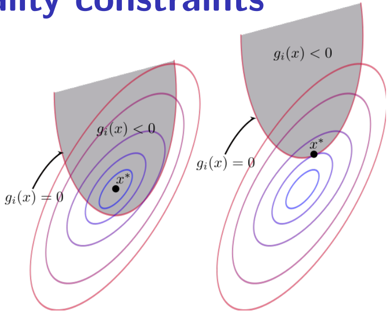
KKT conditions for inequality constraints

Minimize$_x$ $f(x)$

subject to $g_i(x) \leq 0$



Lagrangian function

$$\mathcal{L}(x, \mu) = f(x) + \sum_{i=1}^{m} \mu_i g_i(x)$$

Complementary conditions

$$\mu_i g_i(x) = 0, \text{ for } i = 1, \ldots, m$$

Numerical approximation

Robbins-Munro gradient scheme to approximate the Lagrange multiplier

$$\mu_{k+1} = \mu_k + \gamma_k \mu_k g(x)$$

Just one constraint (to simplify). The index k is the number of iterates.

γ_k must converge to zero, but such that $\sum_k \gamma_k = \infty$, $\sum_k \gamma_k^2$ finite, for the algorithm to converge.

Example: $\gamma_k = \gamma_0 / k$.

If the constraint is active, $g(x) = 0$ and μ_k will approach a value that is not zero.

If the constraint is not active, $g(x) < 0$ and $\mu_k \to 0$.

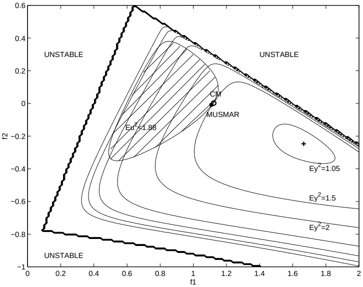
Example: constrained stochastic control

$$\text{Minimize}_u J = E[y^2(t)]$$

$$\text{Subject to} E[u^2(t)] \leq c^2$$

Lagrangian function

$$E[y^2(t)] + \mu E[u^2(t)]$$



Adjust the Lagrange multiplier (weight on the control action) such that

$$\mu(E[u^2(t)] - c^2) = 0$$

Approximate by $\mu_{t+1} + \mu_t + \gamma_t \mu_t (u(t)^2 - c^2)$

Numerical solvers

Interior point methods use the fact that a convex problem can be reduced to a linear programming method.
Linear programming problems can be solved numerically either by searching along the frontiers defined by the constraints or by a sequence of points in the interior of the region defined by these constraints.
Sequential quadratic methods solve a sequence of quadratic problems based on dual optimization.

	Interior-point	SQP
Speed	Slower	Faster
Confidence	More stable for calculating the optimal solution	Has larger variance
Large number of active set changes	Faster	Slower
Computational Complexity	Lower	Higher
Efficiency	Less Efficient	More Efficient

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The fmincon solver

$$\min_x f(x) \; such \; that \begin{cases} c(x) \leq 0, \\ c_{eq}(x) = 0, \\ A \cdot x \leq b, \\ A_{eq} \cdot x = b_{eq}, \\ x_{lb} \leq x \leq x_{ub}, \end{cases}$$

$$x = fmincon(fun, x_0, A, b, A_{eq}, b_{eq}, x_{lb}, x_{ub}, nonlcon, options, P_1, P_2, ...)$$

5
Nonlinear MPC

Nonlinear MPC - general formulation

$$
\begin{aligned}
\min_{\substack{\bar{u}_t,\ldots,\bar{u}_{t+N-1}\\ \bar{x}_t,\ldots,\bar{x}_{t+N}}} \quad & \sum_{i=0}^{N-1} l(\bar{x}_{t+i}, \bar{u}_{t+i}) + V_f(\bar{x}_{k+N}) \\
\text{s.t.} \quad & \bar{x}_t = x_t, \\
& \bar{x}^+ = f(\bar{x}, \bar{u}), \\
& \bar{x}_k \in \mathcal{X}_k, \; k = t, \ldots, t+N, \\
& \bar{u}_k \in \mathcal{U}_k, \; k = t, \ldots, t+N-1
\end{aligned}
$$

LQ MPC with state

$$\underbrace{\begin{bmatrix}\bar{x}_0\\ \bar{x}_1\\ \vdots\\ \bar{x}_N\end{bmatrix}}_{X} = \underbrace{\begin{bmatrix}0 & \dots & \dots & 0\\ A & \dots & 0 & \vdots\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & A & 0\end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix}\bar{x}_0\\ \bar{x}_1\\ \vdots\\ \bar{x}_N\end{bmatrix}}_{X} + \underbrace{\begin{bmatrix}0 & \dots & 0\\ B & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & B\end{bmatrix}}_{\tilde{B}} \underbrace{\begin{bmatrix}\bar{u}_0\\ \bar{u}_1\\ \vdots\\ \bar{u}_{N-1}\end{bmatrix}}_{U} + \underbrace{\begin{bmatrix}I\\ 0\\ \vdots\\ 0\end{bmatrix}}_{E} x_t \quad \tilde{Q} = \begin{bmatrix}Q & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & Q_f\end{bmatrix}, \qquad \tilde{R} = \begin{bmatrix}R & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & R\end{bmatrix}$$

$$\begin{aligned}
\min_{X,U} \quad & X^\top \tilde{Q} X + U^\top \tilde{R} U \\
\text{s.t.} \quad & X = \tilde{A}X + \tilde{B}U + Ex(t), \\
& X \in \tilde{\mathcal{X}}, \\
& U \in \tilde{\mathcal{U}},
\end{aligned}$$

6
Stability constraints

7
State estimation

8
Distributed MPC