Fourier and Wavelets

- Why do we need a Transform?
- Fourier Transform and the short term Fourier (STFT)
- Heisenberg Uncertainty Principle
- The continues Wavelet Transform
- Discrete Wavelet Transform
- Wavelets Transforms in Two dimensions

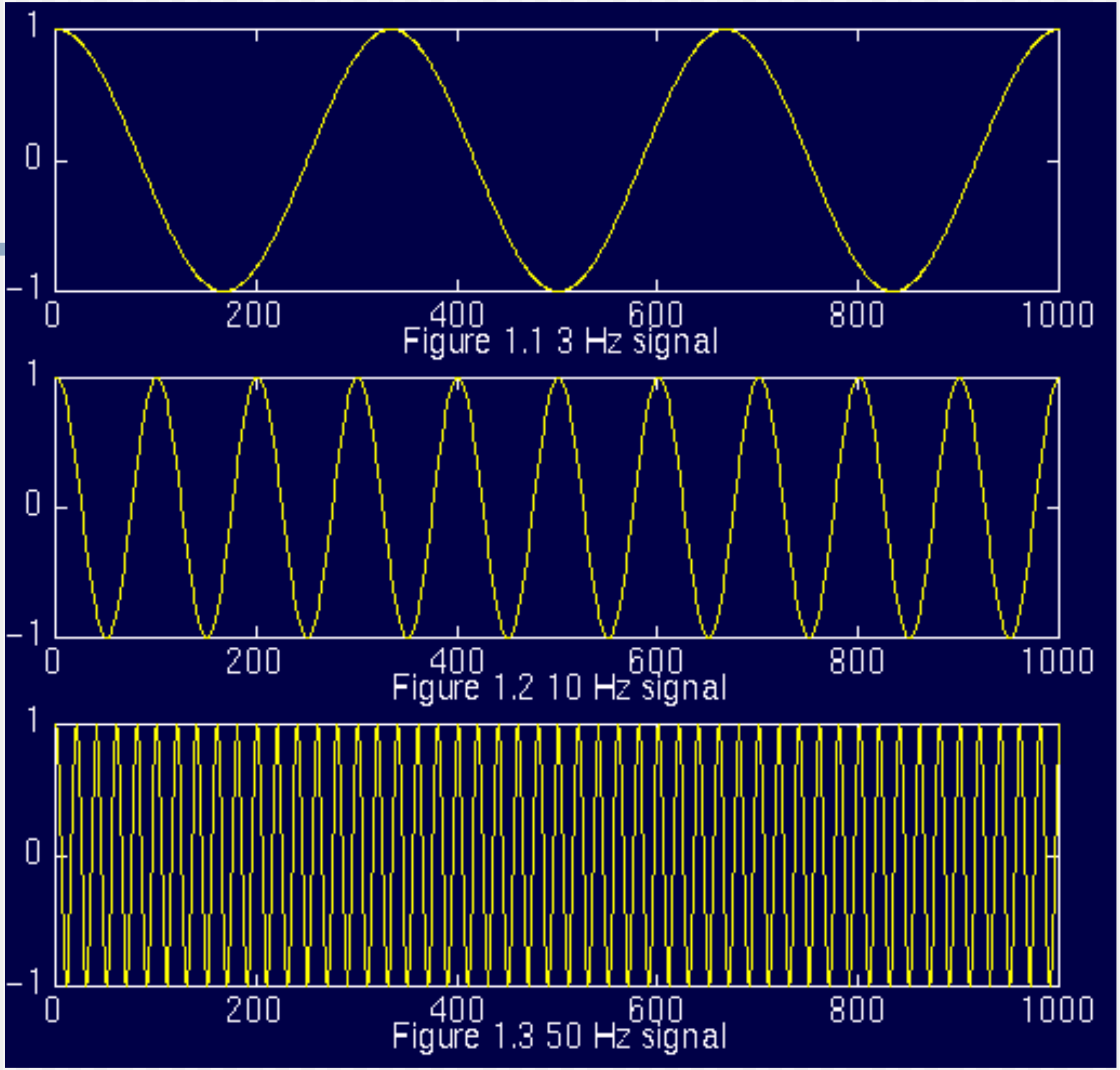
Based on...

- http://users.rowan.edu/~polikar/
- Making Wavelets, Robi Polikar's "The Wavelet Tutorial" featured by the Science Magazine's NetWatch Department, Science, vol. 300, no. 561, pp. 873, May 2003.

I) Why do we need a Transform?

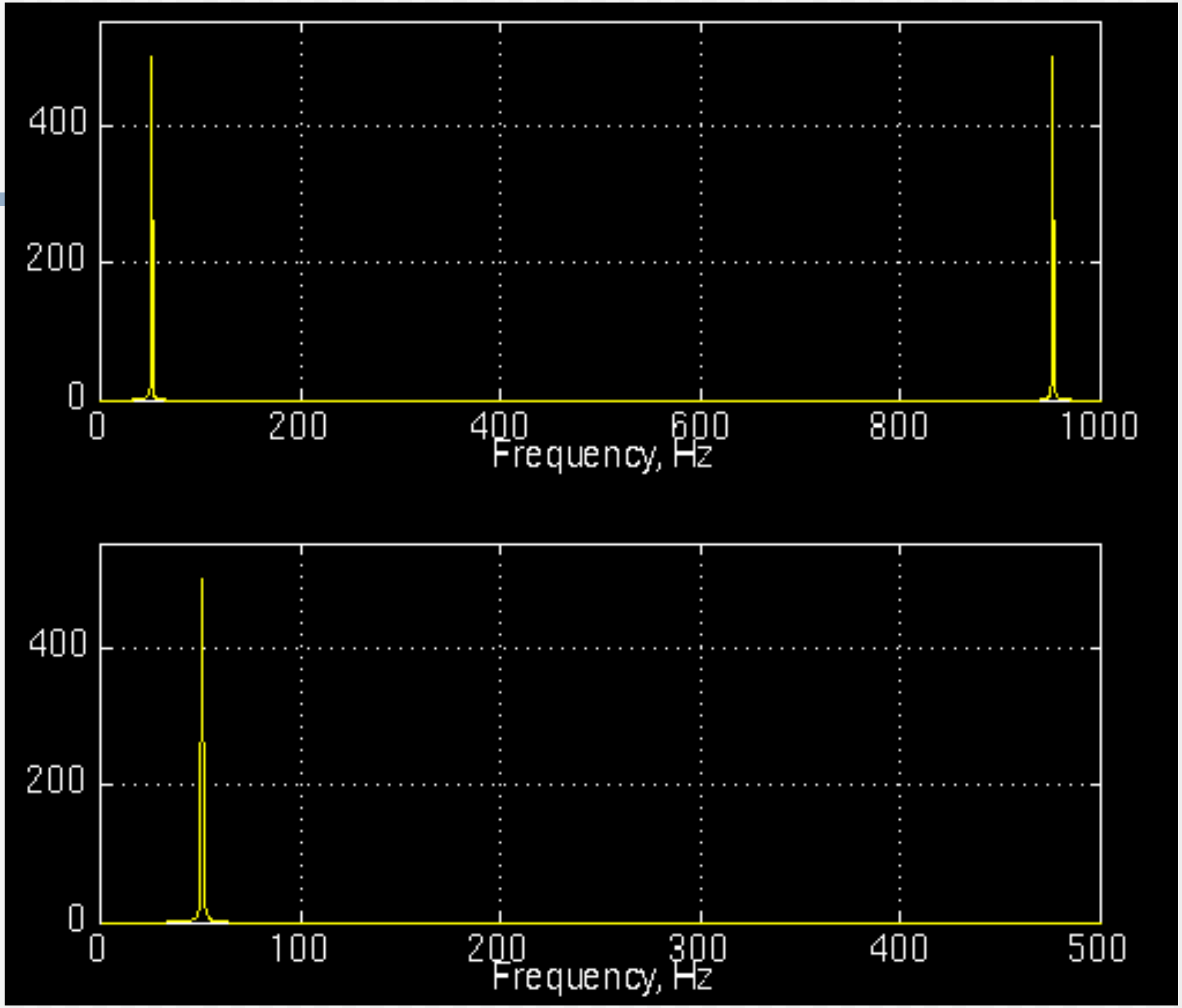
- Transformations are applied to signals to obtain a further information from that signal that is not readily available in the raw signal
- Most of the signals in practice, are TIME-DOMAIN signals in their raw format
- In many cases, the most distinguished information is hidden in the frequency content of the signal

- If something changes rapidly, we say that it is of high frequency
- If this does not change rapidly, i.e., it changes smoothly, we say that it is of low frequency.

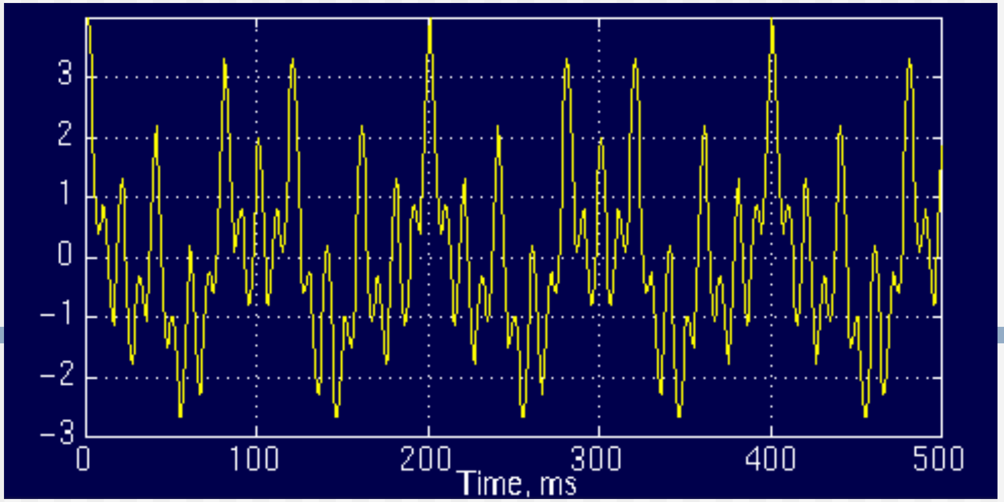

1
0
−1
0
200
400
600
800
1000
Figure 1.1 3 Hz signal
1
0
−1
0
200
400
600
800
1000
Figure 1.2 10 Hz signal
1
0
−1
0
200
400
600
800
1000
Figure 1.3 50 Hz signal

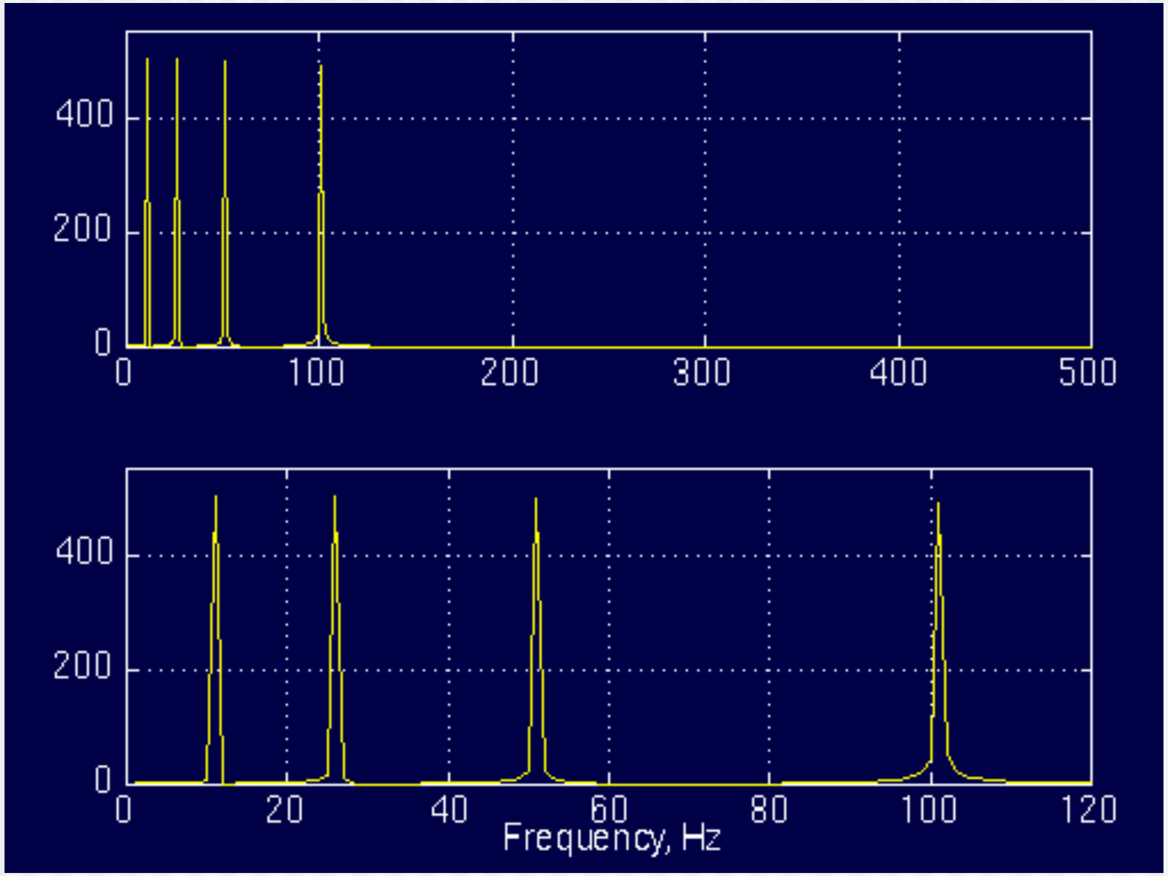
FOURIER TRANSFORM

- For example, if we take the FT of the electric current that we use in our houses,
- We will have one spike at 50 Hz
- Nothing elsewhere, since that signal has only 50 Hz frequency component


400
200
0
0
200
400
600
800
1000
Frequency, Hz
400
200
0
0
100
200
300
400
500
Frequency, Hz

- The frequency spectrum of a real valued signal is always symmetric. The top plot illustrates this point
- However, since the symmetric part is exactly a mirror image of the first part
- This symmetric second part is usually not shown

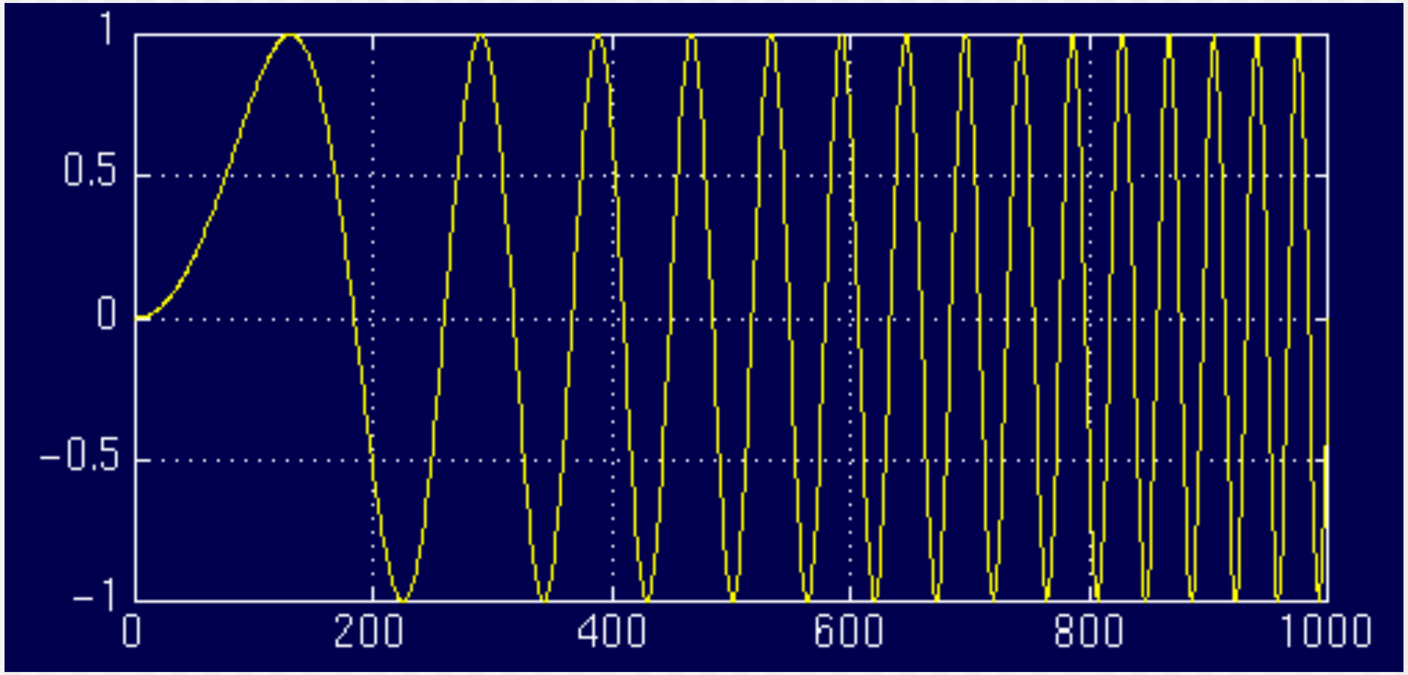

3
2
1
0
−1
−2
−3
0
100
200
300
400
500
Time, ms


400
200
0
0
100
200
300
400
500
400
200
0
0
20
40
60
80
100
120
Frequency, Hz

Stationary Signal

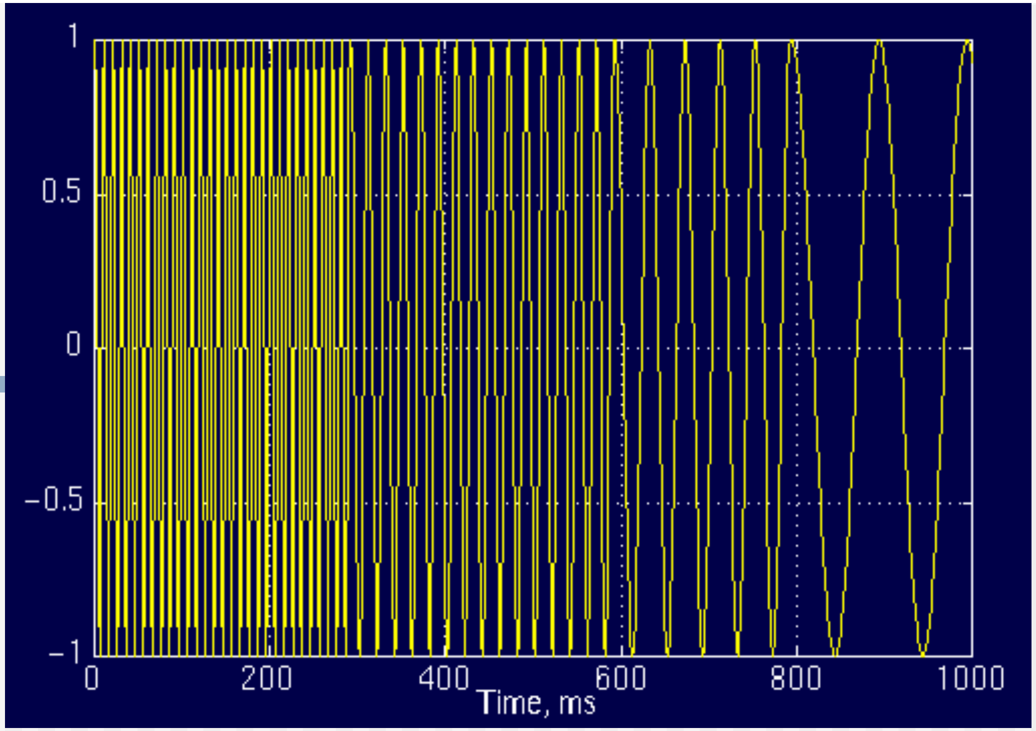
- Signals whose frequency content do not change in time are called stationary signals
- Non stationary signal, frequency content does change over time

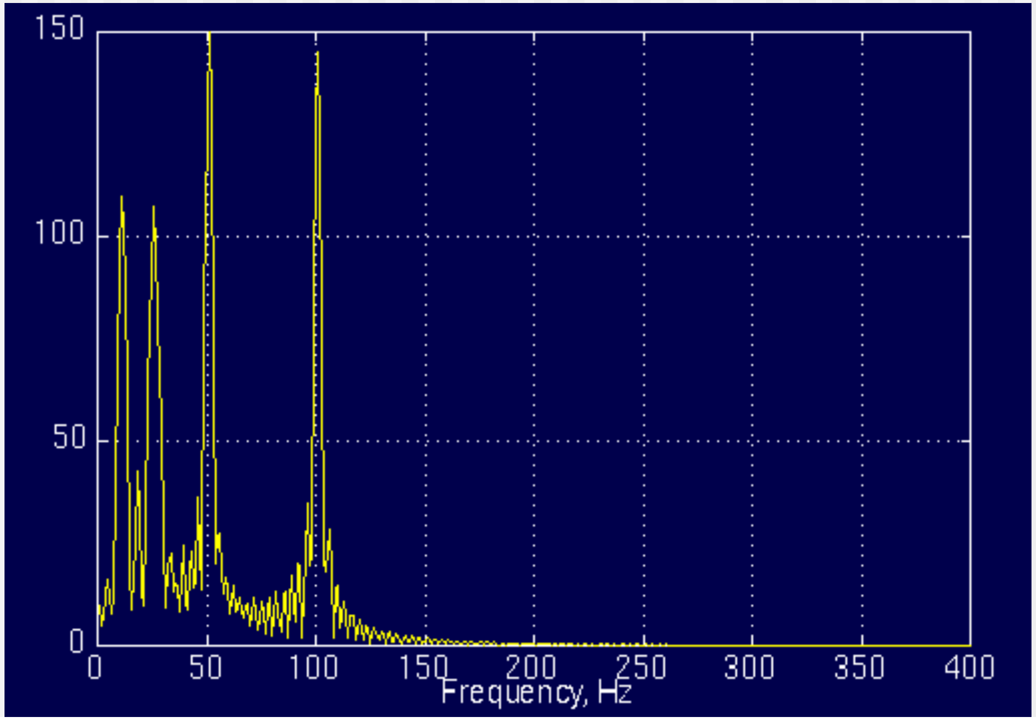
Non stationary signal



- At what times (or time intervals), do these frequency components occur?

