DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Probability Theory
LMAC, MMA

1st. Semester – 2021/2022
2022/02/23 — 08:00
Exam 2

Duration: **120** minutes
- Please justify all your answers.
- This exam has TWO PAGES and TWELVE QUESTIONS. The total of points is 20.0.

Chap. 1 — Probability spaces — 3.5 points

1. Let $\Omega = \mathbb{R}$ and $\mathscr{A}$ be the class consisting of intervals of the type $(x, +\infty)$, for $x \in \Omega$. **(1.0)**

Show that $\mathscr{A}$ is closed under finite unions and finite intersections but not under complements. Comment.

- **Requested proof**

  Let $u = \min\{x, y\}$ and $v = \max\{x, y\}$. Then:

  (i) $(x, +\infty) \cup (y, +\infty) = (u, +\infty) \in \mathscr{A}$ (closure under finite unions ✓);

  (ii) $(x, +\infty) \cap (y, +\infty) = (v, +\infty) \in \mathscr{A}$ (closure under finite intersections ✓);

  (iii) $(x, +\infty)^c = (-\infty, x] \notin \mathscr{A}$ (non-closure under complements ✓).

- **Comment**

  $\mathscr{A}$ cannot be a $\sigma-$algebra on Ω, because it certainly fails the second of the minimal set of three postulates for a non-empty class of subsets $\mathscr{A}$ of Ω to be a $\sigma-$algebra on Ω.

2. Consider $P(A) = \int_{A\cap[0,0.75)} e^{-x}\,dx + e^{-0.75} \times \epsilon_{\{0.75\}}(A)$, for $A \in \mathscr{B}(\mathbb{R})$, where $\epsilon_{\{0.75\}}(A)$ is a point mass at $\{0.75\}$. **(2.5)**

Obtain $P(\{0.75\})$. Moreover, derive the d.f. associated with P, $F_P(x)$, for $x \in \mathbb{R}$, and plot its graph.

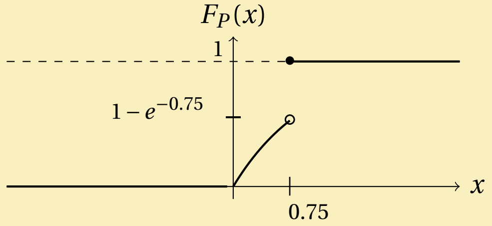
The result is the c.d.f. of a Borel measurable function $X = g(Y)$ of an absolutely continuous r.v. Y. Can you identify g and the distribution of Y?

- **Requested probability**

$$P(\{0.75\}) = \int_{\{0.75\}\cap[0,0.75)} e^{-x}\,dx + e^{-0.75} \times \epsilon_{\{0.75\}}(\{0.75\}) = 0 + e^{-0.75} \times 1 = e^{-0.75}$$

- **D.f. associated with P and its plot**

$$\begin{aligned} F_P(x) &= P((-\infty,x]) = \int_{(-\infty,x]\cap[0,0.75)} e^{-t}\,dt + e^{-0.75} \times \epsilon_{\{0.75\}}((-\infty,x]) \\ &= \begin{cases} 0, & x \le 0 \\ \int_0^x e^{-t}\,dt + e^{-0.75} \times 0 = 1 - e^{-x}, & 0 < x < 0.75 \\ \int_0^{0.75} e^{-t}\,dt + e^{-0.75} \times 1 = 1 - e^{-0.75} + e^{-0.75} = 1, & x \ge 0.75 \end{cases} \end{aligned}$$



- **Identifying the transformation and the distribution of Y**

  $X = \min\{Y, 0.75\}$, where $Y \sim$ exponential(1).

  [X is a mixed r.v. with: a point mass at 0.75; and an absolutely continuous *branch* in $[0, 0.75)$ with a truncated exponential distribution with parameter 1 and p.d.f. given by $\frac{e^{-x}}{1-e^{-0.75}}$, for $x \in [0, 0.75)$.]

3. Let X and Y be two r.v. and prove that $\frac{X}{Y}$ is a r.v. provided that $\{Y=0\}=\emptyset$. **(1.5)**

Hint: Take for granted that the product of two r.v. is also r.v.

