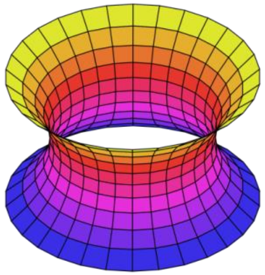
4 - Optimal Control



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[JML-CEE2019] caps. 10 a 13

Syllabus

1. Dynamic optimization problems
2. Pontryagin’s principle
3. Exercises with free end state.
4. Proof of Pontryagin’s principle
5. Equality constraints on the final state
6. The Linear Quadratic problem

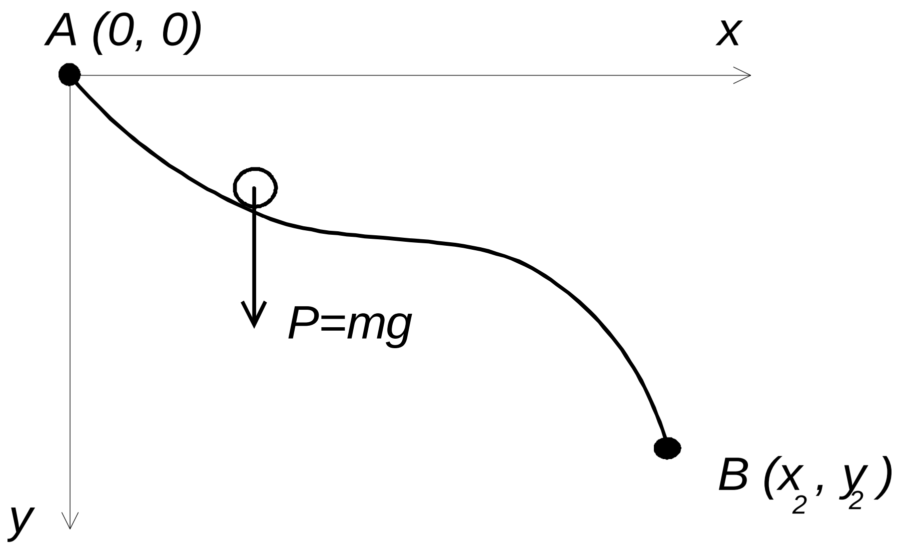
Objectives

Objective: *Introduce a novel class of optimization problems, that are solved with respect to infinite dimensional variables – Optimal Control.*

[JML-CEE2019] cap. 10

A classical problem: The brachistochrone curve

What is the shape of the curve that connects points A and B such data a point mass, under the force of gravity alone, slides (frictionless) from A to B in minimum time?



Which function *y(x)* minimizes the travel time between A and B?

Computing the travel time assuming $y(x) \quad 0 \le x \le x_2$ known

Without friction, the increase of kinetic energy is equal to the loss of potential energy, and $\frac{1}{2}mv^2 = mgy$ or

$$v(x) = \sqrt{2gy(x)}$$

Let s be the arclength. From Pythagoras theorem we get the kinematics relation

$$v(x) = \frac{ds}{dt} = \frac{\sqrt{dx^2 + dy^2}}{dt} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} . \frac{dx}{dt} = \sqrt{1 + \left(y'(x)\right)^2} . \frac{dx}{dt}$$

Energy balance:

$$v(x) = \sqrt{2gy(x)}$$

Kynematics:

$$v(x) = \sqrt{1 + \left(y'(x)\right)^2} \, . \frac{dx}{dt}$$

Eliminate v by equating the r.h.s.:

$$\sqrt{2gy(x)} = \sqrt{1 + \left(y'(x)\right)^2} \, . \frac{dx}{dt}$$

$$\sqrt{2gy(x)} = \sqrt{1+\left(y'(x)\right)^2} \cdot \frac{dx}{dt}$$

or

$$\frac{dt}{dx} = \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}$$

The traveling time is obtained by integration

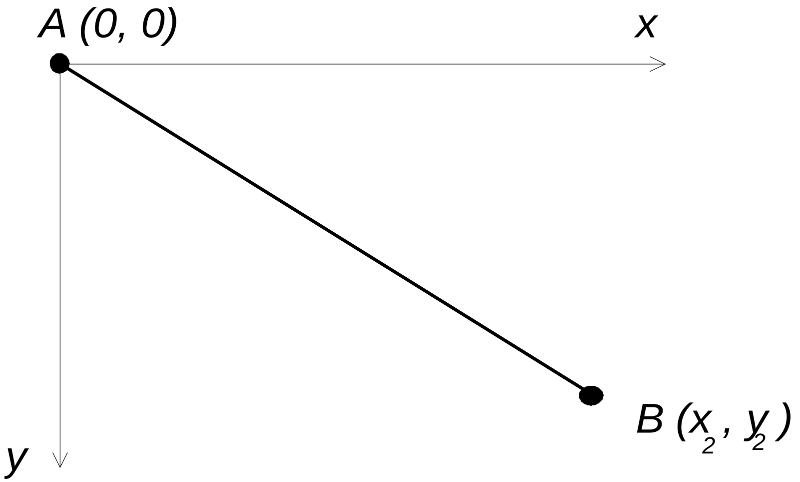
$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}\,dx$$

If we know the function $y(x)$, we can compute the travel time

$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx$$

For instance, if the path to follow is a straight line between A and B,

$$y(x) = \alpha x \text{ with } \alpha = {}^{y_2}\!/_{x_2}$$



The travel time for the rectilinear path is

$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx = \int_0^{x_2} \sqrt{\frac{1+\alpha^2}{2g\alpha}}.x^{-1/2}dx = \sqrt{\frac{1+\alpha^2}{2g\alpha}}.2x_2^{1/2}$$

If we want to compare the travel time for the rectilinear path with the one of another curve (for instance an arc of circle), we can do it, and decide which one leads to the pastest path.

However, the point is that we don’t know the shape of the optimal curve.

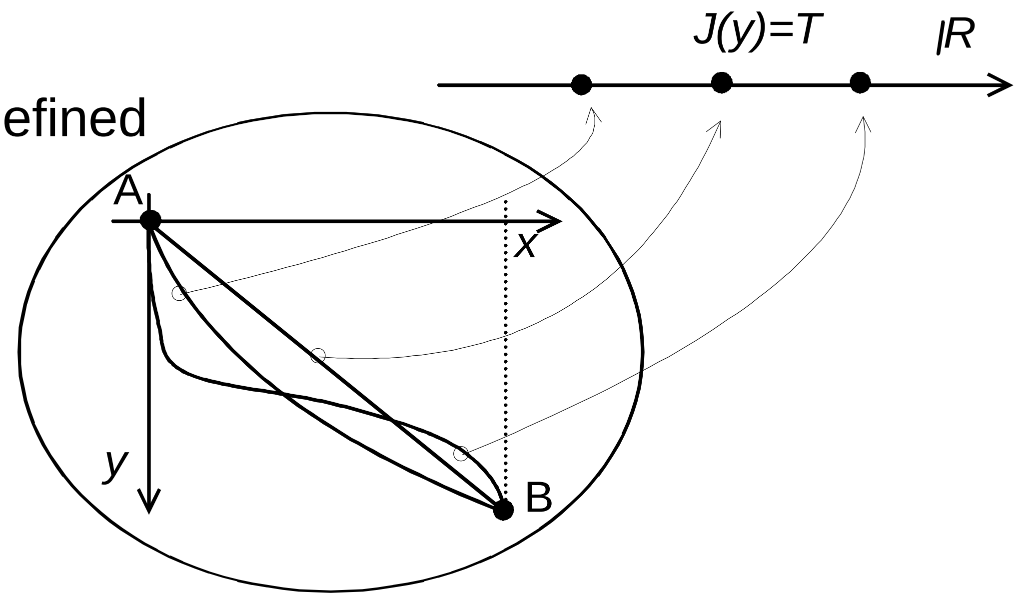
We want to optimize with respect to the curve and this is an infinite dimensional problem, because it depends on the position of the points on the curve (that are infinitely many).