- FT gives the spectral content of the signal, but it gives no information regarding where in time those spectral components appear!





II) FUNDAMENTALS:

FOURIER TRANSFORM

AND

THE SHORT TERM FOURIER TRANSFORM

the Fourier transform of x(t)

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2\pi i t f} dt$$

the inverse Fourier transform of X(f)

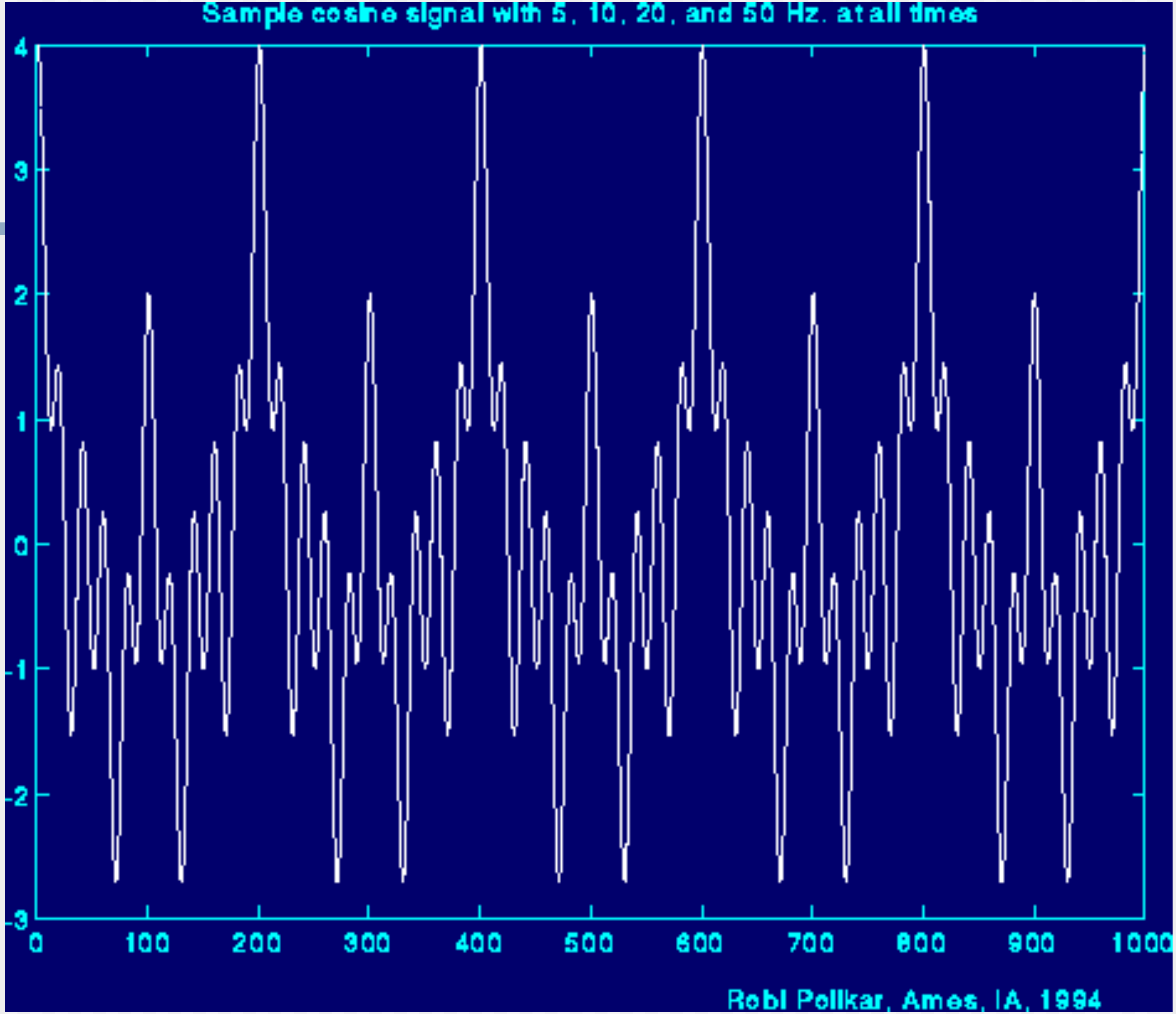
$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot e^{2\pi i t f} df$$

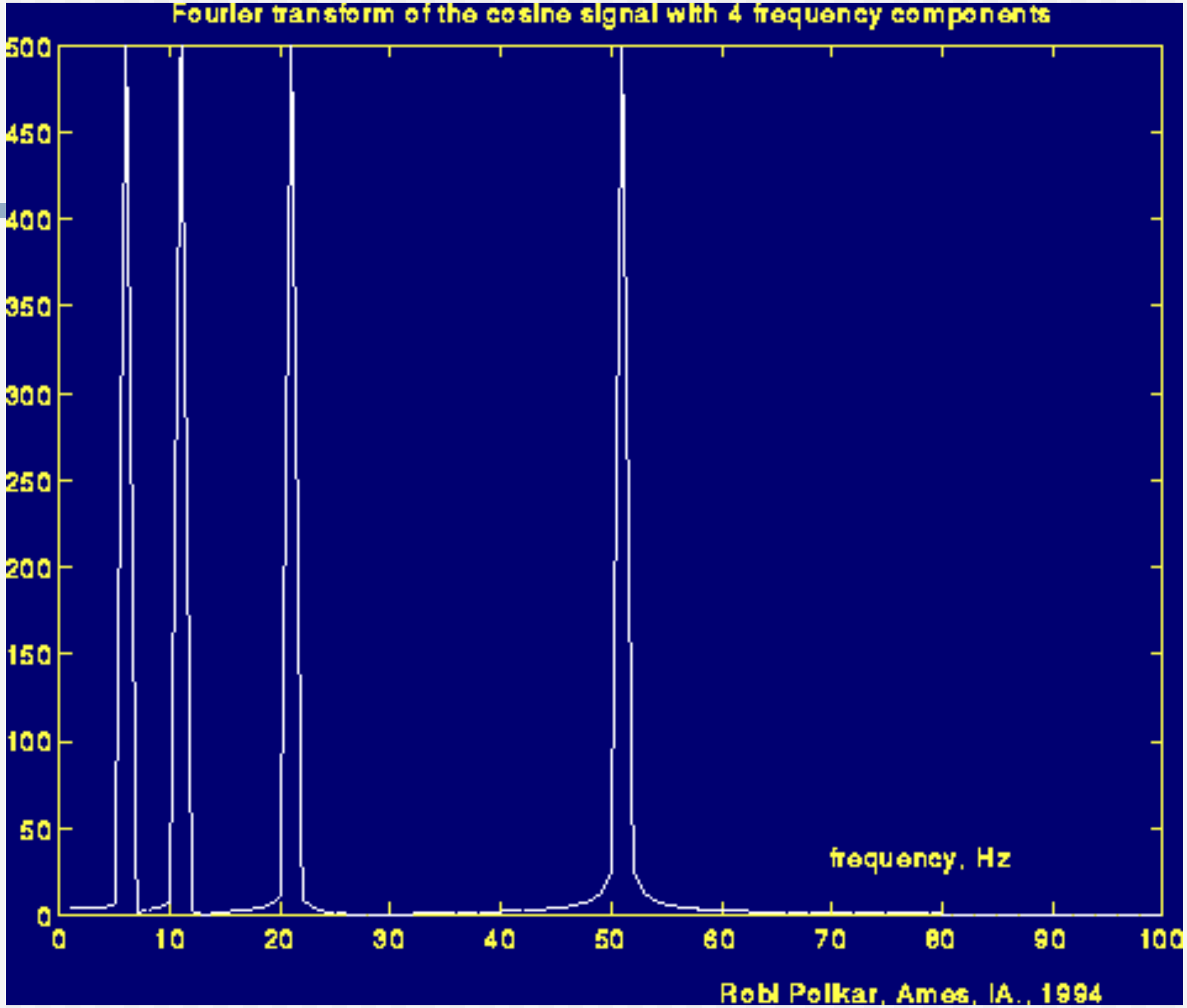
- *t* stands for time, *f* stands for frequency, and *x* denotes the signal
- *x* denotes the signal in time domain and the *X* denotes the signal in frequency domain

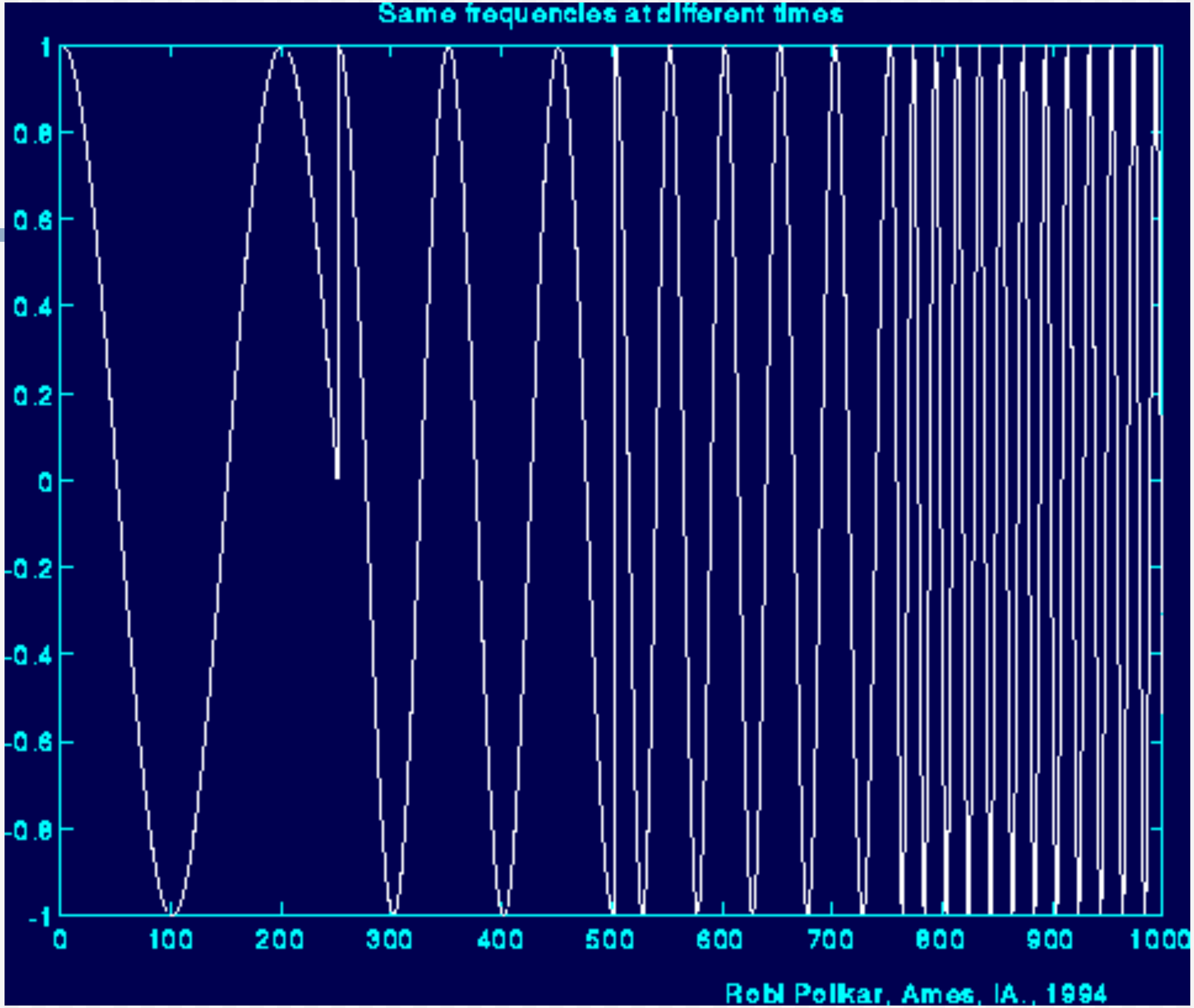
- The signal *x(t),* is multiplied with an exponential term, at some certain frequency "*f*" , and then integrated over **ALL TIMES** !

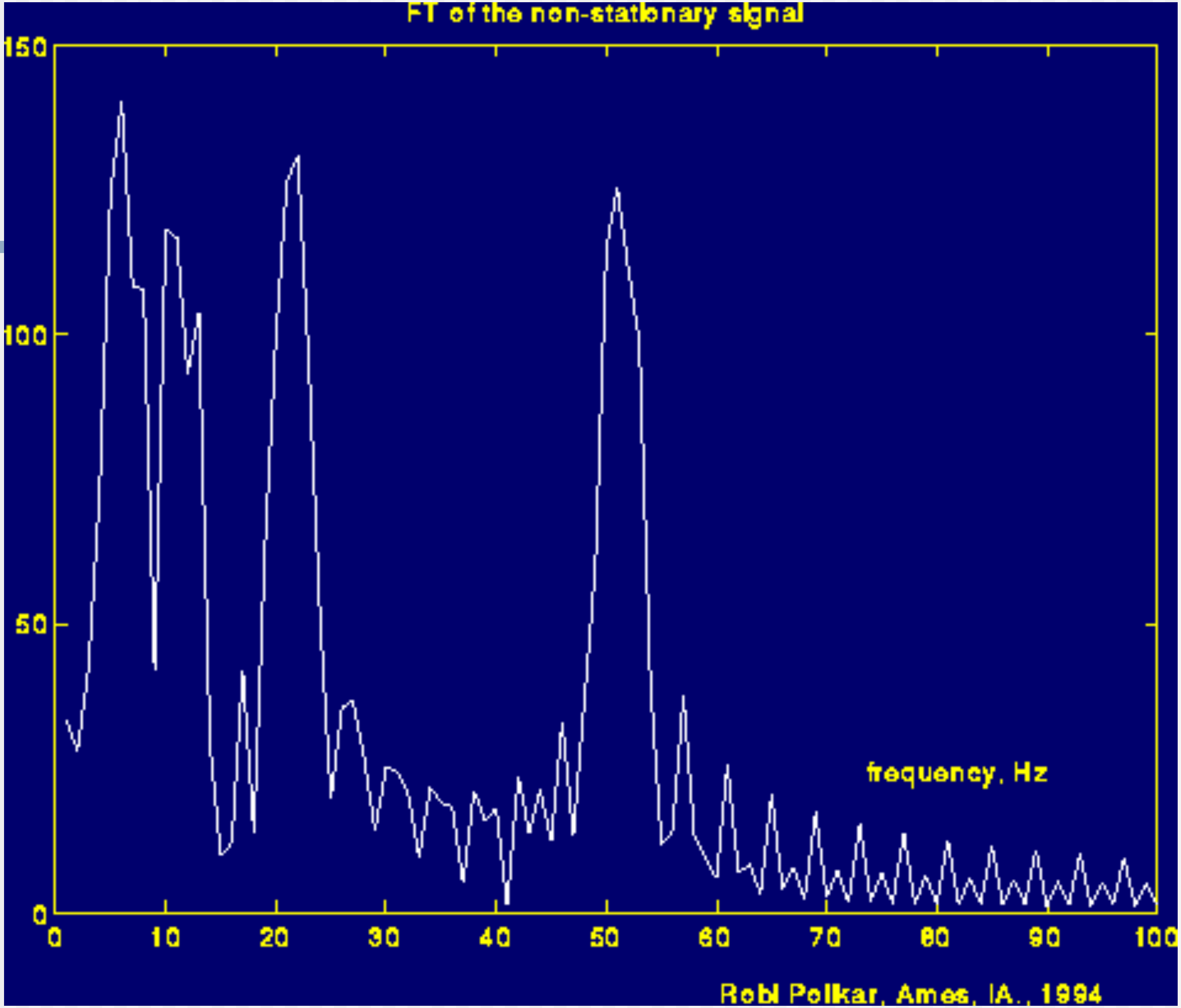
$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot ((\cos 2 \cdot \pi \cdot f \cdot t) + i \cdot \sin(2 \cdot \pi \cdot f \cdot t))dt$$

- Real part of cosine of frequency f, and an imaginary part of sine of frequency f
- If the result of this integration is a large value, then we say that : the signal x(t), has a dominant spectral component at frequency "f“
- The information provided by the integral, corresponds to all time instances
- No matter where in time the component with frequency "f" appears, it will affect the result of the integration equally as well
- Whether the frequency component "f" appears at time t1 or t2 , it will have the same effect on the integration.


Sample cosine signal with 5, 10, 20, and 50 Hz. at all times
4
3
2
1
0
-1
-2
-3
0
100
200
300
400
500
600
700
800
900
1000
Robi Polikar, Ames, IA, 1994


Fourier transform of the cosine signal with 4 frequency components
500
450
400
350
300
250
200
150
100
50
0
0
10
20
30
40
50
60
70
80
90
100
frequency, Hz
Robi Polikar, Ames, IA., 1994


Same frequencies at different times
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
100
200
300
400
500
600
700
800
900
1000
Robi Polikar, Ames, IA., 1994


FT of the non-stationary signal
150
100
50
0
0
10
20
30
40
50
60
70
80
90
100
frequency, Hz
Robi Polikar, Ames, IA., 1994

THE SHORT TERM FOURIER TRANSFORM (STFT)

- If this region where the signal can be assumed to be stationary small...
  - we look at that signal from narrow windows, narrow enough that the portion of the signal seen from these windows are indeed stationary
  - This approach of researchers ended up with a revised version of the Fourier transform, so-called : The Short Time Fourier Transform (STFT)

- There is only a minor difference between STFT and FT
- In STFT, the signal is divided into small enough segments, where these segments (portions) of the signal can be assumed to be stationary
- For this purpose, a window function "*w*" is chosen
- The width of this window must be equal to the segment of the signal where its stationarity is valid...