- **R.v.**
  Let $(\Omega,\mathscr{F})$ and $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ be two measurable spaces. Then, $Y:\Omega\to\mathbb{R}$ and

$$Y^{-1}(B) = \{\omega\in\Omega: Y(\omega)\in B\}\in\mathscr{F}, \quad \forall B\in\mathscr{B}(\mathbb{R}).$$

- **Auxiliary result**
  [A function $g:\mathbb{R}\to\mathbb{R}$ is Borel measurable iff $g^{-1}(B)=\{y\in\mathbb{R}: g(y)\in B\}\in\mathscr{B}(\mathbb{R}), \quad \forall B\in\mathscr{B}(\mathbb{R})$. Moreover,] if

$$g^{-1}((-\infty,z]) = \{y\in\mathbb{R}: g(y)\le z\}\in\mathscr{B}(\mathbb{R}), \quad \forall z\in\mathbb{R},$$

  then $g:\mathbb{R}\to\mathbb{R}$ is Borel measurable.
  Now, let us consider a r.v. Y (such that $\{Y=0\}=\emptyset$) and its transformation $g(Y)=1/Y$. Then:
  - for $z<0$,
  $$g^{-1}((-\infty,z]) = \{y\in\mathbb{R}: g(y)=1/y\le z\} = \left\{y\in\mathbb{R}: 1/z\le y<0\right\} = [1/z,0)\in\mathscr{B}(\mathbb{R});$$
  - for $z\ge 0$,
  $$\begin{aligned} g^{-1}((-\infty,z]) &= \left\{y\in\mathbb{R}: g(y)=1/y\le z\right\} = \left\{y\in\mathbb{R}: y\ge 1/z \text{ or } y<0\right\} \\ &= [1/z,+\infty)\cup(-\infty,0)\in\mathscr{B}(\mathbb{R}). \end{aligned}$$

  As a result, $g(Y)=1/Y$ is a Borel measurable function and therefore a r.v.

- **Requested proof**
  Since we just proved that $1/Y$ is a r.v. and we can take for granted that the product of two r.v. is a r.v., we conclude that $X/Y = X\times\frac{1}{Y}$ is also a r.v. ✓

4. Let X and Y be two r.v. with joint p.d.f. given by $f_{X,Y}(x,y)=y\,e^{-x-y}$, for $x,y>0$. **(2.0)**

Derive (directly) the c.d.f. of $X-Y$.

Note: $1-F_{gamma(2,\lambda)}(x)=e^{-\lambda x}(1+\lambda x)$, for $x>0$ $(\lambda>0)$.

- **Random vector and range**
  (X,Y), $\quad X \perp\!\!\!\perp Y$, $\quad X\sim Y$, $\quad f_{X,Y}(x,y)=y\,e^{-x-y}$, $\quad x,y>0$, $\quad \mathbb{R}_{X,Y}=(\mathbb{R}^+)^2$

- **Transformation of (X,Y) and its range**
  $g(X,Y)=X-Y$, $\quad \mathbb{R}_{X-Y}=g(\mathbb{R}_{X,Y})=\mathbb{R}$

- **C.d.f. of $X-Y$**
  Keep in mind that $\star = P(X-Y\le u) = \int\int_{\{(x,y)\in(\mathbb{R}^+)^2: x\le u+y\}} f_{X,Y}(x,y)\,dx\,dy$.
  For $u>0$,

$$\begin{aligned} \star &= \int_0^{+\infty} y\,e^{-y}\left(\int_0^{u+y} e^{-x}\,dx\right)dy \\ &= \int_0^{+\infty} y\,e^{-y}\left(1-e^{-u-y}\right)dy \\ &= \int_0^{+\infty}\frac{1^2}{\Gamma(2)}\,y^{2-1}\,e^{-y}\,dy - e^{-u}\,\frac{\Gamma(2)}{2^2}\int_0^{+\infty}\frac{2^2}{\Gamma(2)}\,y^{2-1}\,e^{-2y}\,dy \\ &= \int_0^{+\infty} f_{gamma(2,1)}(y)\,dy - \frac{e^{-u}}{4}\int_0^{+\infty} f_{gamma(2,2)}(y)\,dy \\ &= 1-\frac{e^{-u}}{4}. \end{aligned}$$