The expression

$$T = \int_0^{x_2} \sqrt{\frac{1+\left(y'(x)\right)^2}{2gy(x)}}dx$$

defines the functional to minimize.

To each differentiable function $y(x)$ defined on $[0,x_2]$ that satisfies the boundary conditions $y(0)=0$ and $y(x_2)=y_2$ if associates a real number (the travel time).



The Brachistochrone problem was published in 1 January 1667 by Johann Bernouilli, as a challenge to the scientific community: *Nothing is more attractive to intelligent persons than an honest problem that challenges them and which solution brings fame and stays as a lasting monument.*

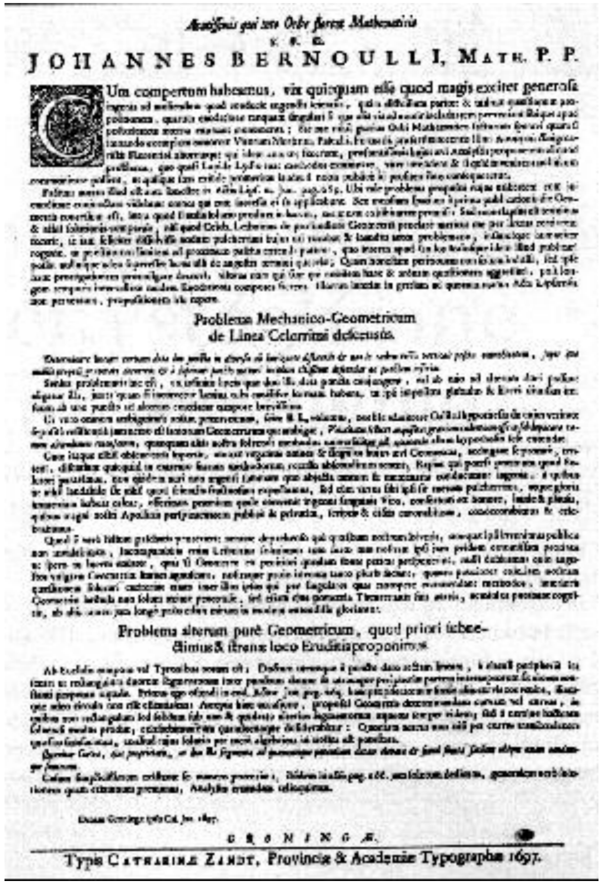
60 years before, Galileo new already that the minimum time trajectory could not be a straight line, although he thought, erroneously, that it was a circumference arc.

This challenge was tackled by six of the most brilliant minds of the time: His elder broither Jacob, Leibniz, Tschirnhaus, l'Hopital and Newton (who published his solution anonymously).

An historical perspective (with technical content) of the Brachistochrone problem and of its relations with Optimal Control may be seen in

Sussmann, H. J. e J. C. Willems (1997). 300 Years of Optimal Control: From the Brachystochrone to the Maximum Principle. *IEEE Control Systems*, 17(3):32-44.

You may access this paper from eduroam using IEEEXPLORE

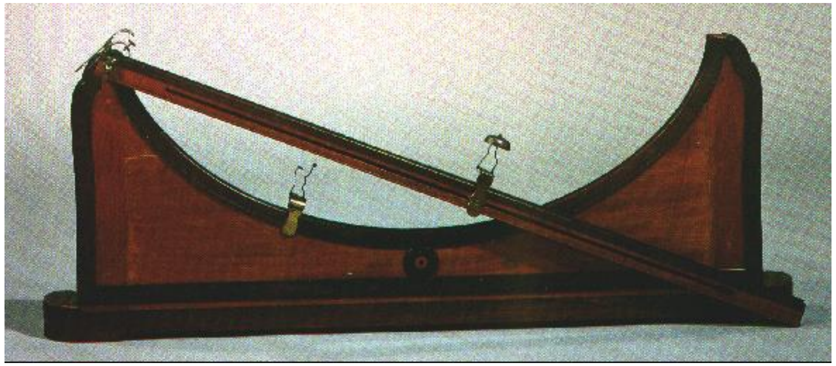


Acutissimis qui toto Orbe florent Mathematicis
S. P. D.
JOHANNES BERNOULLI, MATH. P. P.

Problema Mechanico-Geometricum
de Linea Celerrimi descensus.

Problema alterum purè Geometricum, quod priori subnectimus & strenæ loco Eruditis proponimus.

GRONINGÆ,
Typis CATHARINÆ ZANDT, Provinciæ & Academiæ Typographæ 1697.



A machine to exemplify the brachistochrone, Museu de Física da Universida de Coimbra, Portugal.

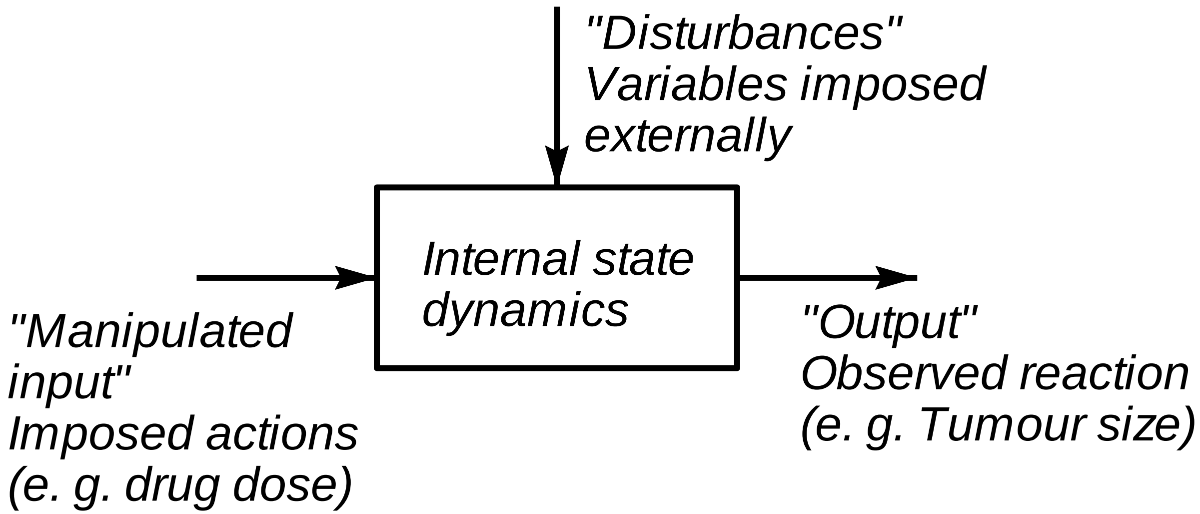
http://Nautilus.fis.uc.pt/museu/index.htm

←The challenge of J. Bernouilli as published in *Acta Eroditorum*

Optimal Control problems

Need for a new tool: Pontryagin's Maximum Principle (1956).

Example: Control approach to therapy design in cancer



Compute the therapy along time that yields the best compromise between

- maximizing the therapeutic effect (minimize tumor size)
- minimizing a measure of toxic effects (minimize total treatment)

Controlled Gompertz model

$$\dot{x} = \alpha\, x\, log\left(\frac{M}{x}\right) - \beta\, x\, u$$

Associated control problem

$$J_2 = x(T) + \rho \int_0^T u(t)\, dt$$

$$0 \leq u \leq u_{max}$$

A basic class of optimal control problems

(Fixed final time, no state constraints)

Let x be the state of a system with manipulated input u, that satisfies

$$\dot{x} = f(x,u) \quad x(0) = x_0 \;\; t \in [0,T] \;\; T \textit{ fixed} \;\; u(t) \in U$$

Find the function u, defined in $[0,T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x,u)dt$$

L is the lagrangian or running cost

Ψ is the terminal cost penalty

Pontriagyn's Maximum Principle

Along an optimal trajectory of x, u, and λ, the following necessary conditions for the maximization of J are verified:

$$\dot{x} = f(x,u) \qquad x(0) = x_0 \qquad t \in [0, T] \;\; u(t) \in U$$

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big)$$

$$\lambda'(T) = \Psi_x(x) \big|_{x=x(T)}$$

Terminal condition on the co-state

At each t, the Hamiltonian function H defined by

$$H(\lambda, x, v) = \lambda' f(x, v) + L(x, v)$$

Is maximum for $v = u$ (the optimal control).