STFT

$$STFT_X^w(t', f) = \int_t x(t) \cdot w^*(t - t') \cdot e^{-i2\pi ft} dt$$

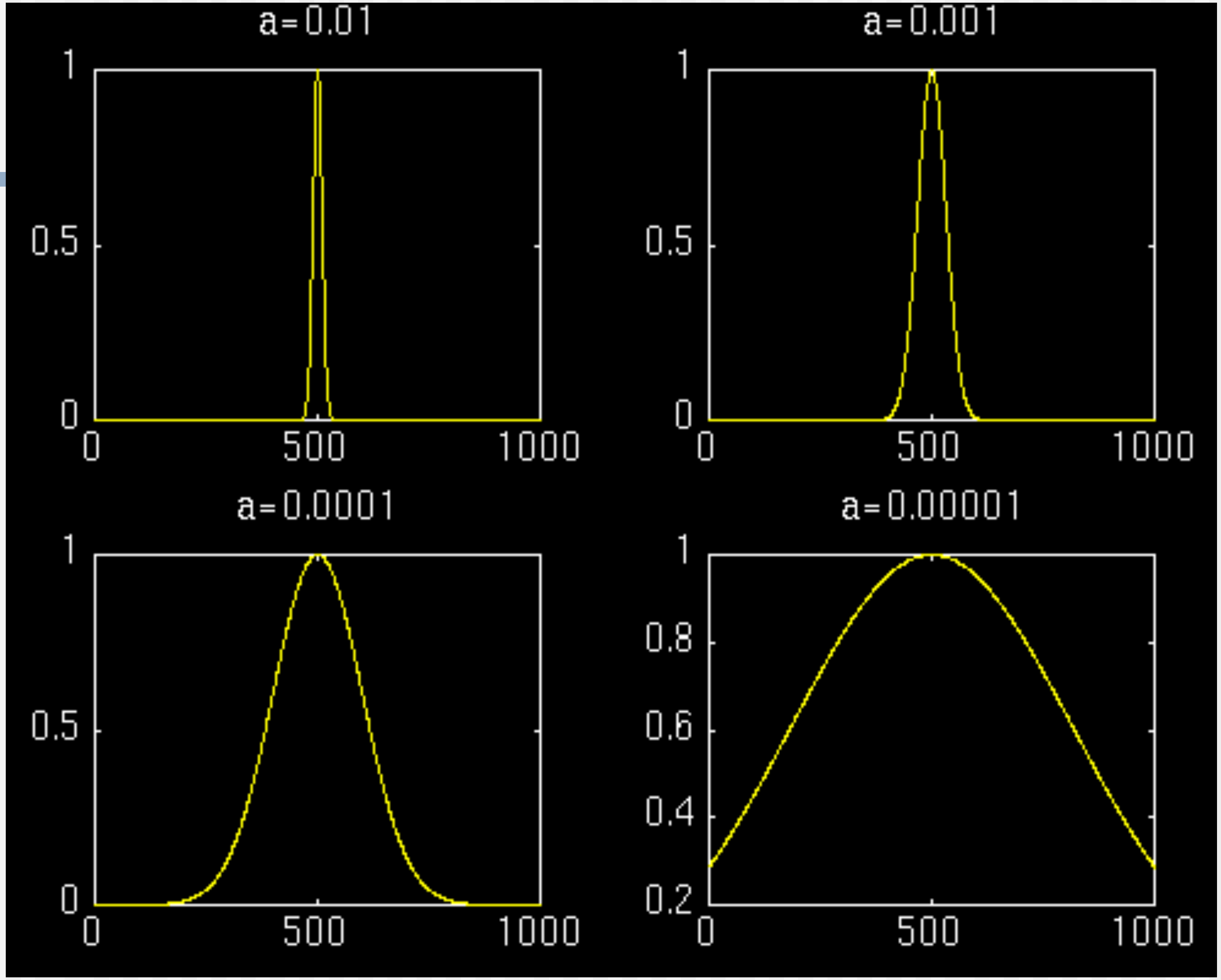
- x(t) is the signal itself, w(t) is the window function, and * is the complex conjugate
- STFT of the signal is nothing but the FT of the signal multiplied by a window function
  - complex conjugate of a complex number is given by changing the sign of the imaginary part
- For every t' and f a new STFT coefficient is computed

Window function

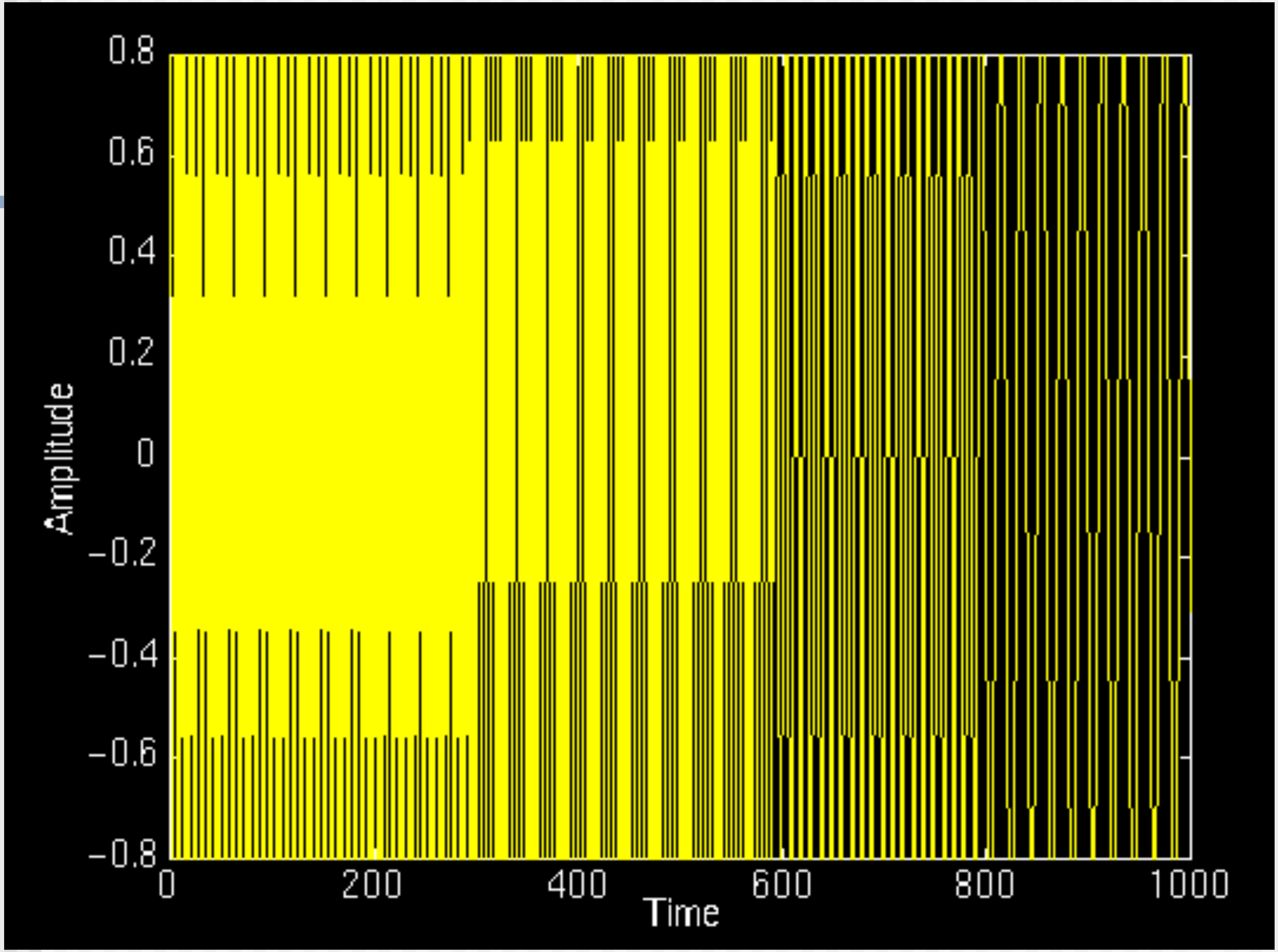
- Gaussian function:

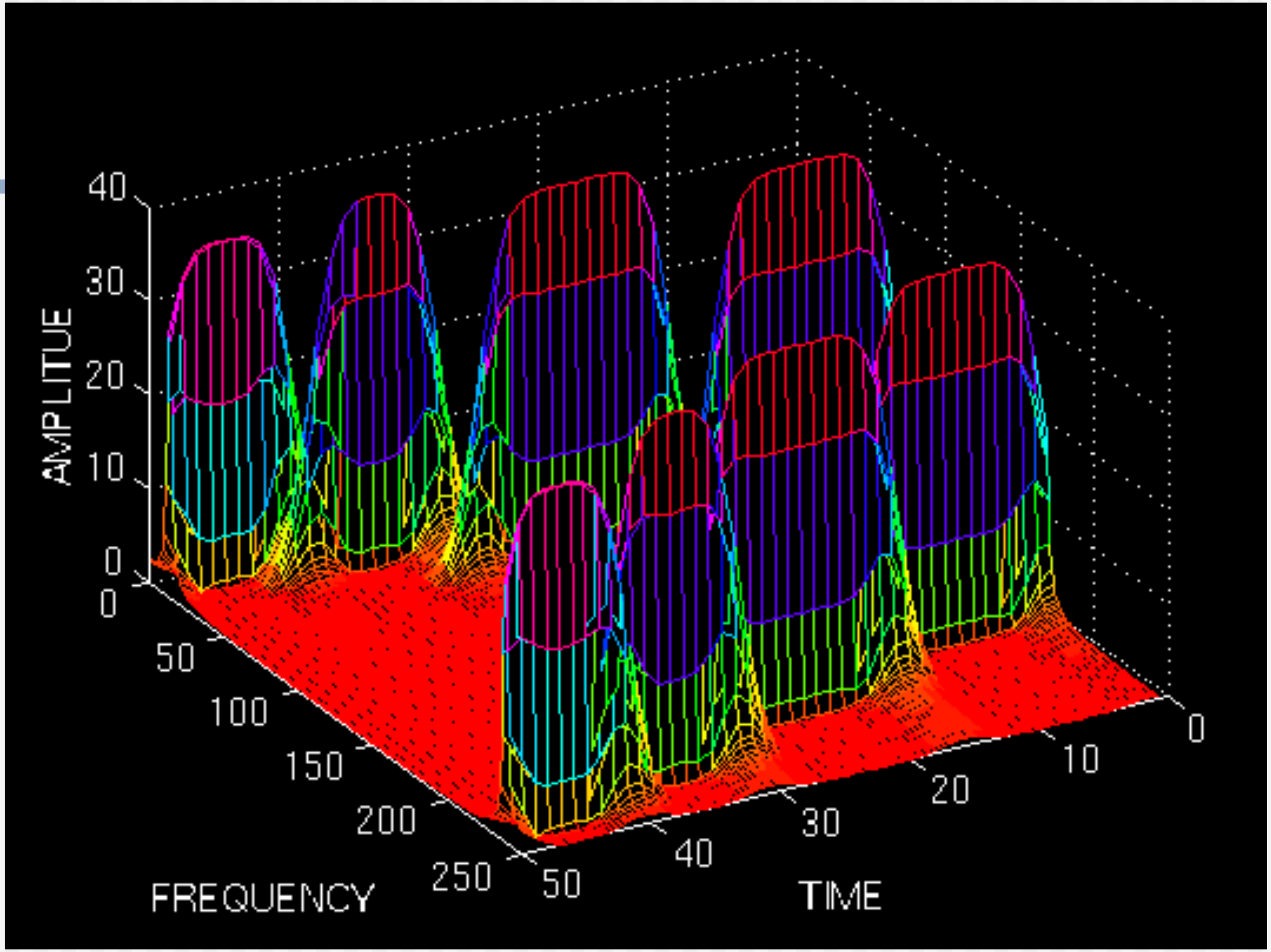
$$w(t) = e^{-a \cdot t^2 / 2}$$

  - *a* determines the length of the window, and *t* is the time



a=0.01
a=0.001
a=0.0001
a=0.00001
1
0.5
0
0.8
0.6
0.4
0.2
500
1000


0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
Amplitude
0
200
400
600
800
1000
Time

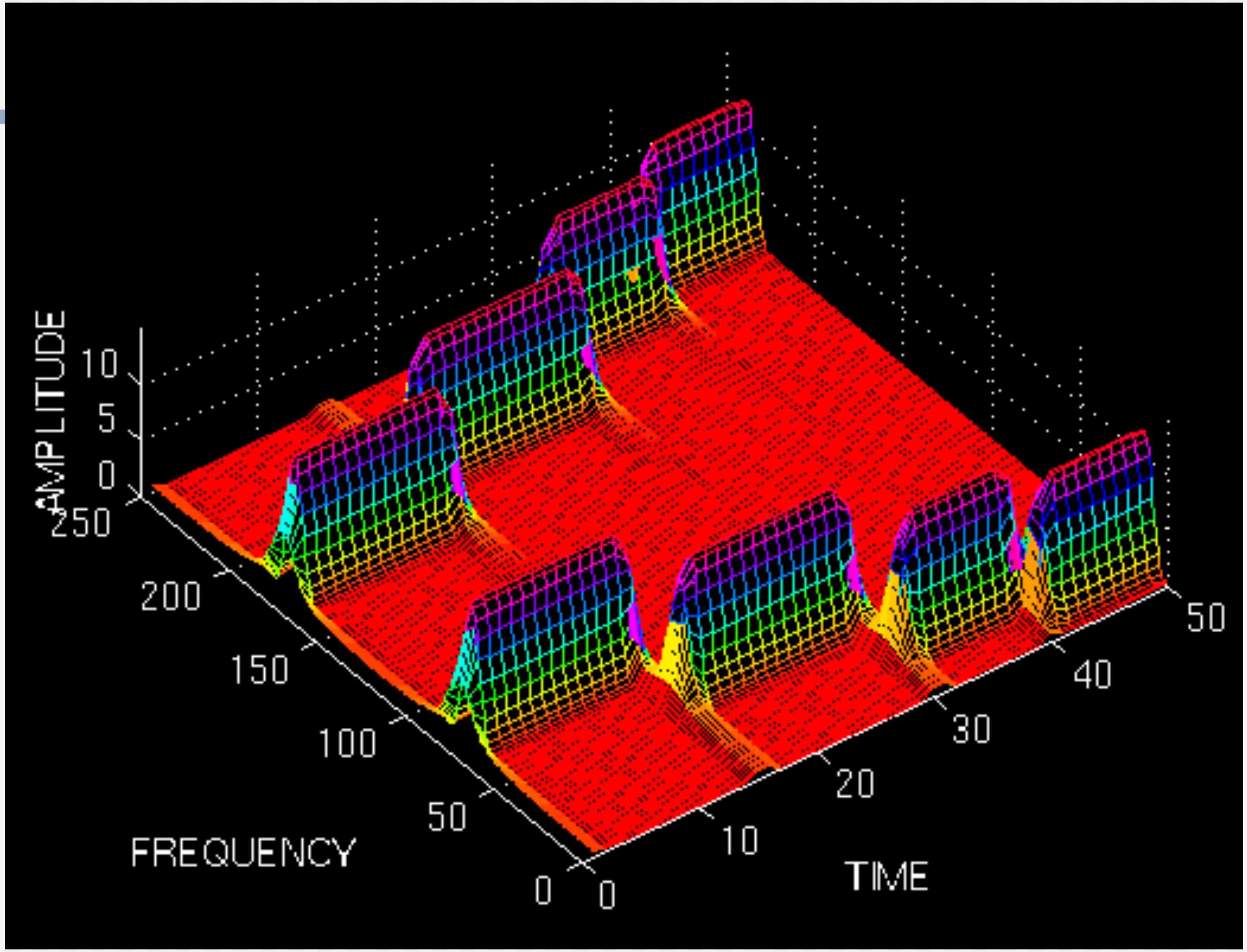

40
30
20
10
0
AMPLITUE
0
50
100
150
200
250
FREQUENCY
50
40
30
20
10
0
TIME

Heisenberg Uncertainty Principle

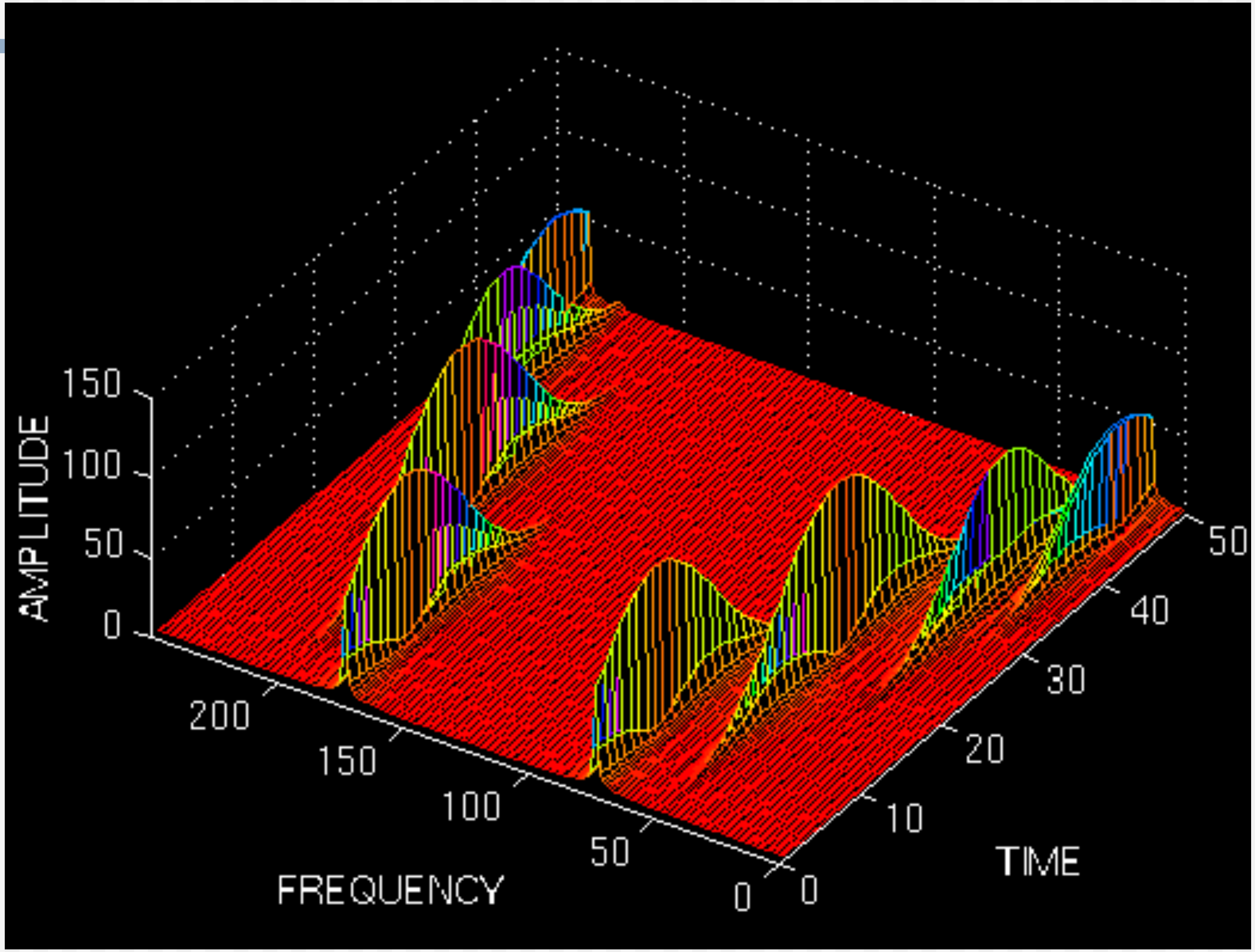
- This principle originally applied to the momentum and location of moving particles, can be applied to time-frequency information of a signal
- This principle states that one cannot know the exact time-frequency representation of a signal
  - One cannot know what spectral components exist at what instances of times
  - What one can know are the time intervals in which certain band of frequencies exist, which is a resolution problem