For $u \le 0$,

$$\begin{aligned}
\star &= \int_{-u}^{+\infty} y\,e^{-y}\left(\int_0^{u+y} e^{-x}\,dx\right)dy = \int_{-u}^{+\infty} y\,e^{-y}\left(1-e^{-u-y}\right)dy \\
&= \int_{-u}^{+\infty} \frac{1^2}{\Gamma(2)}\,y^{2-1}\,e^{-y}\,dy - e^{-u}\frac{\Gamma(2)}{2^2}\int_{-u}^{+\infty}\frac{2^2}{\Gamma(2)}\,y^{2-1}\,e^{-2y}\,dy \\
&= \int_{-u}^{+\infty} f_{gamma(2,1)}(y)\,dy - \frac{e^{-u}}{4}\int_{-u}^{+\infty} f_{gamma(2,2)}(y)\,dy \\
&= 1-F_{gamma(2,1)}(-u) - \frac{e^{-u}}{4}\left[1-F_{gamma(2,2)}(-u)\right] \\
&\overset{Note}{=} e^{u}(1-u) - \frac{e^{-u}}{4}e^{2u}(1-2u) = \frac{e^{-u}}{4}(4-4u-1+2u) = \frac{e^{-u}(3-2u)}{4}.
\end{aligned}$$

Chap. 3 — Independence	4.5 points

5. Let $H=\{\text{heads}\}$ and $T=\{\text{tails}\}$ be the outcomes at tossing a coin with $P(H)=p$ and $P(T)=1-p$, where $p\in[0,1]$. Toss the coin three times independently and consider the events $A=\{\text{at most one tails}\}$ and $B=\{\text{all tosses are the same}\}$. **(1.0)**

After having identified the outcomes in A and in B and the probabilities of these two events, *confirm* that A and B are independent events, when $p=0,\frac{1}{2},1$, and dependent events, for all other values of p in the interval $[0,1]$.

- **Events and probabilities**

  $A=\{\text{at most one tails}\}=\{HHH,HHT,HTH,THH\}$[, where $= HHH = H_1\cap H_2\cap H_3$, etc.]

  $P(A)=P(\{\text{at most one tails}\})=p^3+3p^2(1-p)$

  $B=\{\text{all tosses are the same}\}=\{HHH,TTT\}$

  $P(B)=P(\{\text{all tosses are the same}\})=p^3+(1-p)^3$

- **Requested confirmation**

  Note that A and B are said to be independent events iff

  $$\begin{aligned}
  P(A\cap B) &= P(A)\times P(B)\\
  P(\{HHH\}) &= P(A)\times P(B)\\
  p^3 &= [p^3+3p^2(1-p)]\times[p^3+(1-p)^3]
  \end{aligned}$$

  Since

  $$\begin{aligned}
  0^3 &= [0^3+3\times 0^2(1-0)]\times[0^3+(1-0)^3]=0\\
  1^3 &= [1^3+3\times 1^2(1-1)]\times[1^3+(1-1)^3]=1\\
  0.5^3 &= [0.5^3+3\times 0.5^2(1-0.5)]\times[0.5^3+(1-0.5)^3]\\
  0.125 &= 0.5\times 0.25 = 0.125,
  \end{aligned}$$

  we can *confirm* that A and B are independent events, when $p=0,1/2,1$, and dependent events, for all other values of p in the interval $[0,1]$ ✓

6. Let X and Y two independent r.v. with common gamma$\left(\frac{1}{2},\frac{1}{2}\right)$ distribution. **(1.5)**

Derive (directly) the p.d.f. of $Z=X+Y$ and describe a method to generate pseudorandom numbers from the distribution of Z.

- **Random vector and range**

  (X,Y), $X\perp\!\!\!\perp Y$, $X\sim Y$, $f_X(x)=f_Y(x)=f(x)=\frac{(1/2)^{1/2}}{\Gamma(1/2)}\,x^{1/2-1}\,e^{-x/2}$, $x>0$, $\mathbb{R}_{X,Y}=(0,+\infty)^2$