Notation:

$$\Psi_x(x)\Big|_{x=x(T)} = \begin{bmatrix} \left.\dfrac{\partial \Psi(x)}{\partial x_1}\right|_{x=x(T)} & \cdots & \left.\dfrac{\partial \Psi(x)}{\partial x_n}\right|_{x=x(T)} \end{bmatrix} \qquad L_x(x,u) = \begin{bmatrix} \dfrac{\partial L}{\partial x_1} & \cdots & \dfrac{\partial L}{\partial x_n} \end{bmatrix}$$

$$f_x = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial f_2} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} & \cdots & \dfrac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \dfrac{\partial f_n}{\partial x_2} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}$$

The vector λ is called **co-state**, and its equation is the **adjoint equation**.

Other optimal control problems

- More general problems
  - Free terminal time and minimum time problems
  - Final state constraints
  - Other state constraints
- Important special cases
  - Linear dynamics and quadratic constraints
- Bang-bang control and singular arcs

Bibliographic references on OC for the impatient students

- [L1979] Ch. 11, pp. 394 – 435. This a quick and beautiful introduction to the main points of optimal control and dynamic programming, with a justification using calculus of variations – like arguments of the version of the Pontryagin Principle presented above. The whole book is also a very good, easy to read, and sometimes exhilarant, introduction to dynamic systems and control that is strongly suggested to the students with a lack of background on this subjects.
- [R2015] An introduction to the correct formulation of optimal control problems and solving them with Pontryagin Principle. The emphasis is not on mathematical profs, but on developing skills to correctly formulate OC

problems in such a way that they can be solved with numerical packages such as DIDO, for which a free (limited) version is available. The author, I. M. Ross was one of the developers of a class of numerical methods to solve OC problems known as pseudo-spectral methods.

[L1979] D. G. Luenberger. *Introduction to dynamic Systems*. Wiley, 1979.

[R2015] I. M. Ross. *A primer on Pontryagin's Principle . in Optimal Control*. Collegiate Publishers, 2015.

Exercise 1 (Just to warm up)

Design a curve $x(t)$ that starts at $x(0)=0$, with a maximum slope of 1 and that reaches the maximum height for $t=T$.

The problem may be formulated as an optimal control problem with dynamics

$$\dot{x}(t) = u(t) \qquad x(0) = 0 \qquad U = \{u | u < 1\}$$

and cost functional to be maximized

$$J = x(T)$$

Use Pontryagin's Principle to find the optimal solution.

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big) \qquad \lambda'(T) = \Psi_x(x)\,\big|_{x=x(T)}$$

Since

$$f_x(x,u) = 0 \quad \text{and} \quad L(x,u) = 0$$

The adjoint equation is

$$-\dot{\lambda}(t) = 0$$

With terminal condition

$$\lambda(T) = 1 \quad \text{since} \quad \Psi(x(T)) = x(T)$$

Hence

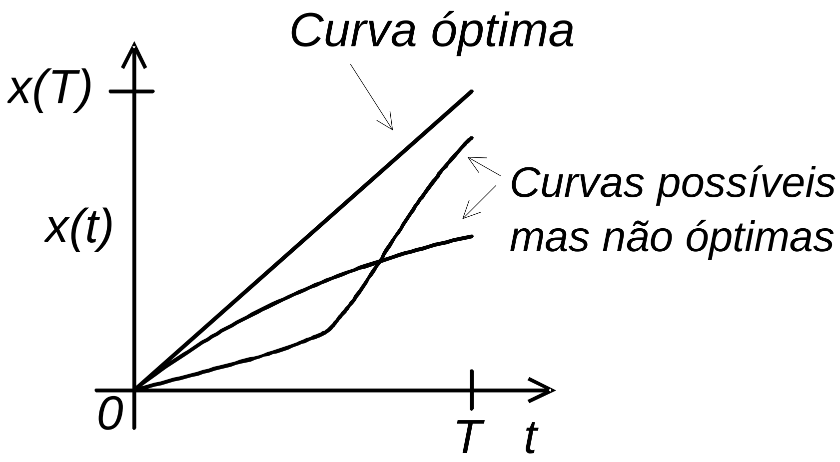
$$\lambda(t) = 1 \qquad 0 \le t \le T$$

The Hamiltonian is

$$H = \lambda' f + L = \lambda u = u$$

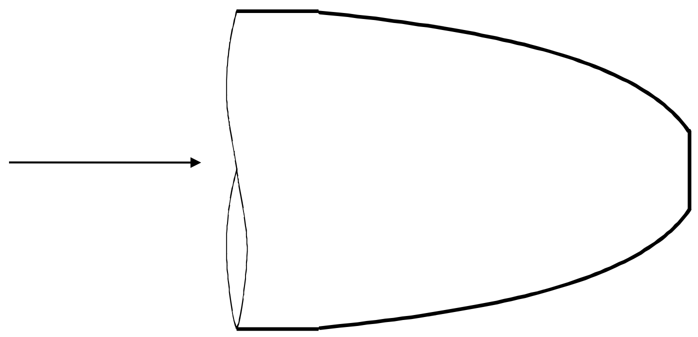
At each *t* the value of *u* that maximes *H* in the set *U* is thus

$$u_{opt}(t) = 1$$

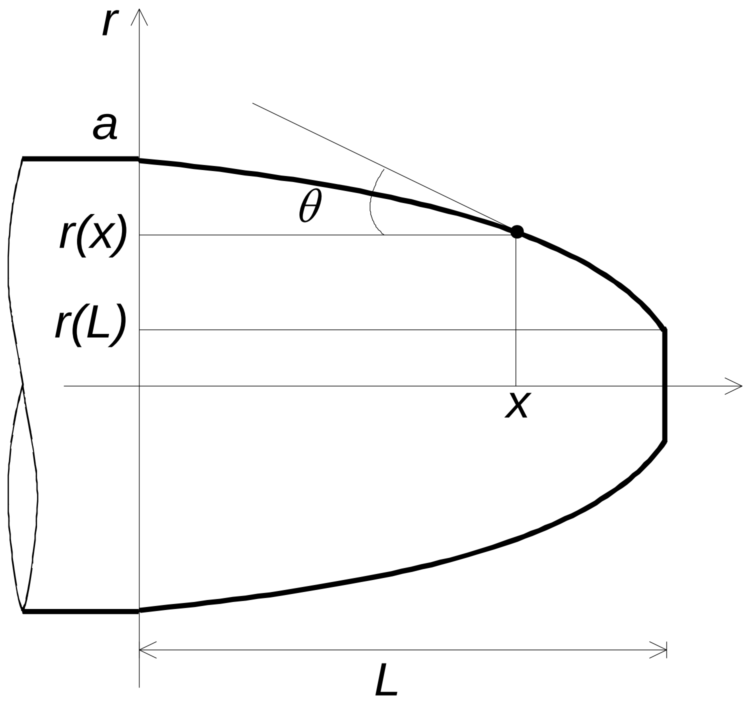


Example: Minimum drag shape of a shell

What is the shape of a shell that leads to a minimum drag?



This problem was solved by Newton in 1686 (10 years before Johann Bernouilli's challenge on the brachistochrone). Newton was aiming an application to ship design but the model he used for the drag force was valid only for very low density atmosphere at a hipersonic velocity.