- The problem with the STFT has to do with the width of the window function that is used
- Narrow window ➔ good time resolution, poor frequency resolution
- Wide window ➔ good frequency resolution, poor time resolution

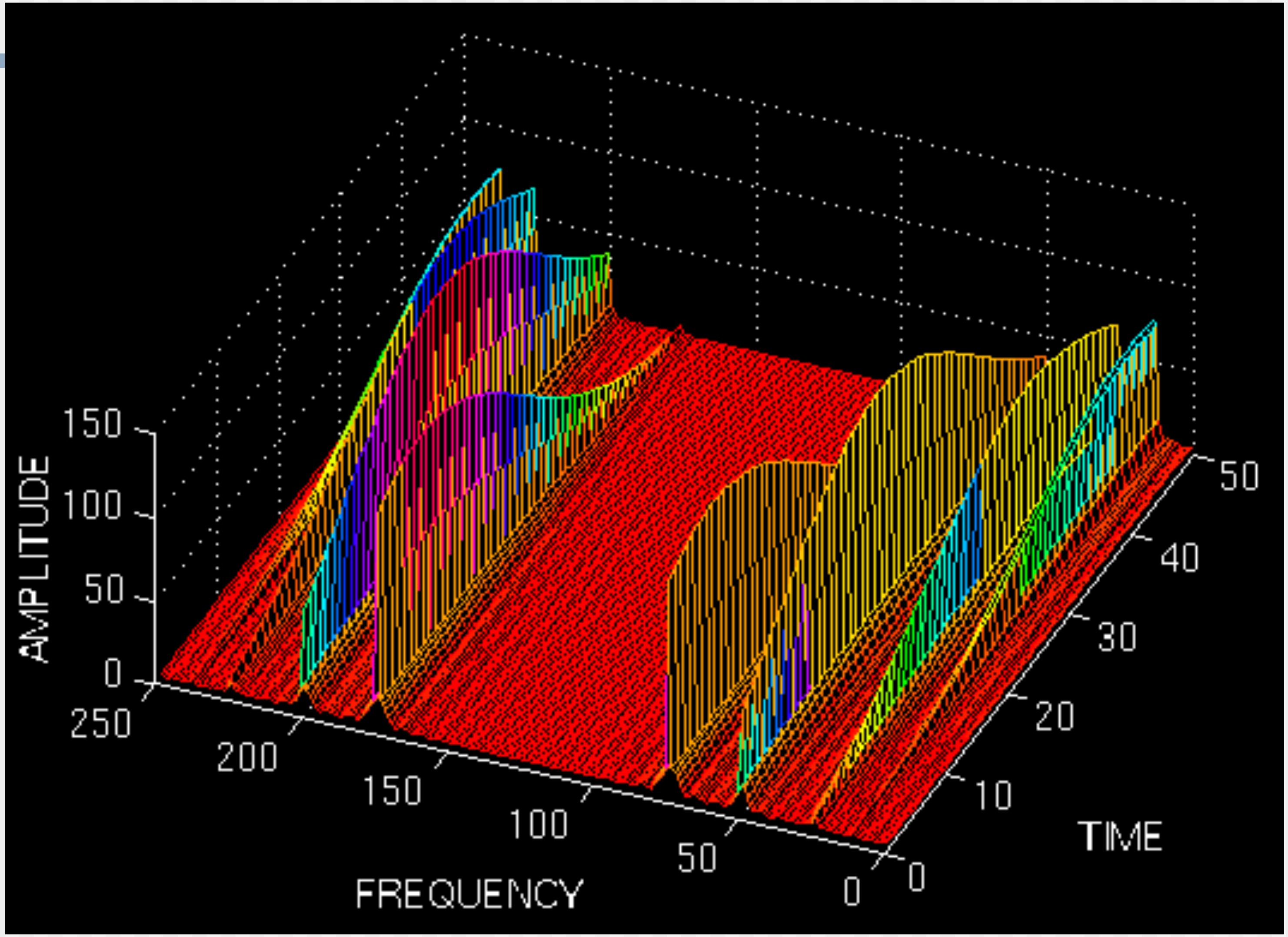
Narrow window good time resolution, poor frequency resolution

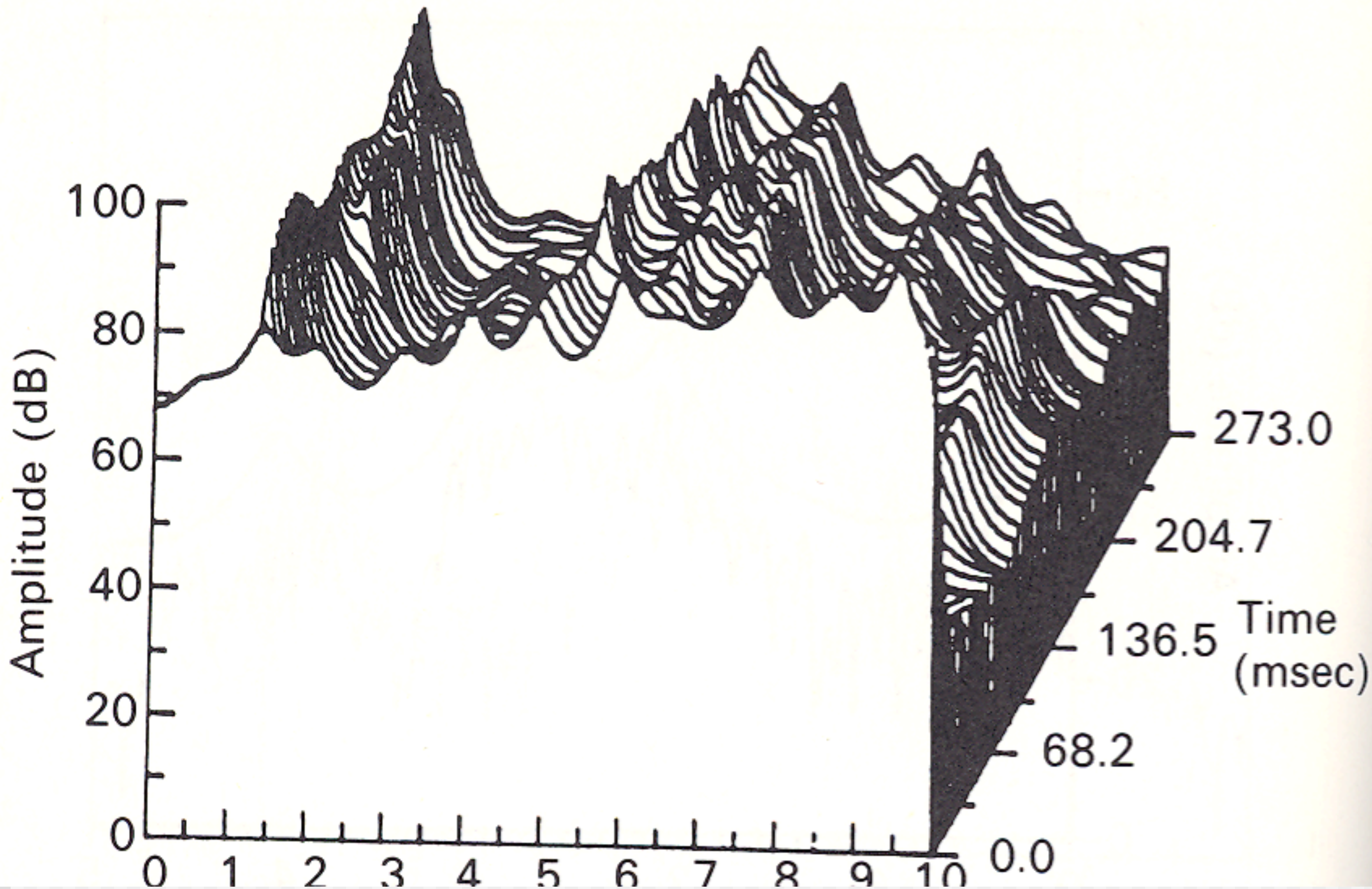


Width window good frquency resolution, poor time resolution



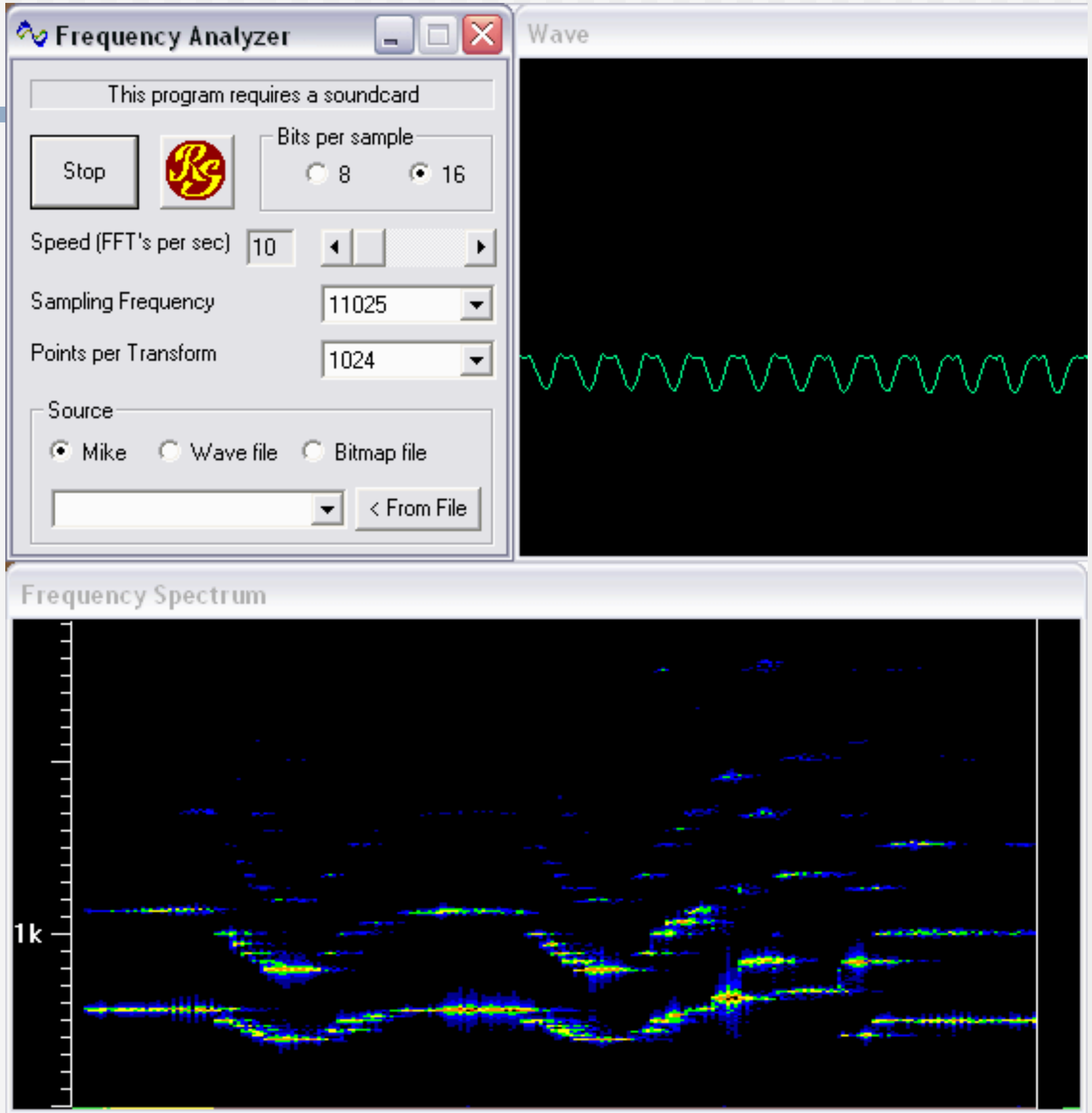
Very width window, very bad time resolution

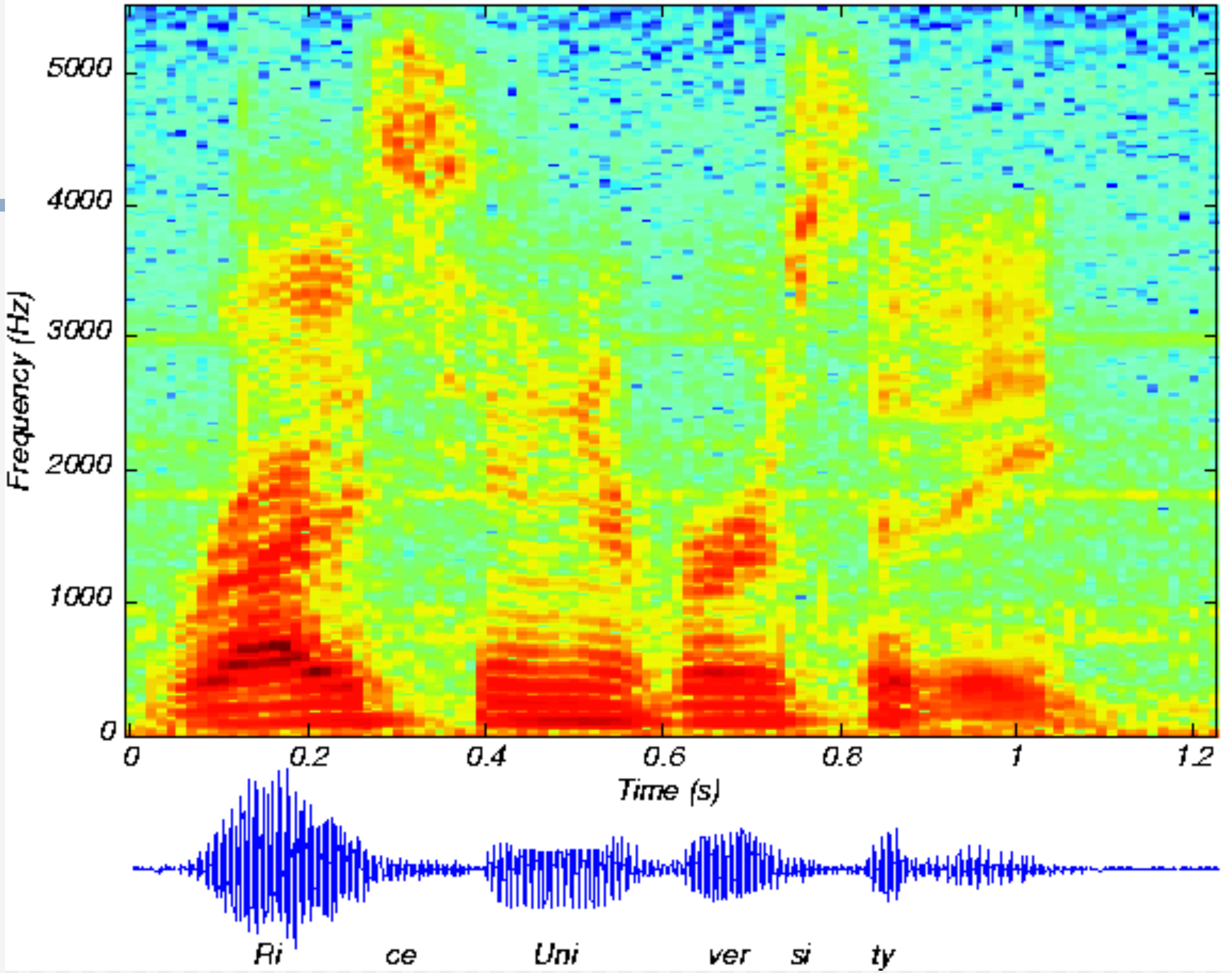




Amplitude (dB)
100
80
60
40
20
0
0
1
2
3
4
5
6
7
8
9
10
273.0
204.7
136.5
68.2
0.0
Time
(msec)

http://www.relisoft.com/freeware/freq.html




5000
4000
3000
2000
1000
0
Frequency (Hz)
0
0.2
0.4
0.6
0.8
1
1.2
Time (s)
Ri
ce
Uni
ver
si
ty

MULTIRESOLUTION ANALYSIS

- Time and frequency resolution problems are results of a physical phenomenon (the Heisenberg uncertainty principle) and exist regardless of the transform used
- Multiresolution analysis (MRA)
  - MRA, as implied by its name, analyzes the signal at different frequencies with different resolutions

III THE CONTINUOUS WAVELET TRANSFORM

$$CWT_x^{\psi}(\tau,s) = \Psi_x^{\psi}(\tau,s) = \frac{1}{\sqrt{|s|}} \int x(t)\psi^{*}\left(\frac{t-\tau}{s}\right)dt$$

- the transformed signal is a function of two variables, τ and s , the translation and scale parameters, respectively
- ψ(t) is the transforming function, and it is called the *mother wavelet*

- The term wavelet means a small wave
  - The smallness refers to the condition that this (window) function is of finite length
  - The wave refers to the condition that this function is oscillatory
  - The term mother implies that the functions with different region of width (support) that are used in the transformation process are derived from the mother wavelet
  - The mother wavelet is a prototype for generating the other window functions

Window versus Wavelet

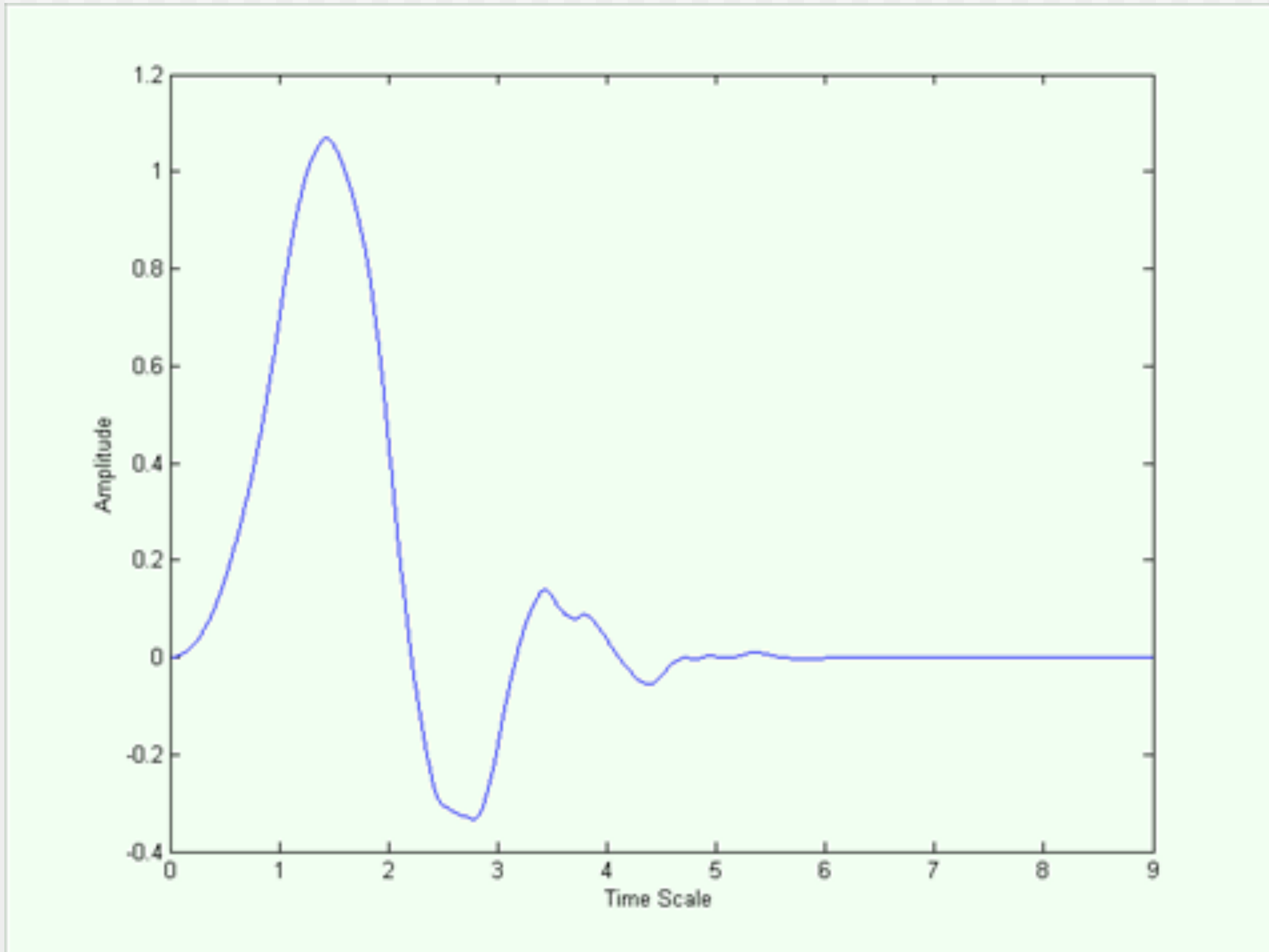
$$STFT_x^{w}(t', f) = \int_t x(t) \cdot w^*(t - t') \cdot e^{-i2\pi ft} dt$$