- **Transformation of (X,Y) and its range**

  $Z = g(X,Y) = X+Y, \qquad \mathbb{R}_Z = g(\mathbb{R}_{X,Y}) = (0,+\infty)$

- **P.d.f. of Z**

  For $z>0$,

$$\begin{aligned}
f_Z(z) &\overset{X,Y\geq 0,\ X \perp\!\!\!\perp Y,\ X\sim Y}{=} \int_0^z f(x)\times f(z-x)\,dx \\
&= \int_0^z \frac{(1/2)^{1/2}}{\Gamma(1/2)}\,x^{-1/2}\,e^{-x/2}\times\frac{(1/2)^{1/2}}{\Gamma(1/2)}\,(z-x)^{-1/2}\,e^{-(z-x)/2}\,dx \\
&\overset{y=x/z,\ x=yz,\ dx=z\,dy}{=} \frac{1}{2}e^{-z/2}\int_0^1 \frac{1}{\Gamma(1/2)\,\Gamma(1/2)}\,[yz(z-yz)]^{-1/2}\,z\,dy \\
&= \frac{1}{2}e^{-z/2}\int_0^1 \frac{\Gamma(1/2+1/2)}{\Gamma(1/2)\,\Gamma(1/2)}\,y^{1/2-1}\,(1-y)]^{1/2-1}\,dx \\
&= \frac{1}{2}e^{-z/2}\int_0^1 f_{beta(1/2,1/2)}(y)\,dy \\
&= \frac{1}{2}e^{-z/2} \qquad [= \frac{(1/2)^{2/2}}{\Gamma(2/2)}\,z^{2/2-1}\,e^{-z/2} \equiv f_{gamma(2/2,1/2)}(z) \equiv f_{exp(1/2)}(z)].
\end{aligned}$$

  [We know that if $X,Y \sim \chi^2_{(1)}$ and $X \perp\!\!\!\perp Y$ then $Z \sim \chi^2_{(2)} \sim$ exponential$(1/2)$.]

- **Generation of a pseudorandom number from Z**

  Note that:

$$\begin{aligned}
F_Z(z) &= P(Z\leq z) = \begin{cases} 0, & z\leq 0 \\ \int_0^x \frac{1}{2}e^{-t/2}\,dt = 1-e^{-z/2}, & z>0; \end{cases} \\
F_Z(z) &= u \quad \Leftrightarrow \quad 1-e^{-z/2} = u \quad \Leftrightarrow \quad F^{-1}(u) = -2\ln(1-u), \quad 0<u<1.
\end{aligned}$$

  Furthermore, by resorting to the quantile transformation, we know that if $U\sim$ uniform$(0,1)$ then $F^{-1}(U) \equiv -2\ln(1-U) \sim Z$.

  Consequently, to generate a pseudorandom number from Z, z, we have to:

  – generate a pseudorandom number, u, from the uniform$(0,1)$ distribution;

  – assign $z = -2\ln(1-u)$.

7. Admit that orders arrive to a depot according to a non-homogeneous Poisson process with mean function $m(t) = \ln(1+t)$, $t\geq 0$. **(2.0)**

Compute the probability that the time between the arrivals of the first and second orders belongs to the interval $[1,2]$.

Note: $\int \frac{1}{(1+s)(1+t+s)}\,ds = \frac{\ln(1+s)}{t} - \frac{\ln(1+t+s)}{t}$.

- **Stochastic process**

  $\{N(t) : t>0\} \sim NHPP$

  $N(t)$ = number of orders arrived to the depot until time t

- **Mean value and intensity functions**

$$\begin{aligned}
m(t) &= \int_0^t \lambda(s)\,ds = \ln(1+t), \quad t\geq 0 \\
\lambda(t) &= \frac{d\,m(t)}{dt} = \frac{1}{1+t}, \quad t\geq 0.
\end{aligned}$$

- **R.v.**

  X_2 = time between the arrivals of the first and second order

- **Requested probability**

Please note that, for $n = 1$ and our particular NHPP,

$$\begin{aligned}
P(X_{n+1} > t) &\overset{form.}{=} \int_0^{+\infty} \lambda(s)\, e^{-m(t+s)} \frac{[m(s)]^{n-1}}{(n-1)!}\, ds \\
&\overset{n=1}{=} \int_0^{+\infty} \frac{1}{1+s}\, e^{-\ln(1+t+s)}\, ds \\
&= \int_0^{+\infty} \frac{1}{(1+s)(1+t+s)}\, ds \\
&\overset{Note}{=} \left. \frac{\ln(1+s)}{t} - \frac{\ln(1+t+s)}{t} \right|_{s=0}^{+\infty} \\
&= \left. \frac{1}{t} \times \ln\left(\frac{1+s}{1+t+s}\right) \right|_{s=0}^{+\infty} \\
&= \ln(1+t)/t.
\end{aligned}$$

Hence, the requested probability can be written as

$$P(X_2 \in [1,2]) \;=\; P(X_{1+1} > 1) - P(X_{1+1} > 2) = \frac{\ln(1+1)}{1} - \frac{\ln(1+2)}{2} \simeq 0.143841.$$

Chap. 4 — Expectation	3.5 points

8. Let X be the mass (in g) of a housemade article and admit that $X \sim \text{exponential}(1)$. **(1.5)**

The only available scale automatically reduces to $1g$ any mass larger than $1g$. Let Y be the mass shown by this scale. Define Y as a function of X and compute $E(Y)$.