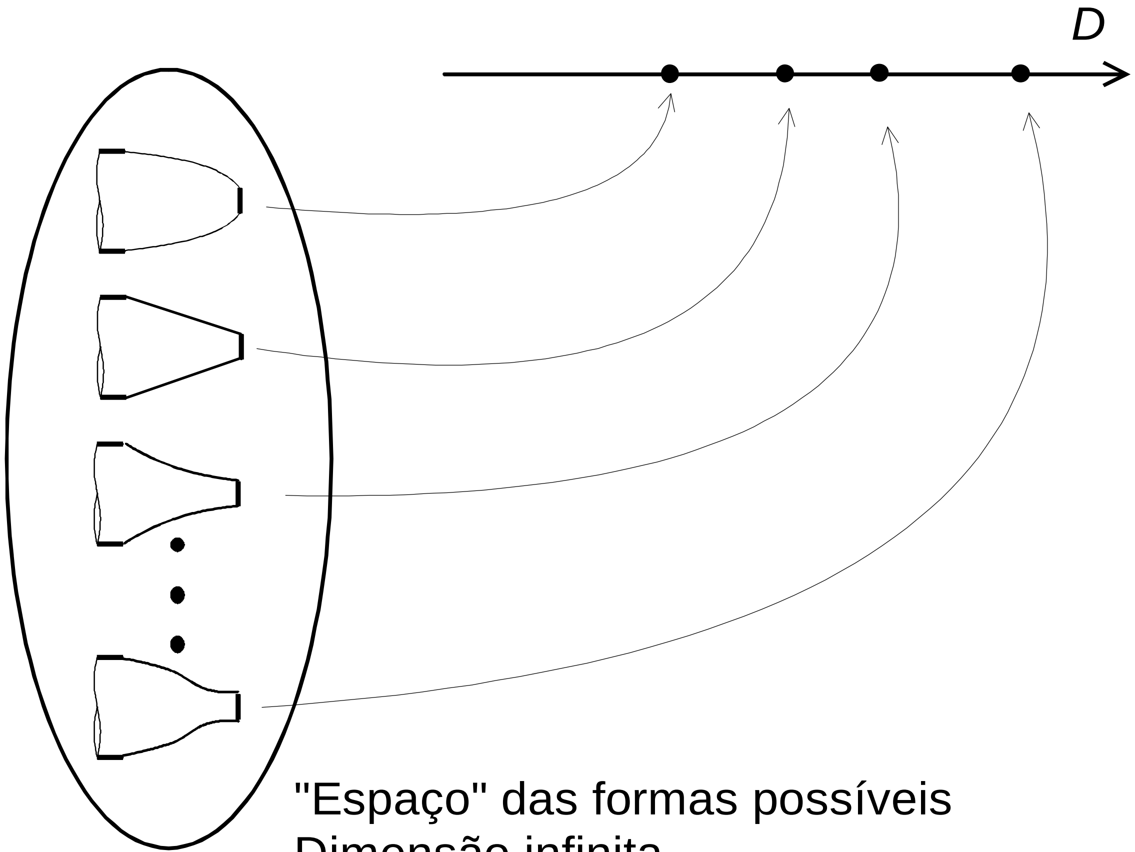
At hypersonic velocities the drag force *D* is approximately given by

$$D = -2\pi q \int_{x=0}^{x=L} C_p(\theta) r dr$$

where q is the dynamic pressure assumed to be constant and

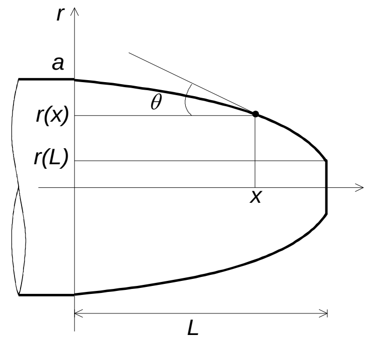
$$C_p = \begin{cases} 2\sin^2\theta \; for \;\; \theta \geq 0 \\ 0 \; for \;\; \theta \leq 0 \end{cases}$$

Each shell shape corresponds to a drag force.



"Espaço" das formas possíveis
Dimensão infinita

$$D = -2\pi g \int_{x=0}^{x=L} 2\sin^2 \theta r dr$$



Can be formulated as an Optimal Control problem:

Minimize:

$$\frac{D}{4\pi q} = \frac{1}{2}\left[r(L)\right]^2 + \int_0^L \frac{ru^3}{1+u^2} dx$$

Subject to the "dynamics"

$$\frac{dr}{dx} = u$$

The previous problem illustrates two significant issues:

- A shape optimization problem of a planar curve may be transformed into an Optimal Control problem by using a dynamic equation that generates the family of curves considered.
- The problem may be formulated as an Calculus of Variation problem. However, it is readily transformed into an Optimal Control problem by using the dynamic equation

$$\frac{dy}{dt} = u$$

where $y = r$ in the shell problem and u is the control variable.

This technique can be applied to transform a CV problem into an equivalent OC problem.

Exercise 2 – Push cart

Objective: find the function $u(t) \quad 0 \le t \le T$ that maximizes

$$J(u) = x_1(T) - \frac{1}{2}\int_0^T u^2(t)dt \quad , \quad (x_1 := z)$$

sendo a dinâmica do carro dada por (condições iniciais nulas):

$$\frac{d^2 z}{dt^2} = u \quad \text{or} \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, u\right)$$

Solution:

$$\begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix} = \begin{bmatrix} x_2 \\ u \end{bmatrix}$$

$$f_x = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \dfrac{\partial f_1}{\partial x_2} \\ \dfrac{\partial f_2}{\partial x_1} & \dfrac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$J(u) = x_1(T) - \frac{1}{2}\int_0^T u^2(t)dt$$

$\Psi\big(x(T)\big) = x_1(T)$ and hence $\Psi_x\big(x(T)\big) = \begin{bmatrix}1 & 0\end{bmatrix}$

$L(x,u) = -\frac{1}{2}u^2(t)$ and hence $L_x(x,u) = \begin{bmatrix}0 & 0\end{bmatrix}$

The adjoint equation is $-\dot{\lambda}' = \lambda' f_x + L_x$ or

$$\begin{bmatrix}-\dot{\lambda}_1 & -\dot{\lambda}_2\end{bmatrix} = \begin{bmatrix}\lambda_1 & \lambda_2\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \qquad \begin{cases}\dot{\lambda}_1 = 0\\ \dot{\lambda}_2 = -\lambda_1\end{cases} \qquad \begin{bmatrix}\lambda_1(T) & \lambda_2(T)\end{bmatrix} = \begin{bmatrix}1 & 0\end{bmatrix}$$

$$\begin{cases} \dot{\lambda}_1 = 0 \\ \dot{\lambda}_2 = -\lambda_1 \end{cases} \quad \begin{bmatrix} \lambda_1(T) & \lambda_2(T) \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

In this case, the adjoint equation can be solved independently of the state and optimal control. Usually it is not so.

Since

$\dot{\lambda}_1(t) = 0$ we conclude that $\lambda_1(t) = C^{te}$

From the terminal condition $\lambda_1(T) = 1$ it is concluded that

$$\lambda_1(t) = 1$$

The equation for $\lambda_2(t)$ is

$$\dot{\lambda}_2(t) = -\lambda_1$$

Since $\lambda_1(t) = 1$, this equation becomes

$$\dot{\lambda}_2(t) = -1$$

And hence

$$\lambda_2(t) = C^{te} - t$$

From the terminal condition $\lambda_2(T) = 0$ we get

$$\lambda_2(t) = T - t$$

Hamiltonian:

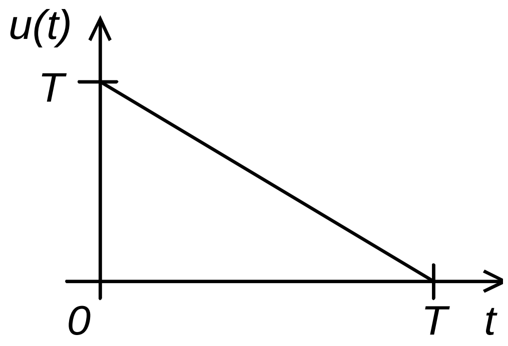
$$H(\lambda, x, u) = \lambda_1 x_2 + \lambda_2 u - \frac{1}{2} u^2$$

In this case there are no control constraints (u may assume values everywhere on $\mathbb{R}$) and the maximum condition for the Hamiltonian is

$$\frac{\partial H}{\partial u} = 0 \quad \text{or} \quad \lambda_2 - u = 0 \quad \text{for each time } t$$

The optimal control is thus

$$u_{opt}(t) = \lambda_2(t) = T - t$$



Exercise 3 – Push cart with minimum fuel

maximize $$J(u) = x_1(T) - \int_0^T u(t)dt$$

s. t. $$\dot{x}_1 = x_2$$ $$\dot{x}_2 = u$$ and $$0 \le u \le \bar{u}$$

Assume $T > 1$.