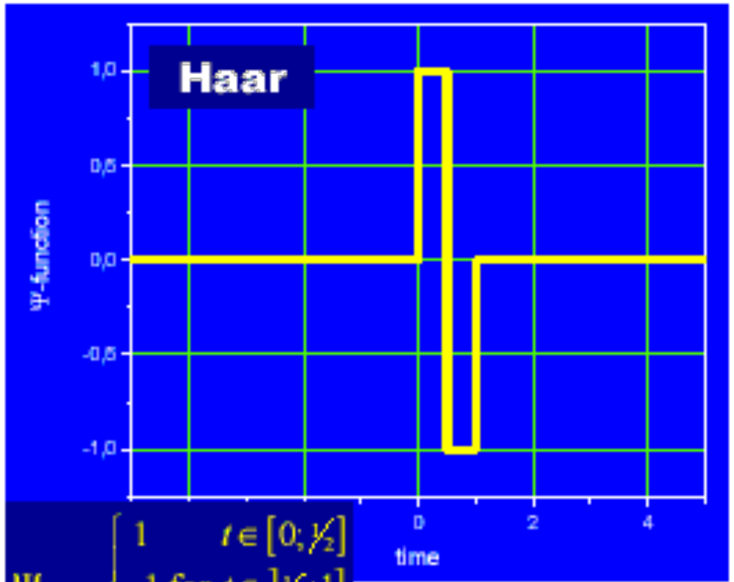
$$w^*(t - t')$$

$$\psi^*\left(\frac{t-\tau}{s}\right)$$

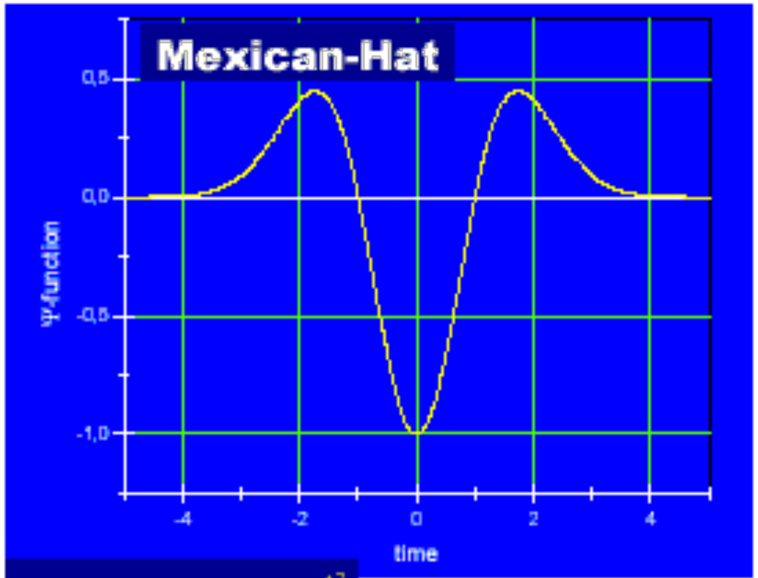
$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int x(t) \psi^*\left(\frac{t-\tau}{s}\right) dt$$

Daubechies No 5 Mother Wavelet

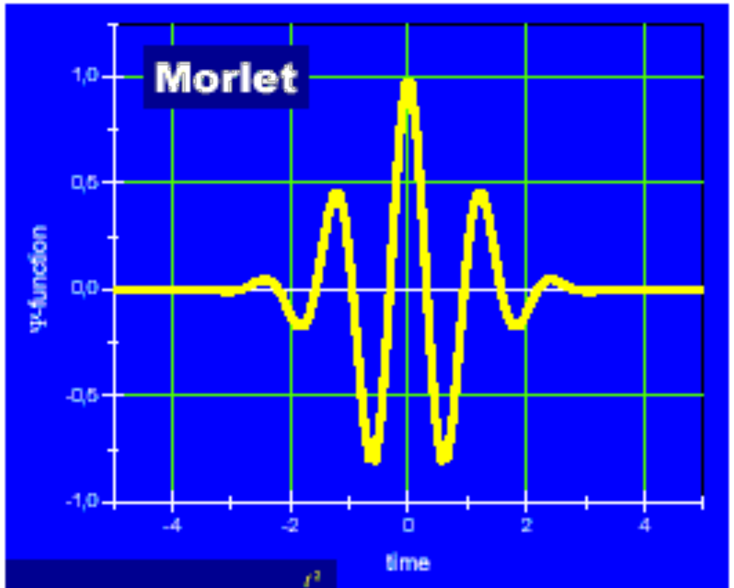




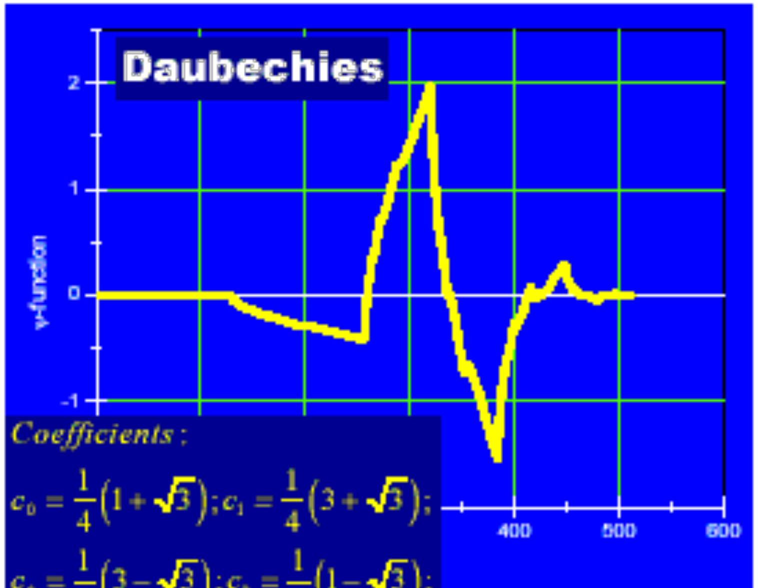
$$\Psi_{(t)} = \begin{cases} 1 & t \in [0; \tfrac{1}{2}] \\ -1 \text{ for } & t \in]\tfrac{1}{2}; 1] \\ 0 & t \in [0; 1] \end{cases}$$



$$\Psi_{(t)} = (t^2 - 1)\, e^{-\frac{t^2}{2}}$$



$$\Psi_{(t)} = \cos(\omega_o t)\, e^{-\frac{t^2}{2\sigma}}$$



Coefficients:

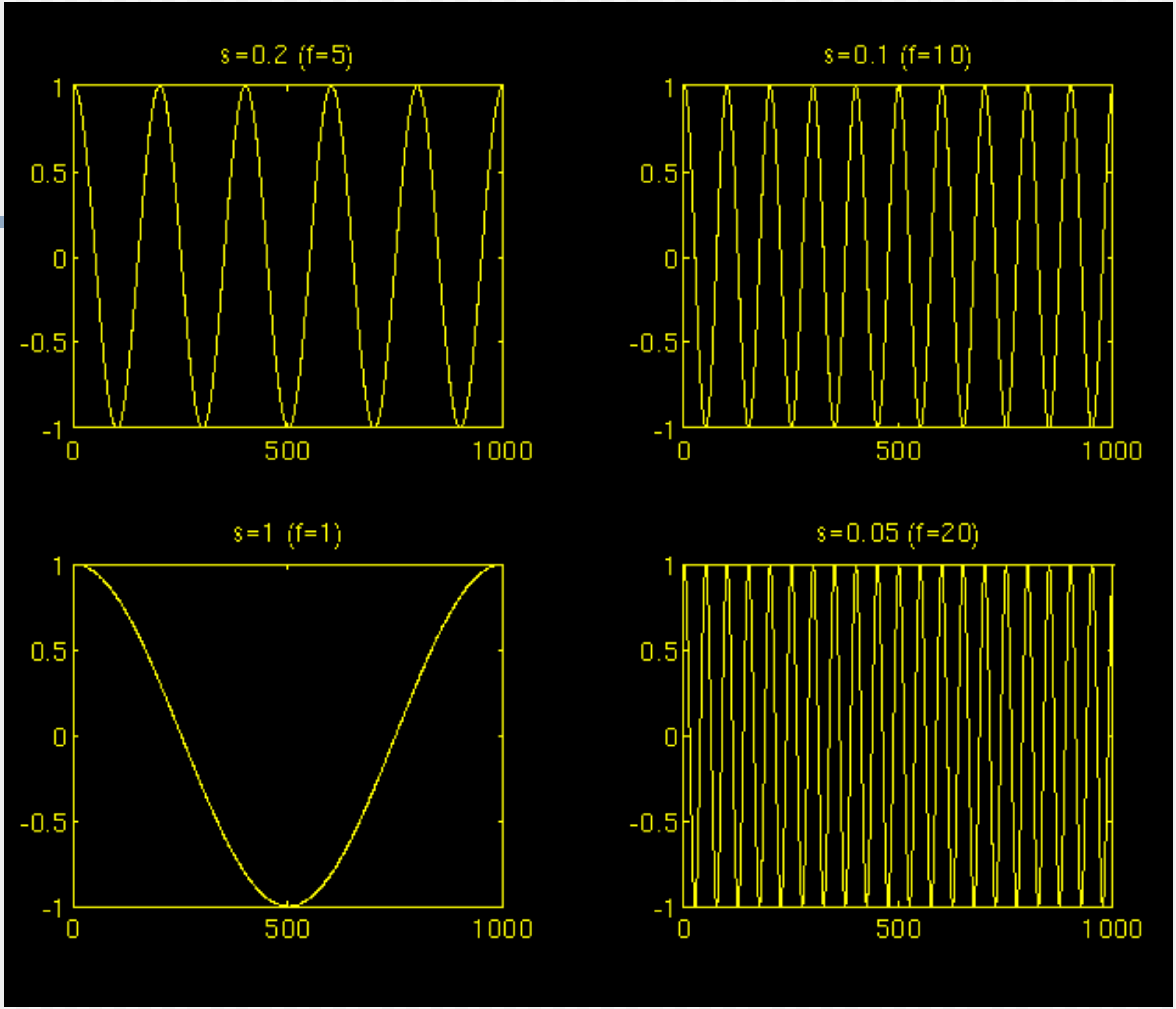
$$c_0 = \frac{1}{4}\left(1+\sqrt{3}\right); c_1 = \frac{1}{4}\left(3+\sqrt{3}\right);$$

$$c_2 = \frac{1}{4}\left(3-\sqrt{3}\right); c_3 = \frac{1}{4}\left(1-\sqrt{3}\right);$$

- The term translation is used in the same sense as it was used in the STFT; it is related to the location of the window, as the window is shifted through the signal
  - This term, obviously, corresponds to time information in the transform domain.
- However, we do not have a frequency parameter, as we had before for the STFT Instead, we have scale parameter which is defined as *1/frequency*

Scale

- The parameter scale in the wavelet analysis is similar to the scale used in maps
  - high scales correspond to a non-detailed global view (of the signal)
  - low scales correspond to a detailed view
- Frequenices:
  - low frequencies (high scales) correspond to a global information of a signal (that usually spans the entire signal)
  - high frequencies (low scales) correspond to a detailed information of a hidden pattern in the signal (that usually lasts a relatively short time)

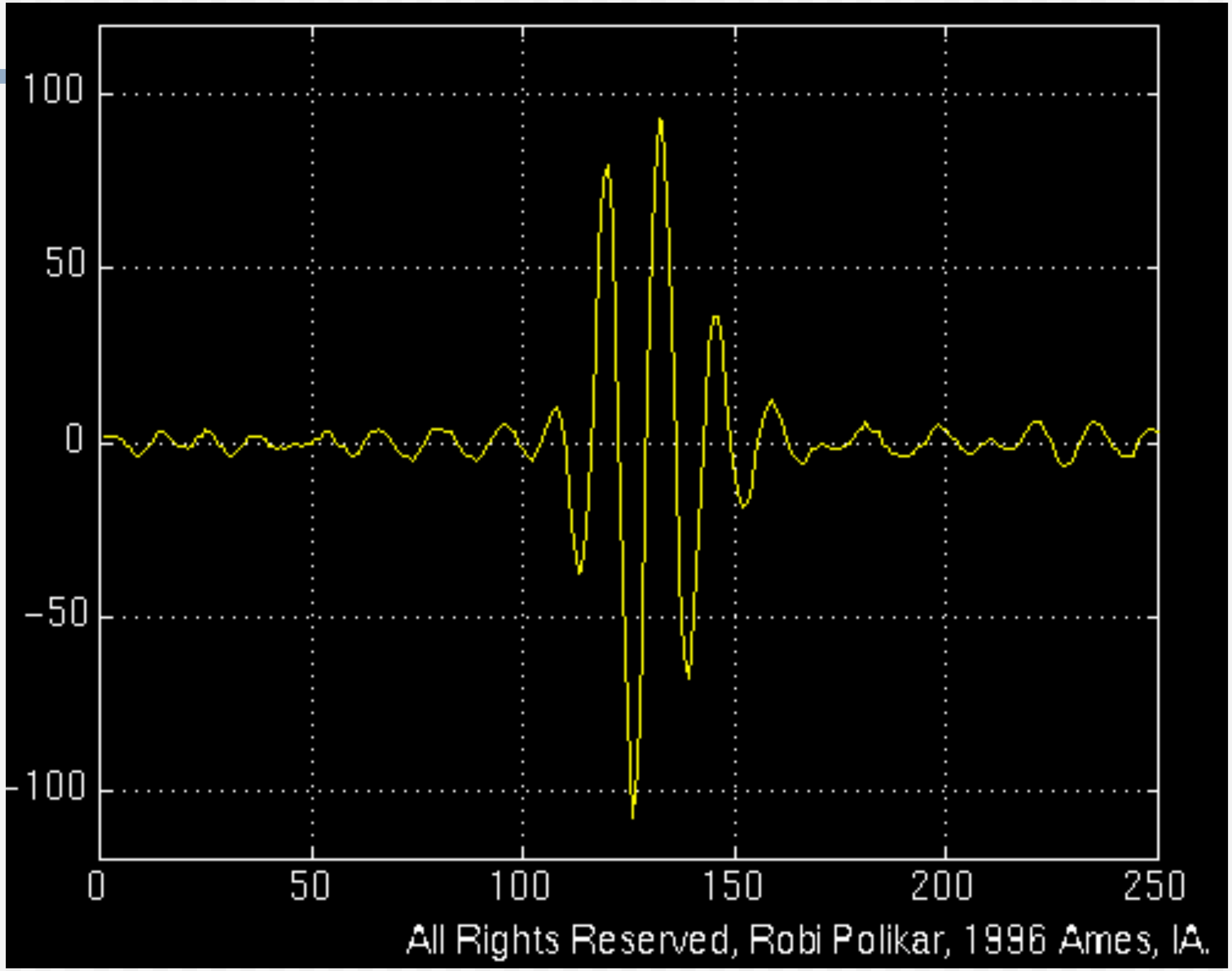


s=0.2 (f=5)
s=0.1 (f=10)
s=1 (f=1)
s=0.05 (f=20)
1
0.5
0
-0.5
-1
0
500
1000

Computation

- The wavelet is placed at the beginning of the signal at the point which corresponds to time=0
- The wavelet function at scale ``1'' is multiplied by the signal and then integrated over all times
  - The result of the integration is then multiplied by the constant number 1/sqrt{s}
    - For energy normalization purposes so that the transformed signal will have the same energy at every scale
- One row of points on the time-scale plane for the scale s=1 is now completed