- **R.v., c.d.f., and range**

X = mass of a housemade article

$$X \sim \text{exponential}(1), \qquad f_X(x) = \begin{cases} 0, & x \le 0 \\ e^{-x}, & x > 0, \end{cases} \qquad \mathbb{R}_X = \mathbb{R}^+$$

- **Relevant r.v. and its range**

Y = mass shown in the scale,

$$Y = \begin{cases} X, & 0 < X < 1 \\ 1, & X \ge 1 \end{cases} = g(X) = \min\{X, 1\}, \qquad \mathbb{R}_Y = g(\mathbb{R}_X) = [0,1]$$

- **Requested expected value**

$$\begin{aligned}
E(Y) &= E(\min\{X,1\}) = \int_{-\infty}^{+\infty} \min\{X,1\} \times f_X(x)\, dx = \int_0^1 x \times e^{-x}\, dx + \int_1^{+\infty} 1 \times e^{-x}\, dx \\
&= F_{gamma(2,1)}(1) + [1 - F_{exponential(1)}(1)] = [1 - F_{Poisson(1\times 1)}(2-1)] + e^{-1} \\
&= (1 - e^{-1} - e^{-1}) + e^{-1} = 1 - e^{-1} \simeq 0.632121
\end{aligned}$$

9. State Hölder's moment inequality. **(2.0)**

Illustrate this inequality, when $p = 3$, $q = \frac{3}{2}$ and the random vector (X, Y) has independent components that are uniformly distributed in the interval $[0,1]$.

- **Statement of the Hölder's moment inequality**

$$X \in L^p,\, Y \in L^q \;(p, q \in (1, +\infty) : \tfrac{1}{p} + \tfrac{1}{q} = 1) \quad \Rightarrow \quad E(|X \times Y|) \le E^{\frac{1}{p}}(|X|^p) \times E^{\frac{1}{q}}(|Y|^q)$$

- **Random vector**

$$(X,Y), \quad X \perp\!\!\!\perp Y \quad X \sim Y \sim \text{uniform}(0,1), \quad f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

- **Requested illustration** $(p=3, q=3/2)$
  Since $X \perp\!\!\!\perp Y$ and $X \sim Y$ we get:

$$\begin{aligned}
E(|X\times Y|) &\leq E^{\frac{1}{p}}(|X|^p)\times E^{\frac{1}{q}}(|Y|^q)\\
\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|x\times y|\times f_X(x)\times f_Y(y)\,dy\,dx &\leq \left[\int_{-\infty}^{+\infty}|x|^p\times f_X(x)\,dx\right]^{\frac{1}{p}}\times\left[\int_{-\infty}^{+\infty}|y|^p\times f_Y(y)\,dy\right]^{\frac{1}{q}}\\
\int_0^1\int_0^1 x\times y\,dy\,dx=\left(\int_0^1 x\,dx\right)^2 &\leq \left[\int_0^1 x^3\,dx\right]^{\frac{1}{3}}\times\left[\int_0^1 y^{3/2}\,dy\right]^{\frac{2}{3}}\\
\left(\frac{x^2}{2}\bigg|_0^1\right)^2 &\leq \left(\frac{x^4}{4}\bigg|_0^1\right)^{\frac{1}{3}}\times\left(\frac{x^{5/2}}{5/2}\bigg|_0^1\right)^{\frac{2}{3}}\\
\left(\frac{1}{2}\right)^2 &\leq \left(\frac{1}{4}\right)^{\frac{1}{3}}\times\left(\frac{2}{5}\right)^{\frac{2}{3}}\\
0.25 &\leq 0.341995\simeq 0.629961\times 0.542884. \qquad \checkmark
\end{aligned}$$

Chap. 5 — Stochastic convergence concepts and classical limit theorems	5.0 points

10. Let $\{X_n : n\in\mathbb{N}\}$ be a sequence of i.i.d. r.v. with common p.d.f. $f_X(x)=\theta\,x^{-2}\times I_{[\theta,+\infty)}(x)$, where θ is an unknown positive constant. **(1.5)**

After having derived the c.d.f. of $Y_n = X_{(1:n)}$, where $X_{(1:n)}=\max_{i=1,\ldots,n} X_i$, show that $Y_n \xrightarrow{P} \theta$.