Solution

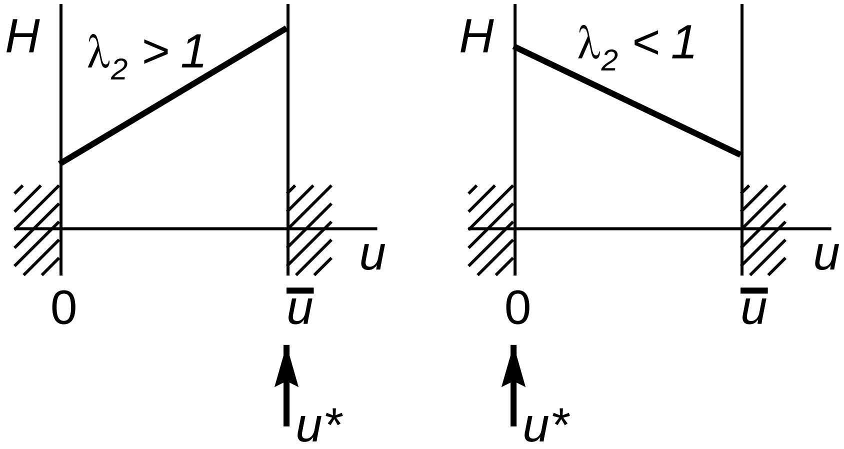
The co-state is as before:

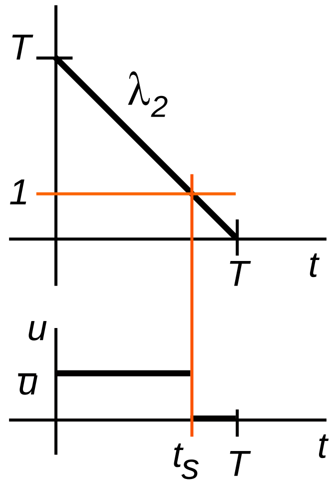
$$\lambda_1(t) = 1, \;\; \lambda_2(t) = T - t$$

The Hamiltonian is now

$$H = [\lambda_1 \quad \lambda_2] \begin{bmatrix} x_2 \\ u \end{bmatrix} - u = \lambda_1 x_2 + (\lambda_2 - 1)u$$

Since the Hamiltonian is linear in u, its maximum is attained at the boundary of the interval of the acceptable values for u.





$$\lambda_2(t_s) - 1 = 0$$

$$T - t_s - 1 = 0$$

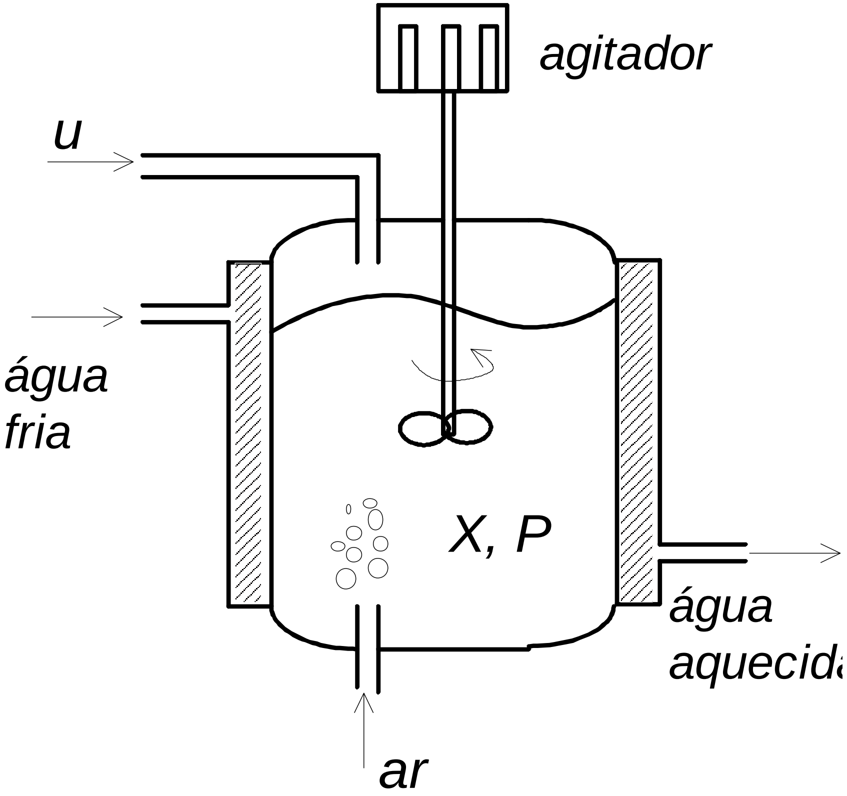
$$t_s = T - 1$$

Penicillin Fermentation reactor

X – Quantity of fungi per unit volume

P – Quantity of penicilin per unit volume

u – Manipulated variable, substract rate (sugar)



Fungi produce penicillin.

A very simplified model of the fermentation

Growth due to
“food

Mortality

$$\dot{X} = buX - \mu X$$

$$\dot{P} = c(1-u)X$$

Fungi
production

Production inibition due
to substract

Fermentation Optimal Control Problem

Model and initial conditions:

$$\dot{X} = uX - 0,5X$$

$$\dot{P} = (1-u)X$$

Initial conditions:

$$X(0) = 1$$

$$P(0) = 0$$

Objective:

Find $u(t)$ $0 \leq t \leq T$, T fixed, so that $J = P(T)$ is maximum given the constraint

$$0 \leq u \leq 1$$

Write the adjoint equation

The cost functional is

$$J = \psi\big(x(T)\big) + \int_0^T L(x,u)dt$$

In this case

$$J_{fermenter} = P(T)$$

Therefore $L(x,u) = 0$

and $\psi(x(T)) = P(T)$, and thus

$$\psi_x(x(T) = \left[\frac{\partial\psi}{\partial x_1} \quad \frac{\partial\psi}{\partial x_2}\right]\Bigg|_{x=x(T)} = \begin{bmatrix}0 & 1\end{bmatrix}$$

The co-state has two components

$$\lambda'(t) = \begin{bmatrix} \lambda_1(t) & \lambda_2(t) \end{bmatrix}$$

Since the Lagrangian is zero:

$$L_x(x,u) = \begin{bmatrix} 0 & 0 \end{bmatrix}$$

Since $f(x,u) = \begin{bmatrix} f_1(x_1, x_2, u) \\ f_2(x_1, x_2, u) \end{bmatrix} = \begin{bmatrix} (u-0.5)x_1 \\ (1-u)x_1 \end{bmatrix}$ it is $f_x(x,u) = \begin{bmatrix} u-0.5 & 0 \\ 1-u & 0 \end{bmatrix}$

Adjoint equation

$$-\dot{\lambda}' = \lambda' f_x(x,u) + L_x(x,u)$$

$$f_x(x,u) = \begin{bmatrix} u-0.5 & 0 \\ 1-u & 0 \end{bmatrix} \qquad L_x(x,u) = 0$$

In this case the adjoint equation is

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + (1-u)\lambda_2$$

$$-\dot{\lambda}_2 = 0$$

With terminal condition

$$\lambda_1(T) = 0 \quad \lambda_2(T) = 1$$

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + (1-u)\lambda_2 \qquad -\dot{\lambda}_2 = 0$$

$$\lambda_1(T) = 0 \quad \lambda_2(T) = 1$$

Considering the terminal conditions

$$\lambda_2(t) = 1 \qquad 0 \leq t \leq T$$

And the equation for the 1st component of the co-state becomes

e a equação para a primeira componente do co-estado reduz-se a

$$-\dot{\lambda}_1 = (u-0.5)\lambda_1 + 1 - u$$

Dificulty: The equation depends on $u(t)$ and $u(t)$ depends on $\lambda(t)$...