Example


100
50
0
−50
-100
0
50
100
150
200
250
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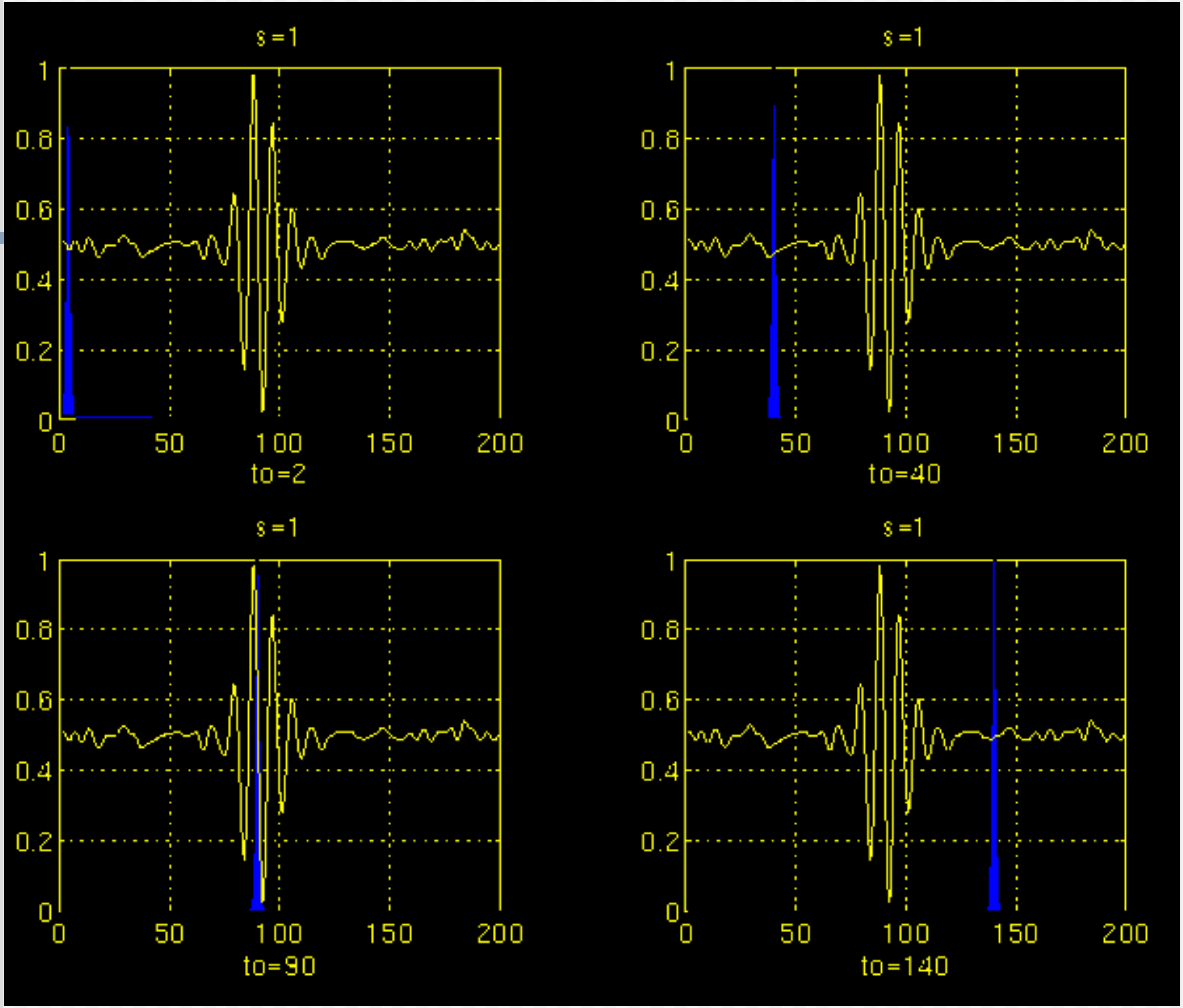
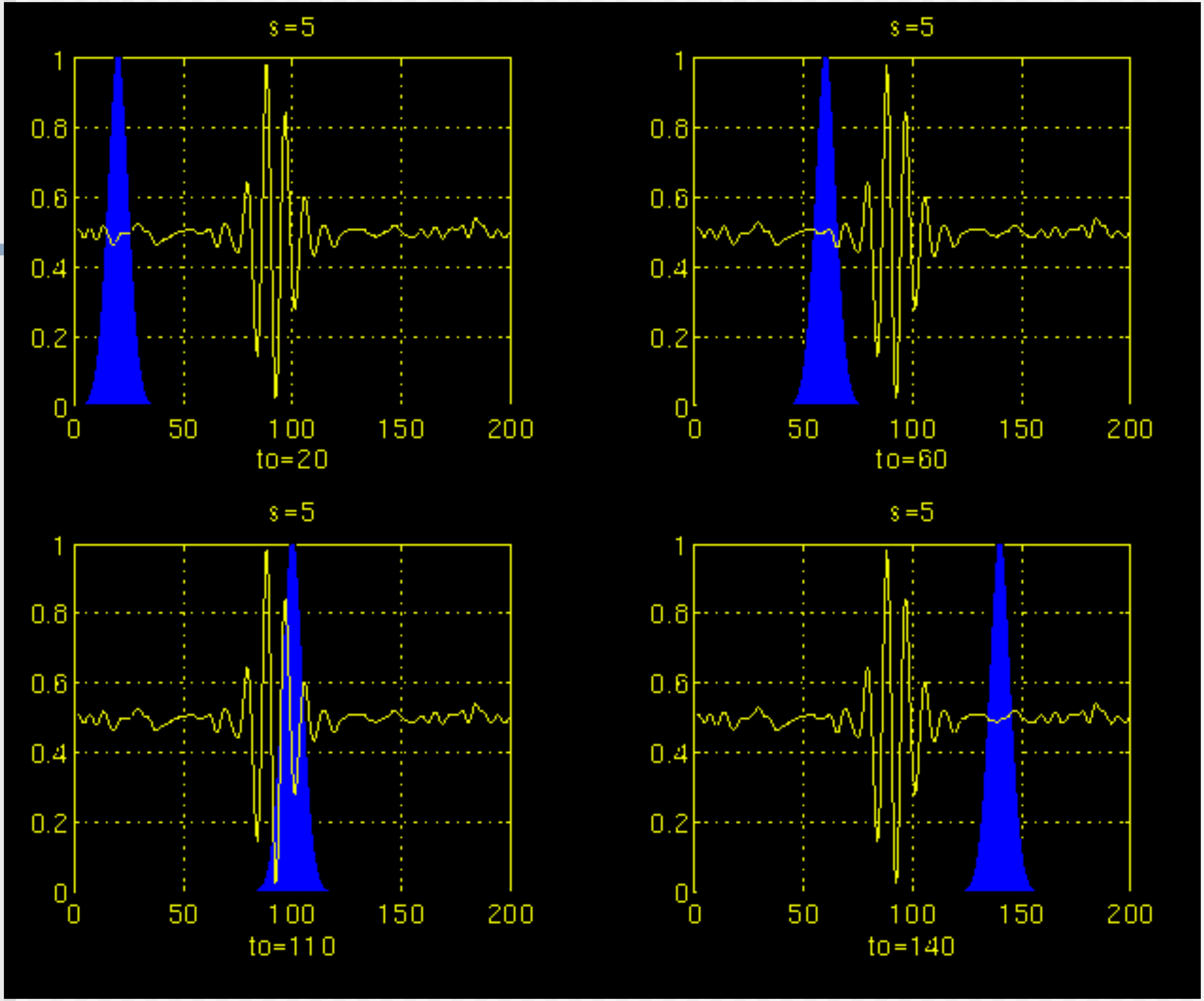
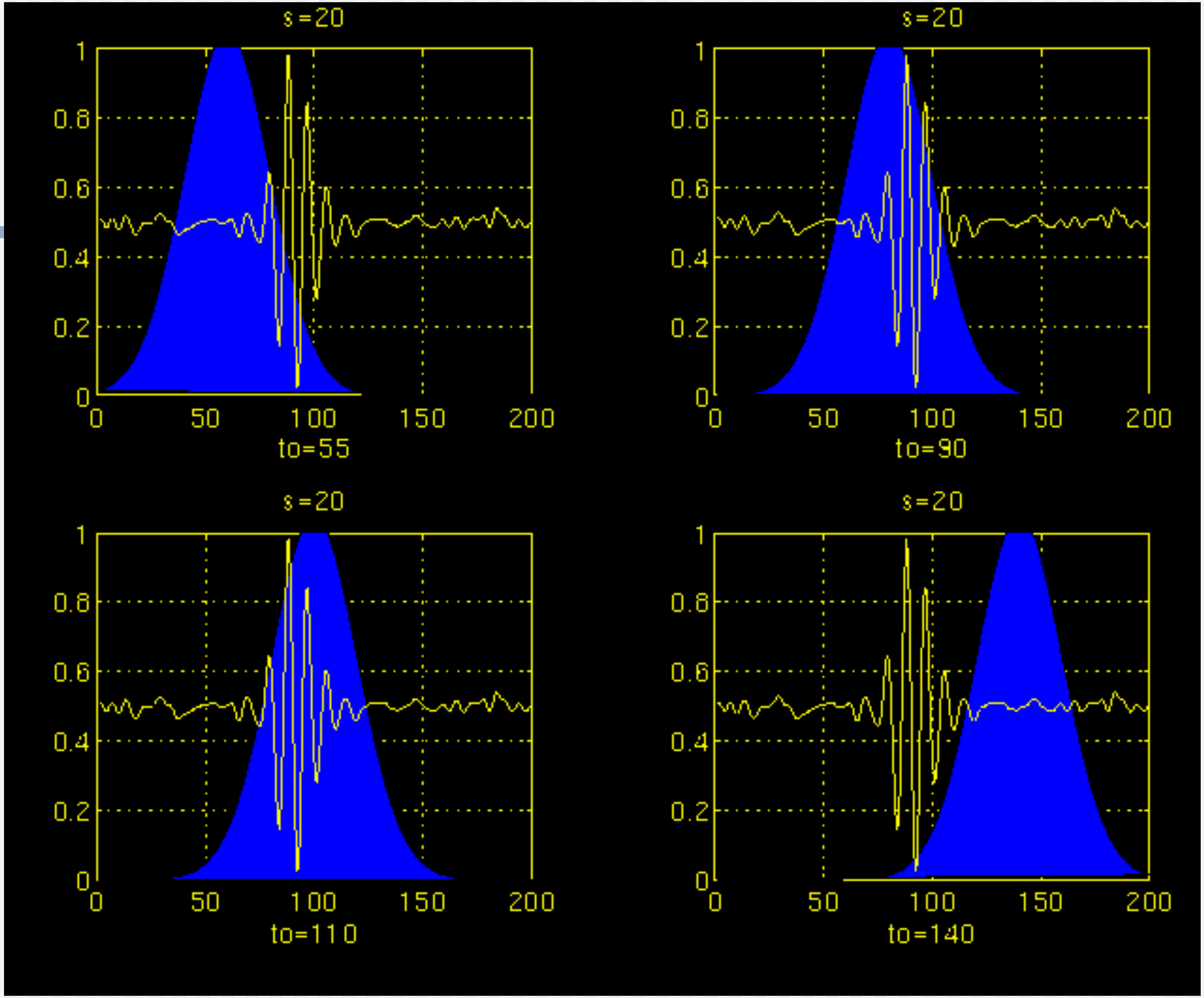

s=1
to=2
s=1
to=40
s=1
to=90
s=1
to=140

- ... process for the scales s=5 and s=20, respectively
- The window width changes with increasing scale (decreasing frequency)
- As the window width increases, the transform starts picking up the lower frequency components



s=5
to=20
s=5
to=60
s=5
to=110
s=5
to=140

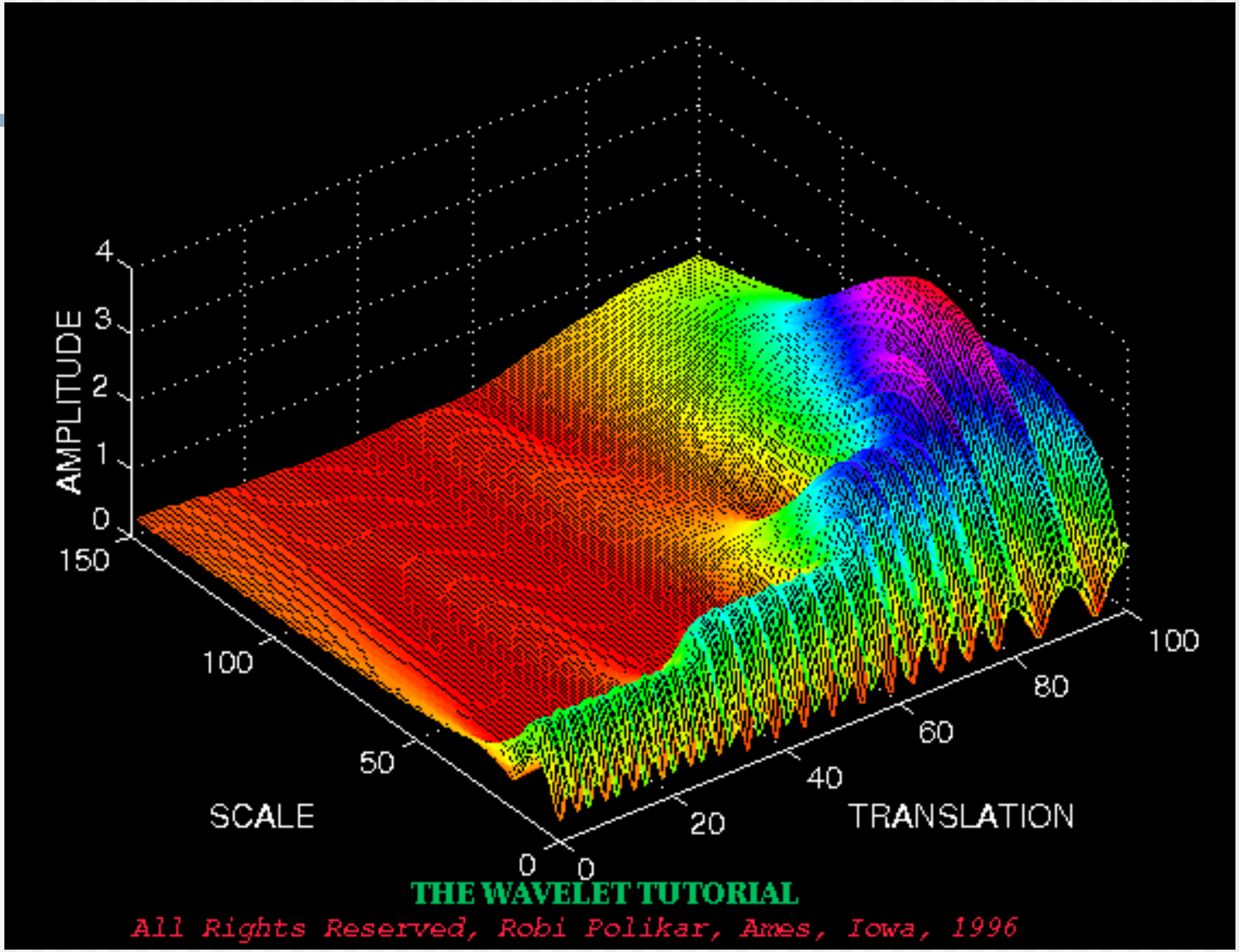

s=20
to=55
s=20
to=90
s=20
to=110
s=20
to=140

- As a result, for every scale and for every time (interval), one point of the time-scale plane is computed
- The computations at one scale construct the rows of the time-scale plane, and the computations at different scales construct the columns of the time-scale plane

Example 2

30 Hz
20 Hz
10 Hz
5 Hz
1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
0
200
400
600
800
1000
Time, ms

CWT


AMPLITUDE
0
1
2
3
4
150
100
50
0
SCALE
0
20
40
60
80
100
TRANSLATION
THE WAVELET TUTORIAL
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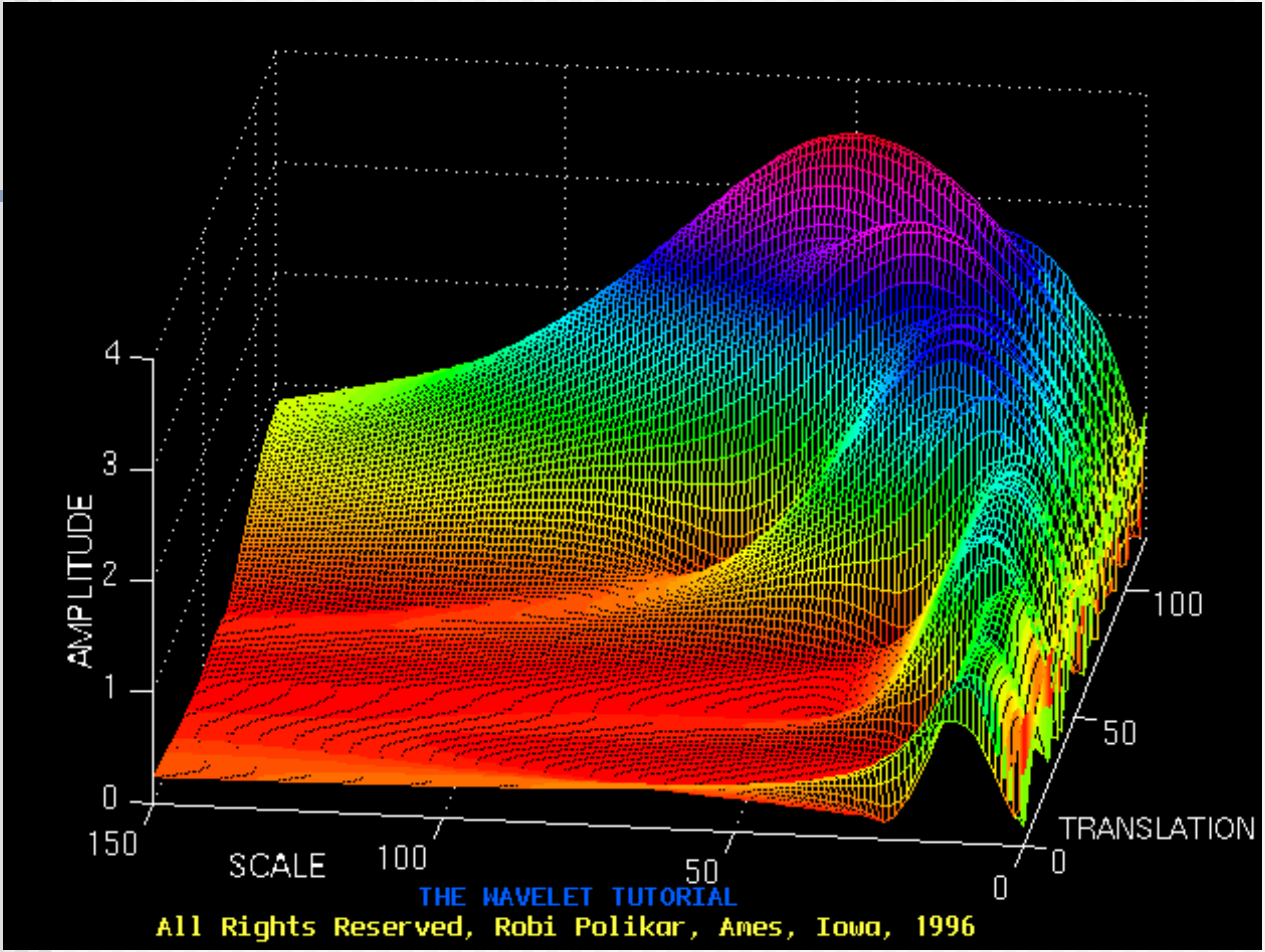
- Continuous wavelet transform (CWT) of signal
- The axes are translation and scale, not time and frequency.
- Translation is strictly related to time, since it indicates where the mother wavelet is located
- The scale is actually inverse of frequency