- **Sequence of r.v.**

$$\{X_n : n\in\mathbb{N}\},\qquad X_n \overset{i.i.d.}{\sim} X,\quad n\in\mathbb{N},\qquad f_X(x)=\begin{cases}\theta\,x^{-2}, & x\geq\theta\quad(\theta>0)\\ 0, & \text{otherwise}\end{cases}$$

- **Another sequence of r.v.**
  $\{Y_n : n\in\mathbb{N}\}$

  $Y_n = X_{(1:n)}$

- **Requested c.d.f.**
  For $y\in[\theta,+\infty)$, we have

$$\begin{aligned}
F_{X_{(1:n)}}(x) &= 1-P\left(\min_{i=1,\ldots,n} X_i > x\right)=1-P(X_i>x,\, i=1,\ldots,n)\overset{X_i\overset{i.i.d.}{\sim}X}{=}1-[P(X>x)]^n\\
&=\begin{cases}0, & x\leq\theta\\ 1-\left(\int_x^{+\infty}\theta\,\frac{1}{t^2}\,dt\right)^n=1-\left(-\frac{\theta}{t}\Big|_x^{+\infty}\right)^n=1-\left(\frac{\theta}{x}\right)^n, & x>\theta\end{cases}
\end{aligned}$$

- **Requested proof**
  Since $\frac{\theta}{y}\in(0,1)$, when $y>\theta$, we have

$$\lim_{n\to+\infty} F_{Y_n}(y) = \begin{cases}0, & y\leq\theta\\ 1, & y>\theta\end{cases}$$

  which is equal to the c.d.f. of a degenerate r.v. at θ, $F_\theta(x)=I_{[\theta,+\infty)}(y)$, for all points at which $F_\theta(x)$ is continuous. Hence, $Y_n\xrightarrow{d}\theta$, that is, $Y_n\xrightarrow{P}\theta$. $\checkmark$

11. Let $\{X_n : n\in\mathbb{N}\}$ be a sequence of independent r.v. such that: **(1.5)**

- $P(X_n=-1)=P(X_n=+1)=\frac{1}{2}-\frac{1}{2^{n+1}}$;
- $P(X_n=-2^n)=P(X_n=+2^n)=\frac{1}{2^{n+1}}$.

After having identified convenient centering and norming constants, a_n and b_n, prove that this sequence of r.v. obeys the weak law of large numbers.

- **Sequence of r.v.**

  $\{X_n : n \in \mathbb{N}\}$

  X_n independent (thus uncorrelated) r.v., with the following symmetric p.f.:

  - $P(X_n = -1) = P(X_n = +1) = \frac{1}{2} - \frac{1}{2^{n+1}}$;
  - $P(X_n = -2^n) = P(X_n = +2^n) = \frac{1}{2^{n+1}}$.

  $E(X_n) = 0$ (p.f. symmetric around 0)

  $V(X_n) \stackrel{E(X_n)=0}{=} E(X_n^2) = 2 \times (-1)^2 \times \left(\frac{1}{2} - \frac{1}{2^{n+1}}\right) + 2 \times (-2^n)^2 \times \frac{1}{2^{n+1}} = 1 - \frac{1}{2^n} + 2^n < +\infty$

  $X_n \in L^2$

- **Centering and norming constants**

  Let $S_n = \sum_{i=1}^n X_i$. Then

$$\begin{aligned} a_n &= E(S_n) = \sum_{i=1}^n E(X_i) = 0 \\ b_n &= V(S_n) \stackrel{X_i\ indep.}{=} \sum_{i=1}^n V(X_i) = \sum_{i=1}^n \left(1 - \frac{1}{2^i} + 2^i\right) = n - \frac{1}{2} \times \frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}} + 2 \times \frac{1-2^n}{1-2} \\ &= n + 2^{n+1} + 2^{-n} - 3 \end{aligned}$$