Hints

a) Write the Hamiltonian for this special case. Remember that

$$H(\lambda, x, u) = \lambda' f + L$$

b) Assume that you know $\lambda(t)$. Find $u(t)$ that maximizes H for each t.
Remember the constraint $0 \le u \le 1$ and assume that $X > 0$

c) From b) you know the shape of $u(t)$ as a function of t. In particular, what is the value of $u(t)$ for t close to T? And the corresponding equation for $\lambda_1(t)$ during thgis time period?

d) Go “backwards” in time. What happens to $\lambda_1(t)$? And $u_{optim}(t)$?

$$H = \lambda' f + L$$

$$H = \lambda_1 f_1(X, P) + \lambda_2 f_2(X, P) + 0$$

$$H = \lambda_1 (u - 0.5) X + (1 - u) X$$

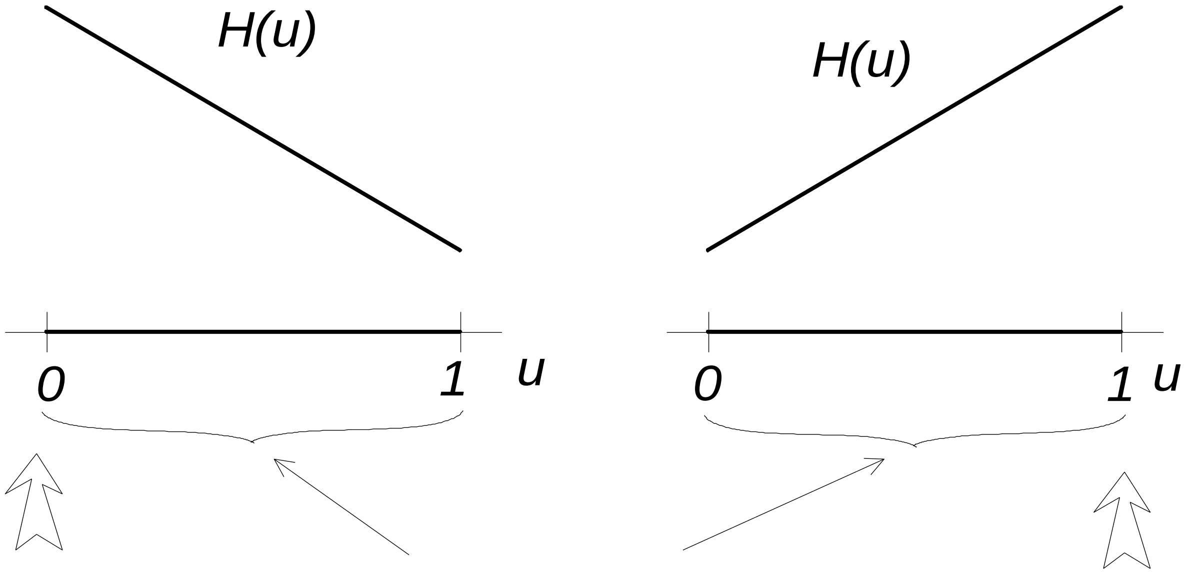
Can be written as

$$H = \left[(\lambda_1 - 1) u + (1 - 0.5 \lambda_1) \right] X$$

The Hamiltonian H is a linear function of u.

Assuming $X > 0$, H growing or decreasing depends just on $\lambda_1 - 1$.

$$H = \left[(\lambda_1 - 1)u + (1 - 0.5\lambda_1)\right]X$$



Neste caso $u_{opt}=0$

Intervalo de valores admissíveis para u

Neste cas $u_{opt}=1$

$\lambda_1 < 1$ $\lambda_1 > 1$

Since

$$\lambda_1(T) = 0$$

for t close to T, $\lambda_1(t) = 0$. Thus, since $\lambda_1(T) < 1$, the corresponding optimal control is

$$u_{opt}(t) = 0$$

Close to T, the adjoint equation becomes

$$-\dot{\lambda}_1 = (\underbrace{u}_{=0} - 0.5)\lambda_1 + 1 - \underbrace{u}_{=0}$$

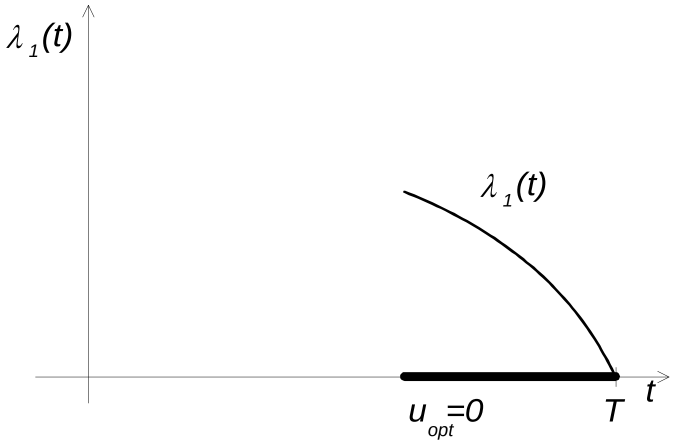
$$\dot{\lambda}_1(t) = 0.5\lambda_1 - 1$$

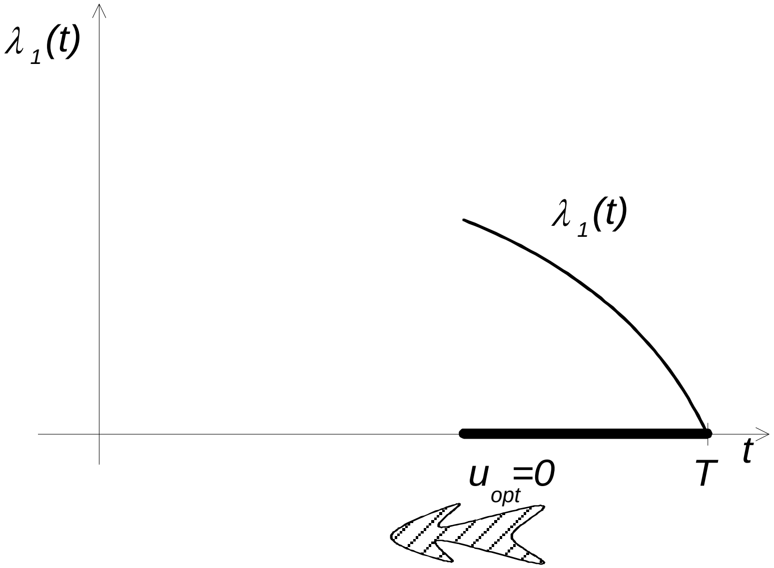
Near the end of the optimization interval the adjoint equation becomes