AMPLITUDE
4
3
2
1
0
150
SCALE
100
50
0
TRANSLATION
0
50
100
THE WAVELET TUTORIAL
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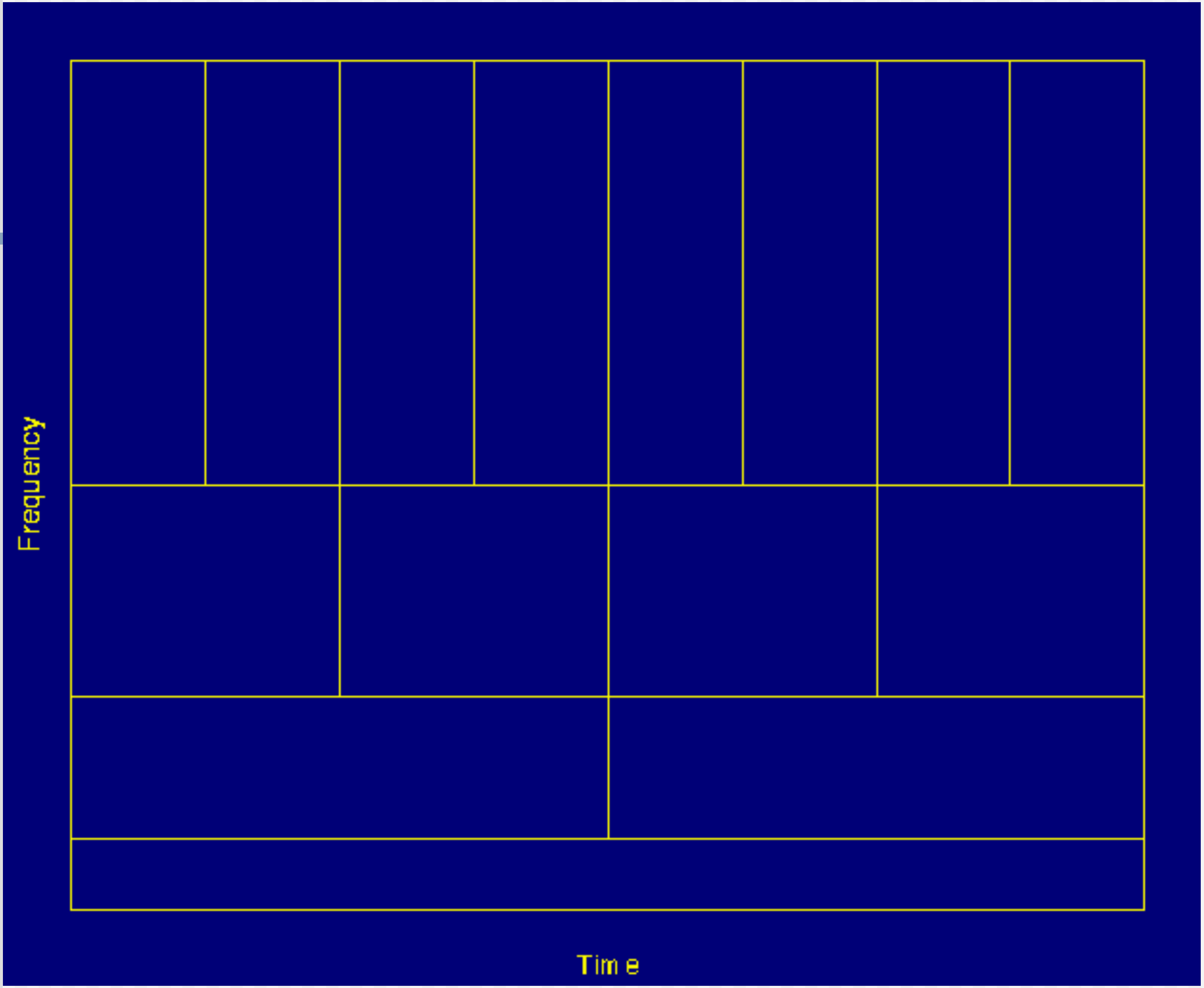
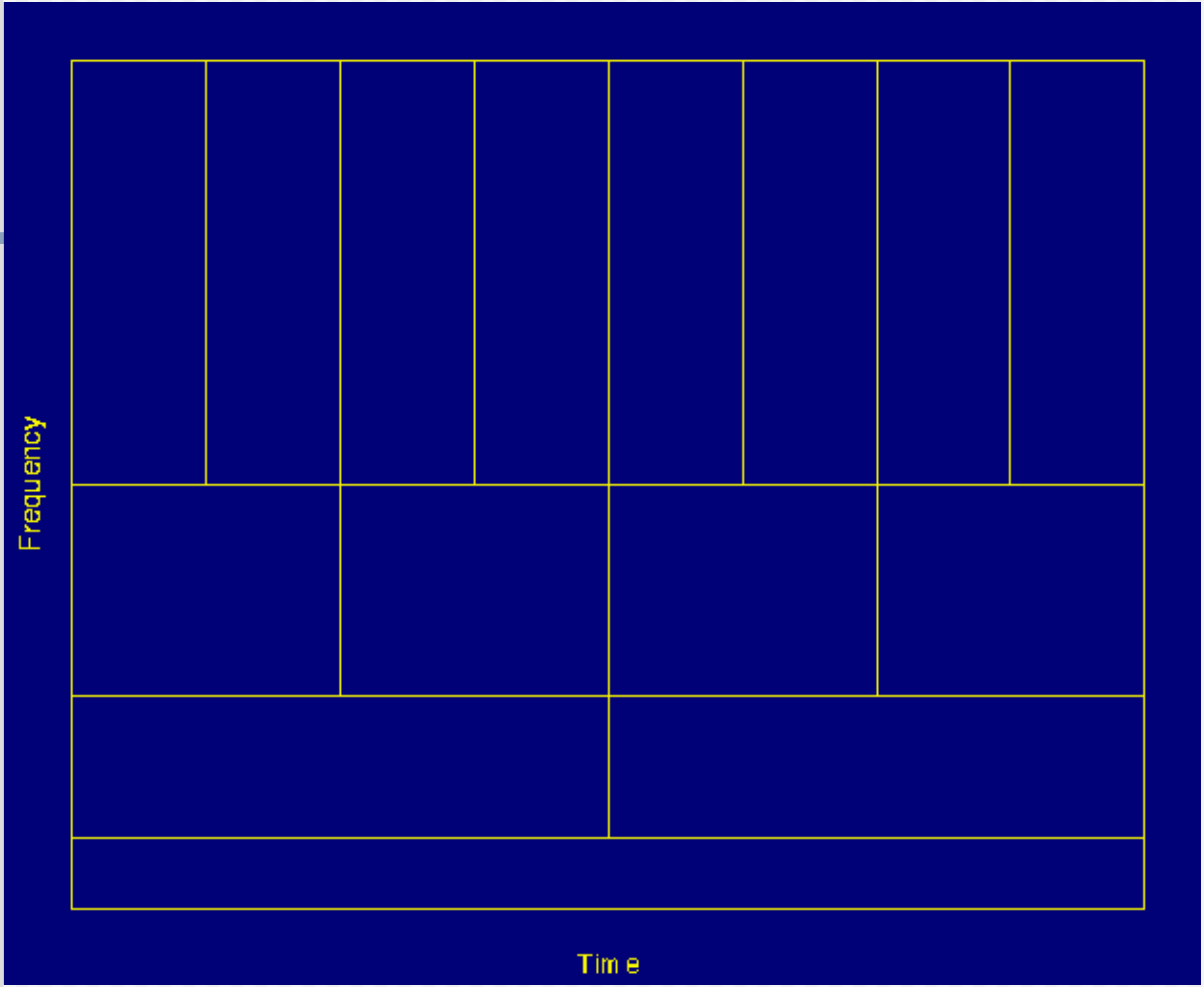
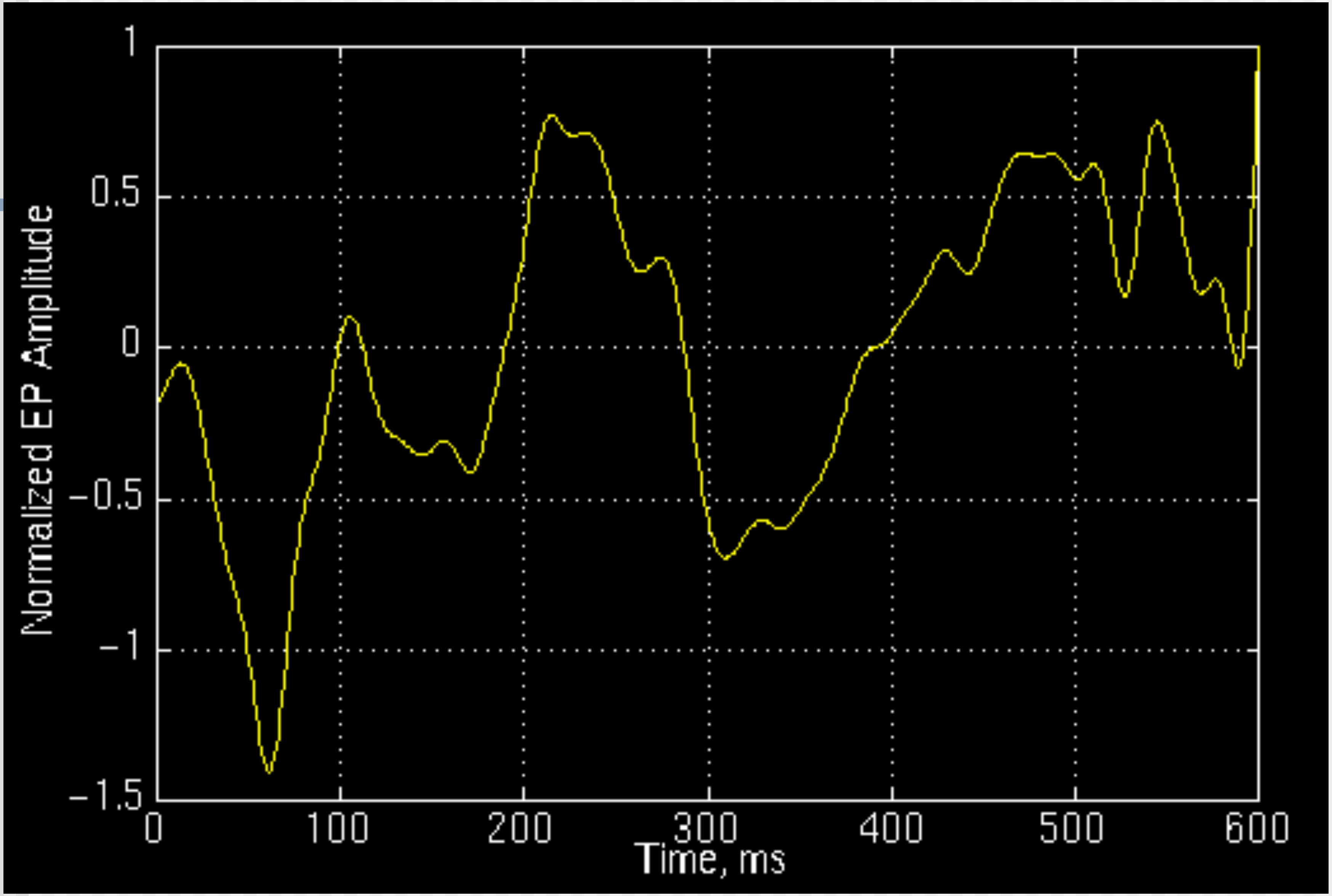
- Good time and poor frequency resolution at high frequencies (lower scales)
- Good frequency and poor time resolution at low frequencies (high scales)

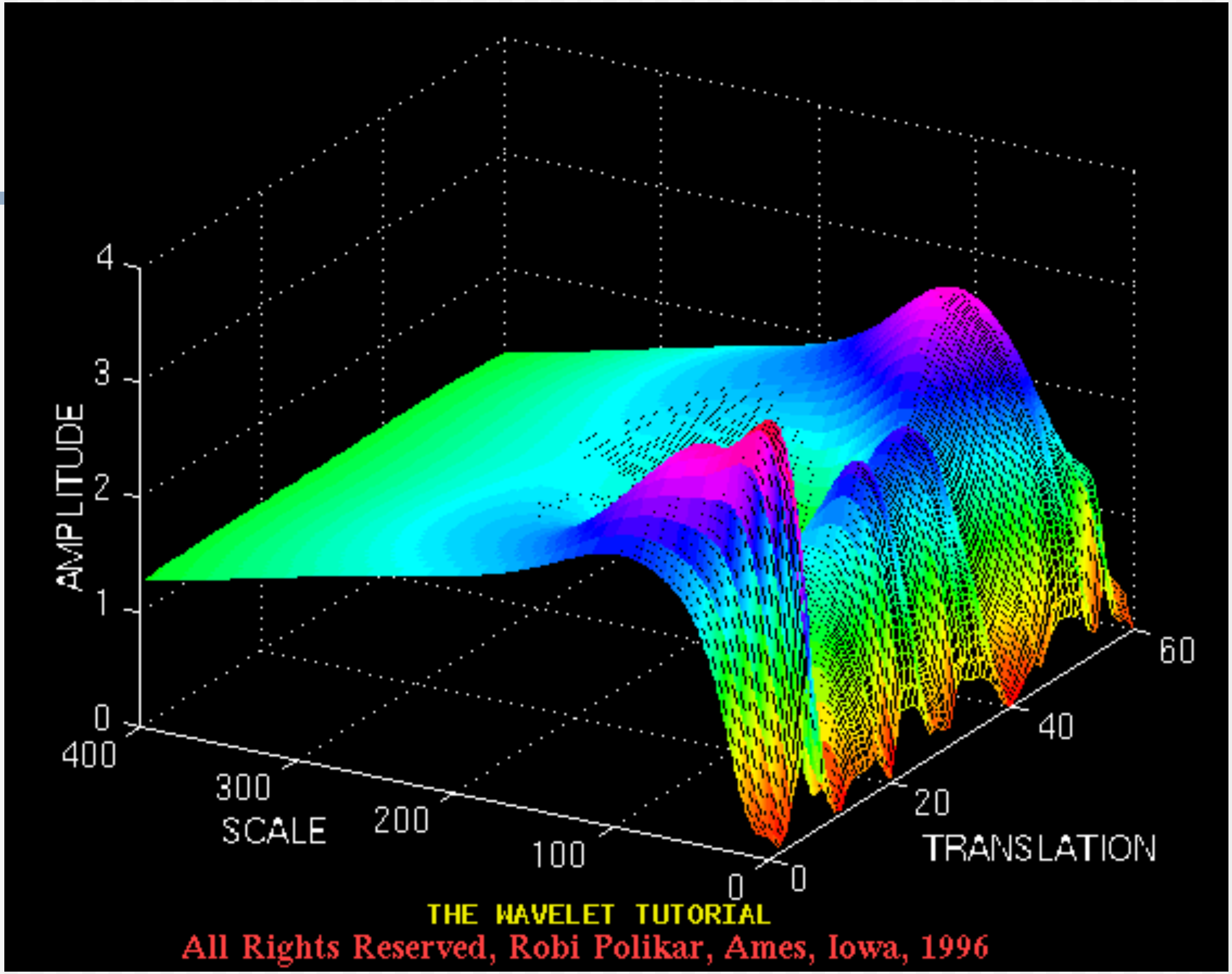

Frequency
Tim e

- Every box corresponds to a value of the wavelet transform in the time-frequency plane
- At low frequencies, the height of the boxes are shorter (which corresponds to better frequency resolutions), but their widths are longer (which correspond to poor time resolution)
- At higher frequencies the width of the boxes decreases, i.e., the time resolution gets better, and the heights of the boxes increase, i.e., the frequency resolution gets poorer


Frequency
Time



Normalized EP Amplitude
Time, ms
1
0.5
0
−0.5
−1
−1.5
0
100
200
300
400
500
600


AMPLITUDE
4
3
2
1
0
400
300
SCALE
200
100
0
0
20
40
60
TRANSLATION
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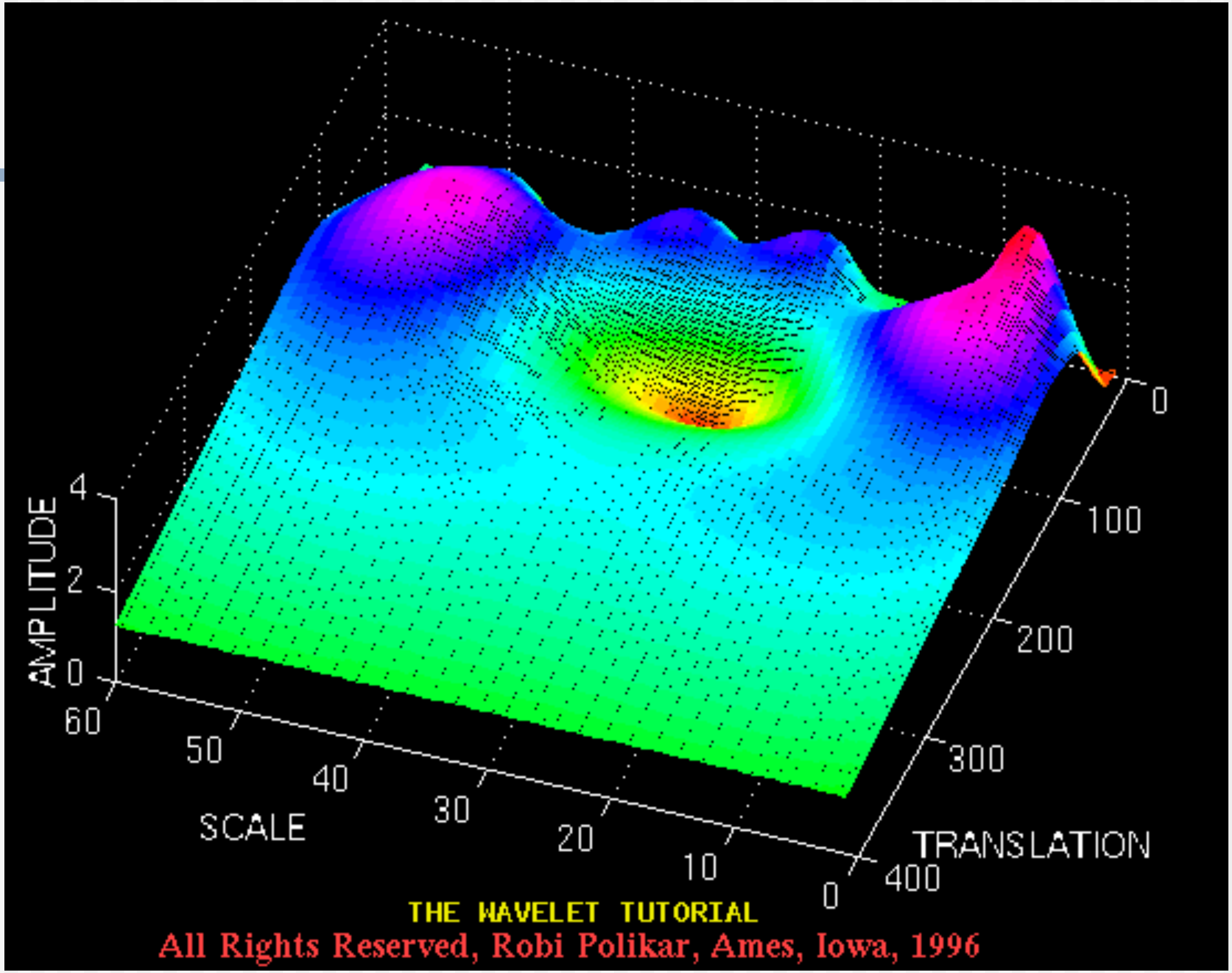
AMPLITUDE
4
2
0
60
50
40
30
20
10
0
SCALE
0
100
200
300
400
TRANSLATION
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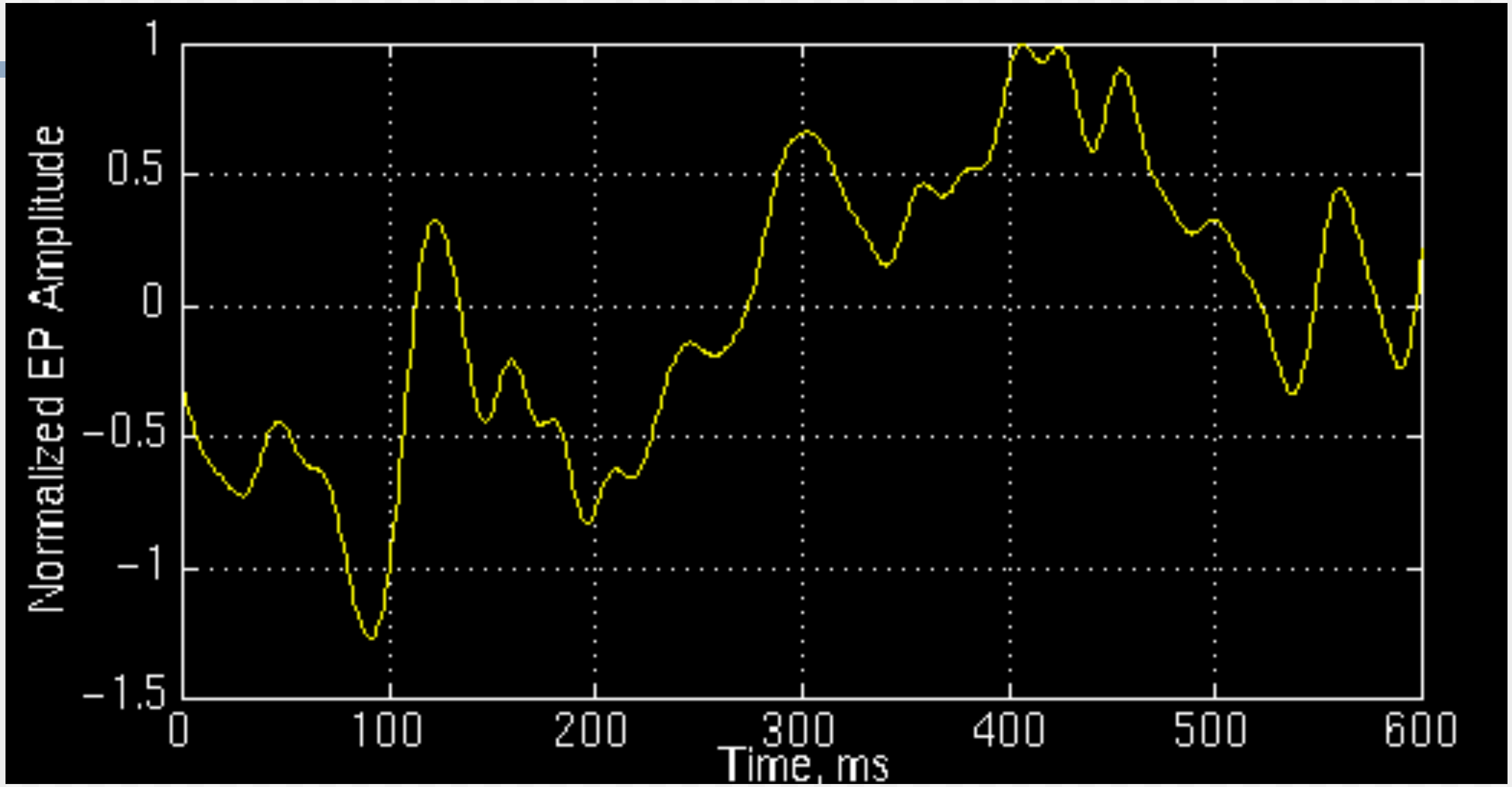
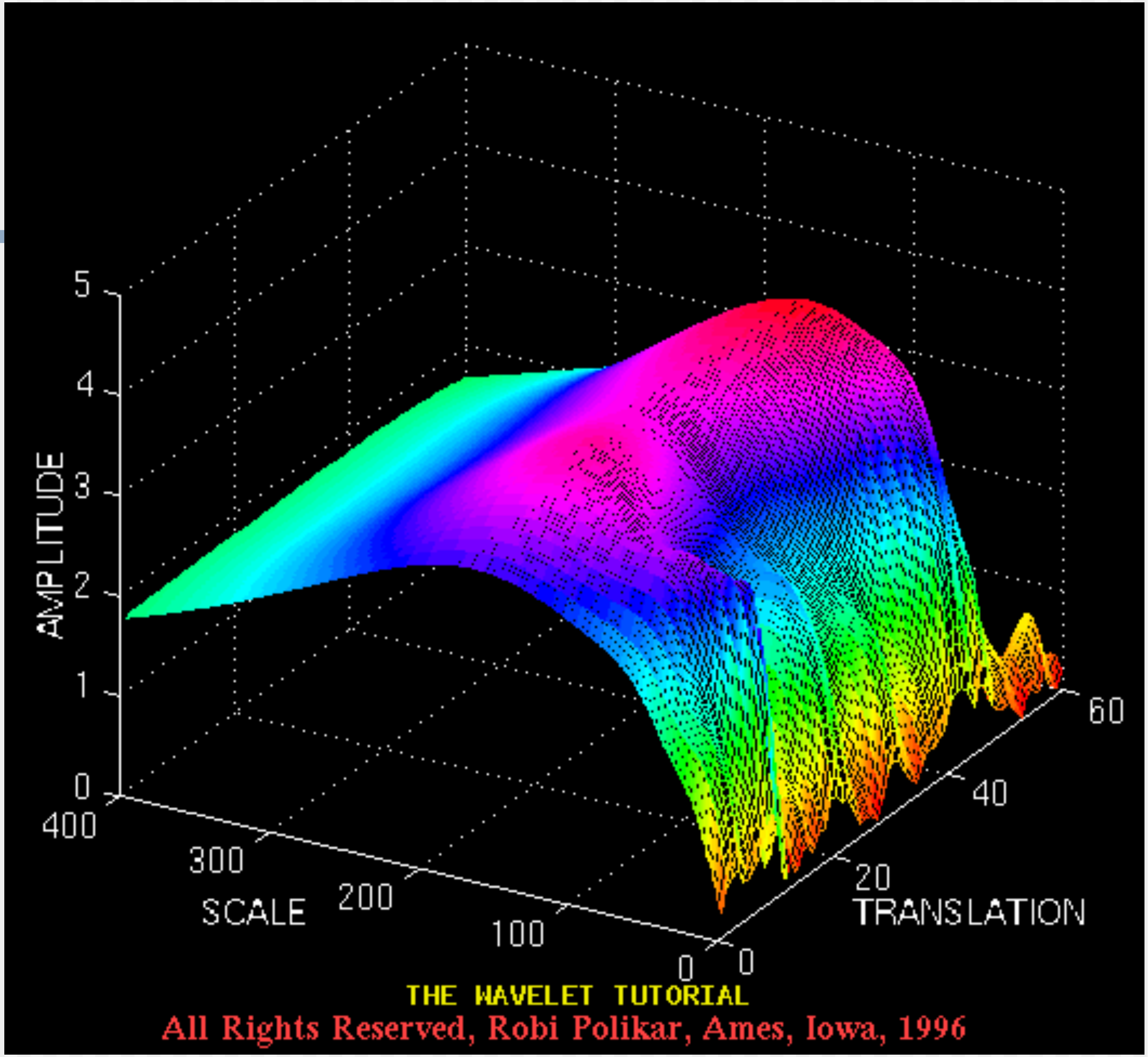