- **Requested proof**

  Since $b_n \to +\infty$, we can invoke the WLLN for pairwise uncorrelated r.v. in L^2 to conclude that $\frac{S_n - a_n}{b_n} \stackrel{P}{\to} 0$, i.e., $\{X_n : n \in \mathbb{N}\}$ obeys the WLLN with respect to the norming constants b_n (and the centering constants a_n). ✓

12. Let: **(2.0)**

- $\{X_n : n \in \mathbb{N}\}$ and $\{Y_n : n \in \mathbb{N}\}$ be two independent sequences of i.i.d. r.v. to $X \sim \text{Bernoulli}(p_X)$ and $Y \sim \text{Bernoulli}(p_Y)$, respectively;
- $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the mean of the first n terms of $\{X_n : n \in \mathbb{N}\}$;
- $\bar{Y}_n$ is defined similarly.

Show that $\frac{(\bar{X}_n - \bar{Y}_n) - (p_X - p_Y)}{\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{\bar{Y}_n(1-\bar{Y}_n)}{n}}} \stackrel{d}{\to} \text{normal}(0,1)$.

- **Sequence of r.v.**

  $\{X_n : n \in \mathbb{N}\}$, where $X_n \stackrel{i.i.d.}{\sim} X \sim \text{Bernoulli}(p_X)$, $\quad n \in \mathbb{N}$

  $\perp\!\!\!\perp$

  $\{Y_n : n \in \mathbb{N}\}$, where $Y_n \stackrel{i.i.d.}{\sim} Y \sim \text{Bernoulli}(p_Y)$, $\quad n \in \mathbb{N}$

- **Other r.v.**

  $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i, \quad n \in \mathbb{N}$

  $\perp\!\!\!\perp$

  $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i, \quad n \in \mathbb{N}$

- **Relevant sequence of r.v.**

  $\{Z_n : n \in \mathbb{N}\}$

  $Z_n = \frac{(\bar{X}_n - \bar{Y}_n) - (p_X - p_Y)}{\sqrt{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{\bar{Y}_n(1-\bar{Y}_n)}{n}}}$

- **Auxiliary results**

  $E(\bar{X}_n) = p_X, \quad E(\bar{Y}_n) = p_Y, \quad E(\bar{X}_n - \bar{Y}_n) = p_X - p_Y$

  $V(\bar{X}_n) = \frac{p_X(1-p_X)}{n}, \quad V(\bar{Y}_n) = \frac{p_Y(1-p_Y)}{n}, \quad V(\bar{X}_n - \bar{Y}_n) \overset{\bar{X}_n \perp\!\!\!\perp \bar{Y}_n}{=} \frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{n} < +\infty$

- **Convergence I**

  Combining the auxiliary results and the mere application of the Lindeberg-Lévy CLT leads to the conclusion that

$$U_n = \frac{(\bar{X}_n - \bar{Y}_n) - (p_X - p_Y)}{\sqrt{\frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{n}}} = \frac{(\bar{X}_n - \bar{Y}_n) - E(\bar{X}_n - \bar{Y}_n)}{\sqrt{V(\bar{X}_n - \bar{Y}_n)}} \xrightarrow{d} \text{normal}(0,1).$$

- **Convergence II**

  We can invoke the WLLN for i.i.d. r.v. in L^2 and state the following convergences in probability:

$$\bar{X}_n \xrightarrow{P} p_X;$$
$$\bar{Y}_n \xrightarrow{P} p_Y.$$

  Capitalizing on these two results and on the closure of convergence in probability under product, addition, and continuous mappings, we get:

$$V_n = \sqrt{\frac{\frac{\bar{X}_n(1-\bar{X}_n)}{n} + \frac{\bar{Y}_n(1-\bar{Y}_n)}{n}}{\frac{p_X(1-p_X)}{n} + \frac{p_Y(1-p_Y)}{n}}} = \sqrt{\frac{\bar{X}_n(1-\bar{X}_n) + \bar{Y}_n(1-\bar{Y}_n)}{p_X(1-p_X) + p_Y(1-p_Y)}} \xrightarrow{P} 1.$$

- **Convergence III**

  Finally, we apply Slutsky's theorem to justify the preservation of the convergence in distribution under (restricted) division to obtain the desired result:

$$Z_n = \frac{U_n}{V_n} \xrightarrow{d} \text{normal}(0,1). \qquad \checkmark$$