$$\dot{\lambda}_1(t) = 0.5\lambda_1(t) - 1 \qquad \lambda_1(T) = 0$$

It has the solution

$$\lambda_1(t) = \frac{1}{0.5}\left(1 - e^{0.5(t-T)}\right)$$





$$\frac{1}{0.5}\left(1-e^{0.5(t_s-T)}\right)=1$$

$$e^{0.5(t_s-T)} = 0.5$$

$$\log 0.5 = 0.5(t_s - T)$$

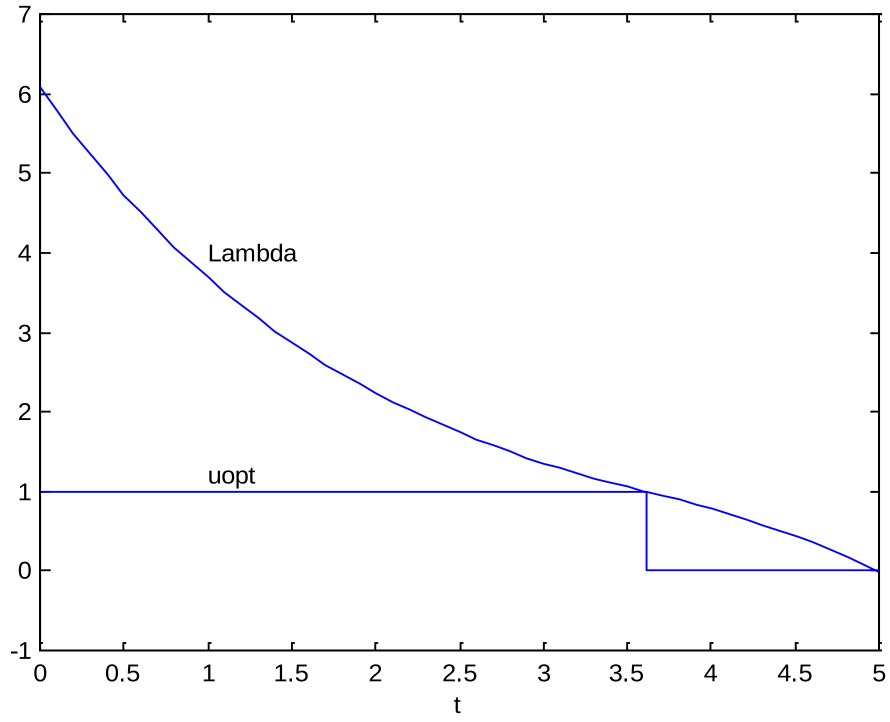
$$t_s = T + 2\log o.5 \cong T - 1.39$$

"Moving" in this sense *u*

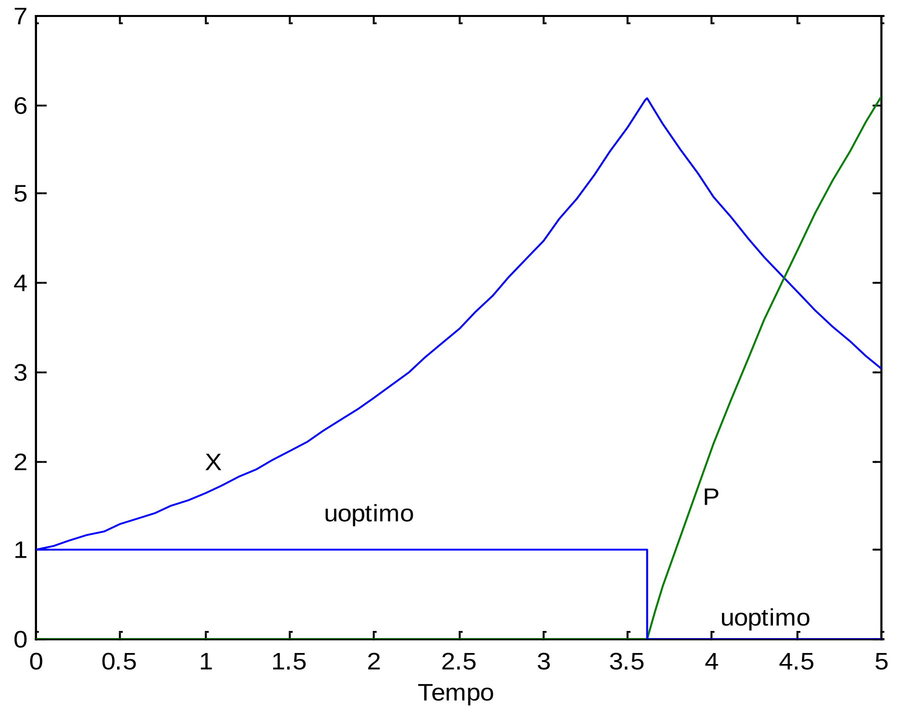
becomes 1 at instant t_s in which

$$\lambda_1(t_s)=1$$

Example for the situation in which *T=5*

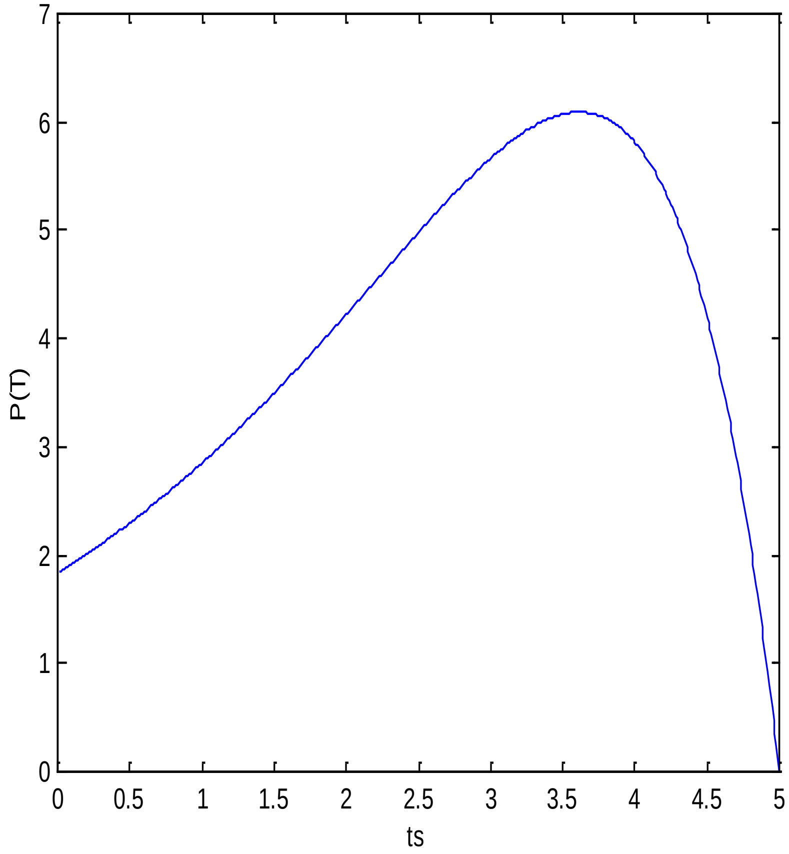


The optimal solution admits the following interpretation: Initially, all the effort is to make the fungi colony to grow. Due to the inhibition effect of the substrate there is no penicillin production. After the switching instant, the control variable is adjusted to maximize the penicillin production.



Assuming a bang-bang shape for the control function, the switching instant corresponds to the maximum.

It is remarked that Pontryagin's Principle yields not only the switching instant but also the shape of the control function.



Proof of Pontryagin's Principle

Objective:

Proof of Pontryagin's Principle necessary conditions for fixed time problems with free end state, using a variational method.

The optimal control problem

Let x be the state of a system with manipulated input u, that satisfies

$$\dot{x} = f(x,u) \quad x(0) = x_0 \;\; t \in [0,T] \;\; T \text{ fixed} \;\; u(t) \in U$$

Find the function u, defined in $[0,T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x,u)dt$$

Proof strategy

If u_{opt} is a function that maximizes $J(u)$, any “small” δu of the control function leads to a decrease on $J(u)$:

$$\delta J = J(u_{opt} + \delta u) - J(u_{opt}) < 0$$

Passos na demonstração do Princípio de Pontryagin

- Modificação do funcional de custo através de uma funcional de custo por forma a simplificar o cálculo da sua variação quando o **controlo é perturbado**
- Cálculo da relação existente entre uma variação "pequena" no controlo óptimo e a correspondente **variação no funcional**. Retêm-se apenas termos de 1ª ordem
- Exprimir a condição de que a variação do funcional é negativa através de **uma condição de máximo na Hamiltoniana para cada instante de tempo**.

Modified objective function

$$\bar{J} = J - \int_0^T \lambda'(t)\left[\dot{x}(t) - f(x(t), u(t))\right]dt$$

Along the plant trajectories, the term inside the square brackets is zero and $\bar{J} = J$, and hence the u that optimizes $\bar{J}$ is the same that optimizes J.