Normalized EP Amplitude
1
0.5
0
−0.5
−1
−1.5
0
100
200
300
400
500
600
Time, ms



5
4
3
2
1
0
AMPLITUDE
400
300
200
100
0
SCALE
0
20
40
60
TRANSLATION
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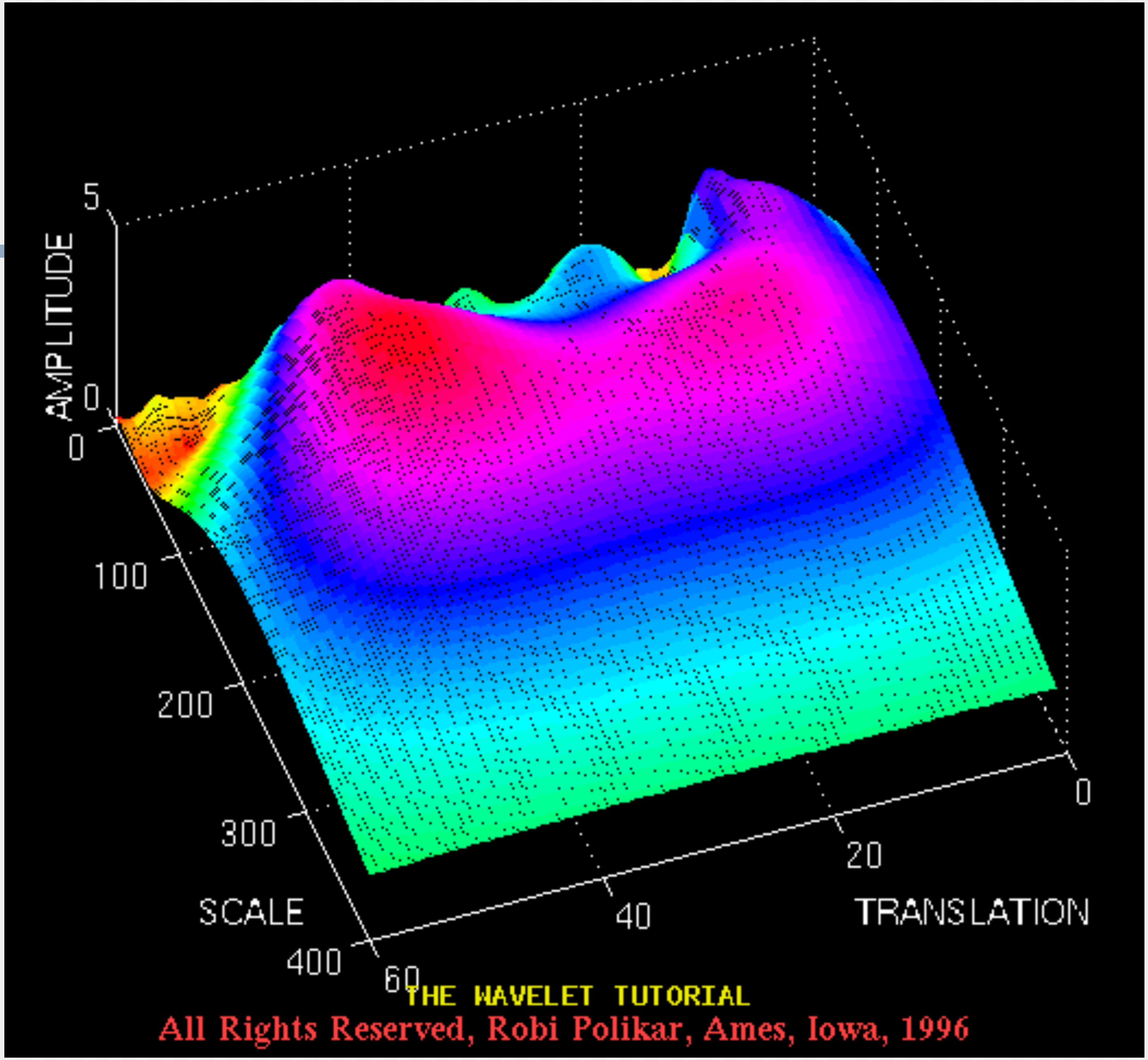
5
AMPLITUDE
0
0
100
200
300
SCALE
400
60
40
20
0
TRANSLATION
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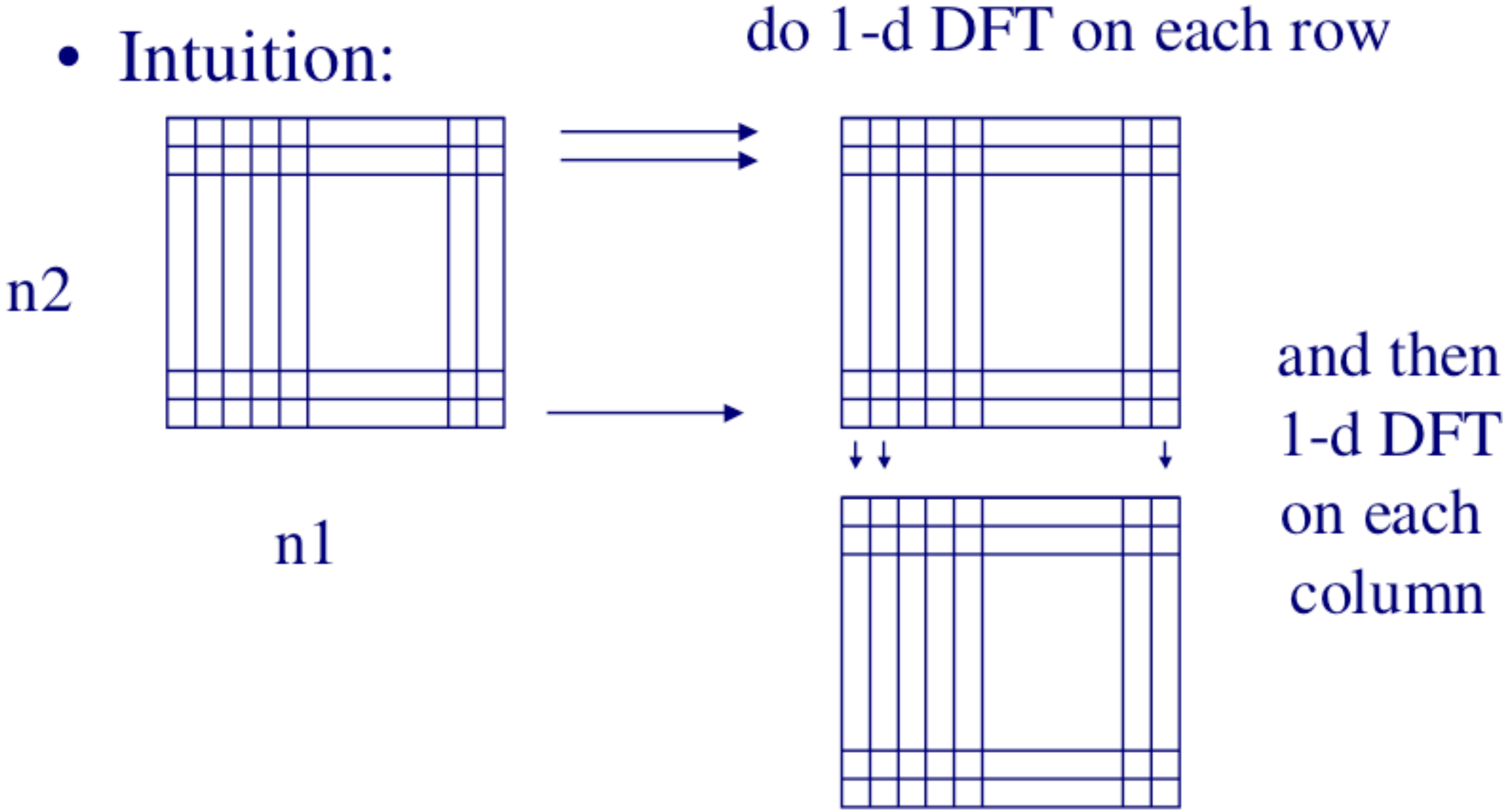
Discrete Wavelet Transform

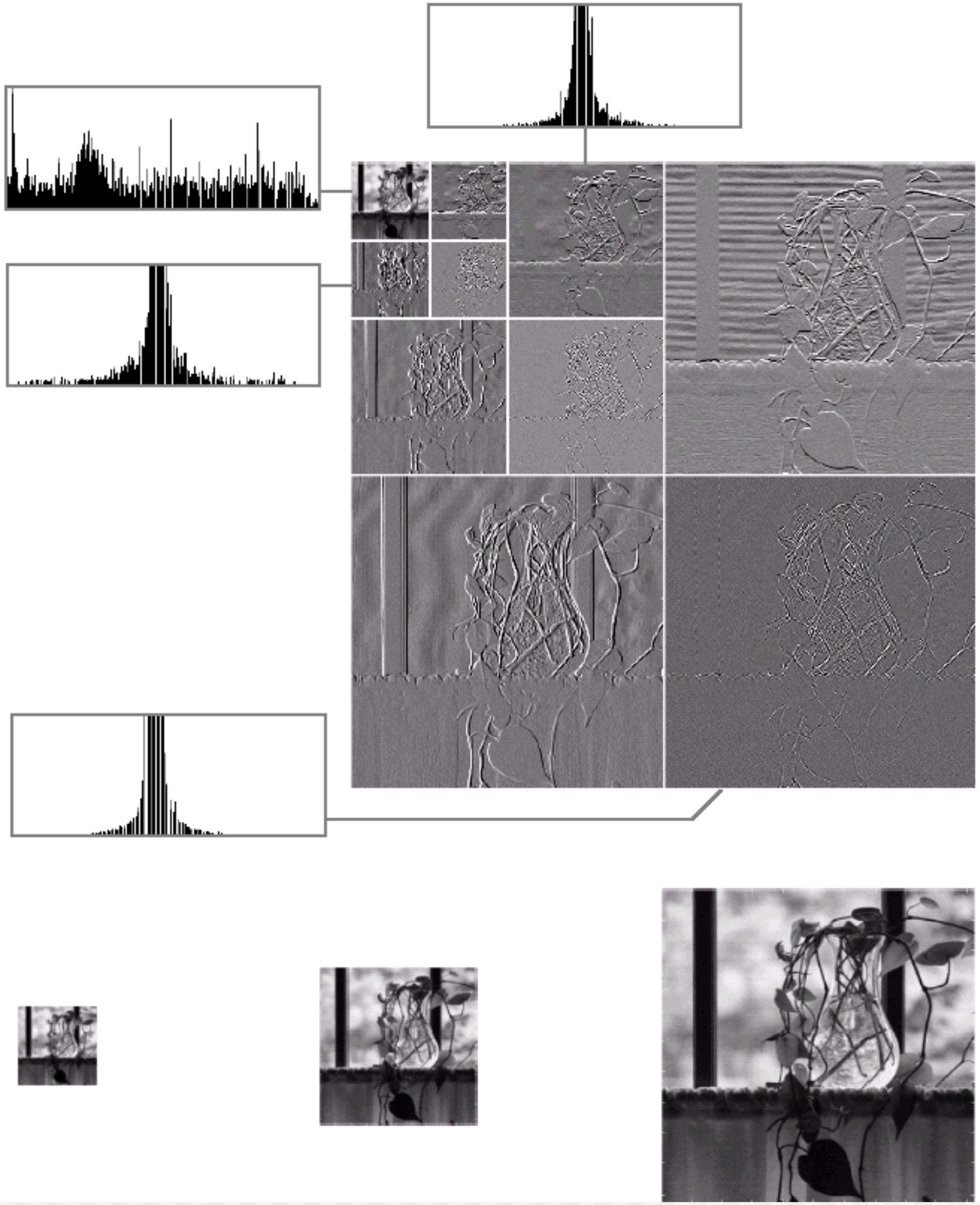
- In the discrete case, filters of different cutoff frequencies are used to analyze the signal at different scales
  - The signal is passed through a series of high pass filters to analyze the high frequencies, and it is passed through a series of low pass filters to analyze the low frequencies
- The resolution of the signal is changed by the filtering operations, and the scale is changed by upsampling and downsampling (subsampling) operations.

- Intuition:



Wavlets Transforms in Two dimensions

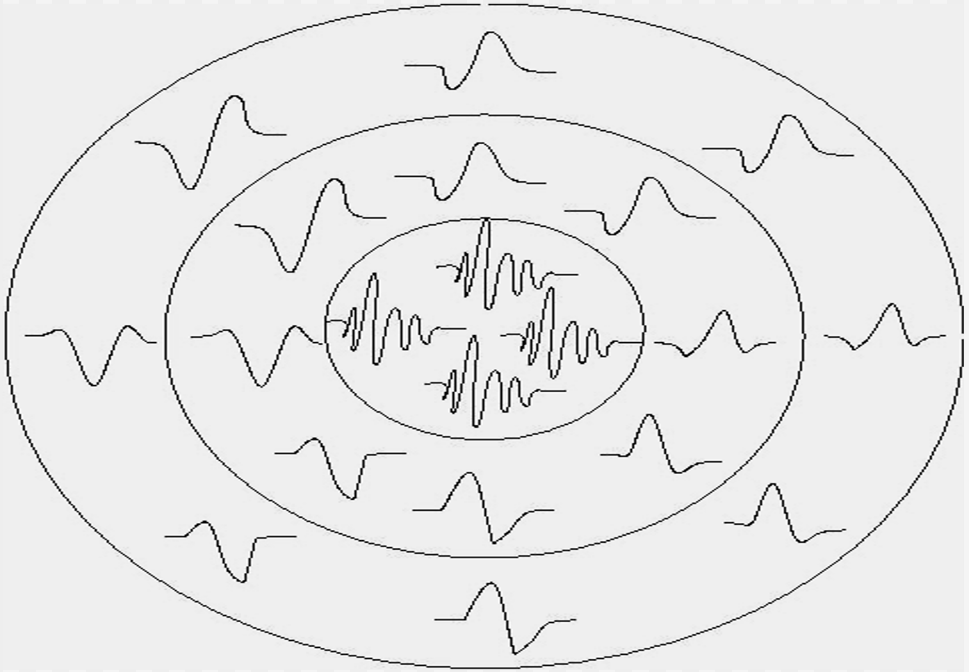
- We have three directionaly sensitive wavelets
  - Variations along columns
  - Variations along rows
  - Variation along diagonals



a
b c d

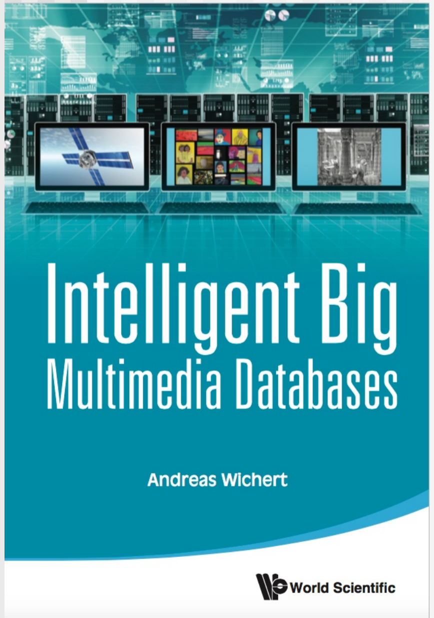
FIGURE 7.8 (a) A discrete wavelet transform using Haar basis functions. Its local histogram variations are also shown; (b)–(d) Several different approximations (64×64, 128×128, and 256×256) that can be obtained from (a).

Brain - Visual cells....



- Why do we need a Transform?
- Fourier Transform and the short term Fourier (STFT)
- Heisenberg Uncertainty Principle
- The continues Wavelet Transform
- Discrete Wavelet Transform
- Wavelets Transforms in Two dimensions

Literature



- Intelligent Big Multimedia Databases, A. Wichert, World Scientific, 2015
  - *DFT, Wavelets etc*