Therefore, we can select λ such as to simplify the problem.

The Hamiltonian

Define the Hamiltonian by

$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$$

With this definition

$$\bar{J} = J - \int_0^T \lambda'(t)\left[\dot{x}(t) - f(x(t), u(t))\right] dt = \Psi(x(T) + \int_0^T \left[L(x, u) + \lambda' f(x, u) - \lambda' \dot{x}\right] dt$$

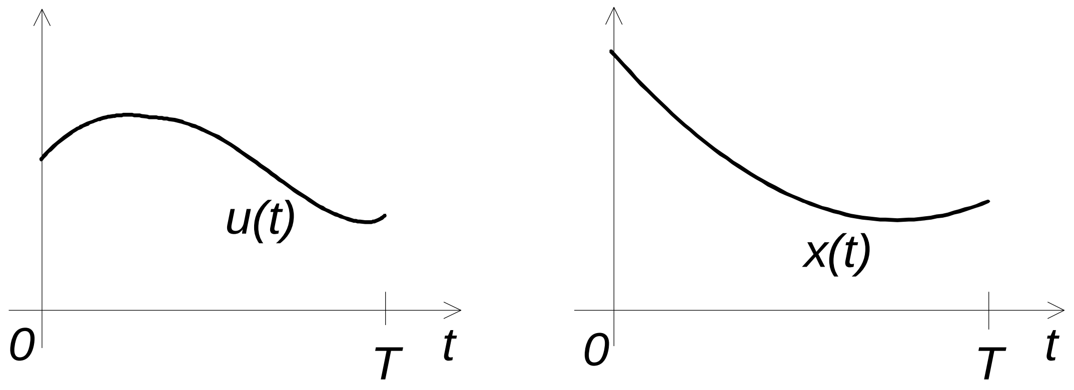
or

$$\bar{J} = \Psi(x(T)) + \int_0^T \left[H(\lambda(t), x(t), u(t)) - \lambda'(t)\dot{x}(t)\right] dt$$

The optimal profile

Let $\{u(t), 0 \le t \le T\}$ be the optimal control function

Together with the initial condition, it determines the state function along the optimal profile $\{x(t), 0 \le t \le T\}$.



Optimal control variation

Perturb the optimal control u to obtain a perturbed control v

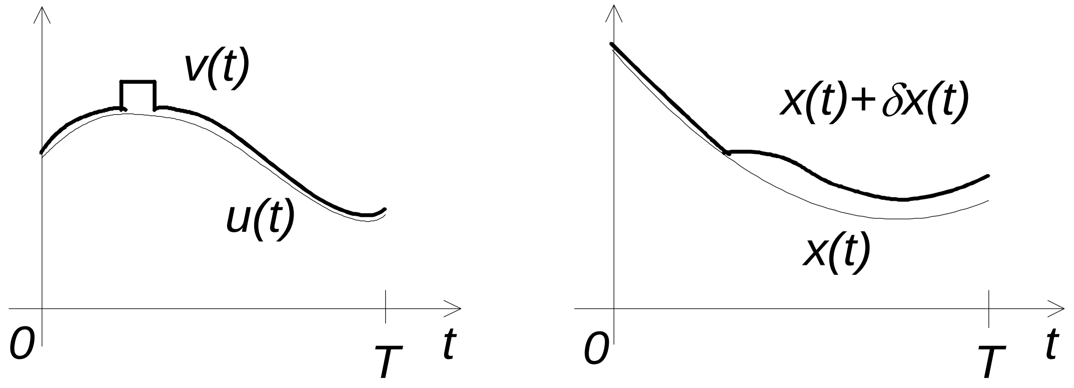
The perturbatuion is small in the sense that for all the components u_i and v_i

$$\int_0^T |u_i(t) - v_i(t)| dt < \varepsilon$$

where ε is a small positive real number.

The state trajectory that corresponds to v deviates little from the optimal one

Let $\delta x(t)$ be this state variation.



Let $\delta\bar{J}$ be the corresponding variation in the objective function

$$\delta\bar{J} = \bar{J}(v) - \bar{J}(u)$$

If u is optimal, this variation is negative.

Computation of the functional variation

Remember that

$$\bar{J} = \Psi\big(x(T)\big) + \int_0^T \big[H\big(\lambda(t), x(t), u(t)\big) - \lambda'(t)\dot{x}(t)\big]dt$$

The variation is thus

$$\delta\bar{J} = \Psi\big(x(T) + \delta x(T)\big) - \Psi\big(x(T)\big) + \int_0^T \big[H\big(\lambda, x + \delta x, v\big) - H\big(\lambda, x, u\big) - \lambda'\delta\dot{x}\big]dt$$

Integration by parts formula:

Since $\frac{d}{dt}(ab) = \dot{a}b + a\dot{b}$ it is $\int_0^T (\dot{a}b)dt = (ab)\Big|_0^T - \int_0^T (a\dot{b})dt$

Apply this rule with

$$a = \delta x \qquad b = \lambda'$$

$$\int_0^T \lambda' \delta\dot{x} dt = \lambda'(T)\delta x(T) - \lambda'(0)\delta x(0) - \int_0^T \dot{\lambda}' \delta x dt$$

Remark that $\delta x(0) = 0$ because the control variation does not change the initial condition.

$$\int_0^T \lambda' \delta\dot{x} dt = \lambda'(T)\delta x(T) - \int_0^T \dot{\lambda}' \delta x dt$$

We have concluded that

$$\delta\bar{J} = \Psi\left(x(T) + \delta x(T)\right) - \Psi\left(x(T)\right) + \int_0^T \left[H\left(\lambda, x + \delta x, v\right) - H\left(\lambda, x, u\right) - \lambda' \delta\dot{x}\right] dt$$

Therefore:

$$\delta\bar{J} = \Psi\left(x(T) + \delta x(T)\right) - \Psi\left(x(T)\right) - \lambda'(T)\delta x(T) + \int_0^T \left[H\left(\lambda, x + \delta x, v\right) - H\left(\lambda, x, u\right) + \dot{\lambda}' \delta x\right] dt$$

$$\delta \bar{J} = \Psi\big(x(T)+\delta x(T)\big) - \Psi\big(x(T)\big) - \lambda'(T)\delta x(T) + \int_0^T \Big[H\big(\lambda, x+\delta x, v\big) - H\big(\lambda, x, u\big) + \dot{\lambda}'\delta x\Big]dt$$

Make 1st orderTaylor series approximations:

$$\Psi\big(x(T)+\delta x(T)\big) \approx \Psi\big(x(T)\big) + \Psi_x\big(x(T)\big)\delta x(T)$$

$$H\big(\lambda, x+\delta x, v\big) \approx H\big(\lambda, x, v\big) + H_x\big(\lambda, x, v\big)\delta x$$

Up to 1st order terms:

$$\delta\bar{J} = \left[\Psi_x\big(x(T)\big) - \lambda'(T)\right]\delta x(T) + \int_0^T \left[H_x\big(\lambda, x, u\big) + \dot{\lambda}'\right]\delta x dt + \int_0^T \left[H\big(\lambda, x, v\big) - H\big(\lambda, x, u\big)\right]dt$$

If λ is selected to satisfy

$$-\dot{\lambda}'(t) = H_x\big(\lambda(t), x(t), u(t)\big)$$

With the final condition

$$\lambda'(T) = \Psi_x\big(x(T)\big)$$

The expression for the variation of the functional becomes

$$\delta\bar{J} = \int_0^T \left[H\big(\lambda(t), x(t), v(t)\big) - H\big(\lambda(t), x(t), u(t)\big)\right]dt$$

$$\delta \bar{J} = \int_0^T \Big[H\big(\lambda(t), x(t), \underset{\substack{\uparrow \\ \text{Perturbed}}}{v(t)}\big) - H\big(\lambda(t), x(t), \underset{\substack{\uparrow \\ \text{Optimal}}}{u(t)}\big) \Big] dt$$

This expression shows the effect of a control variation on the objective function.

Remark that λ, x and u are known and independent of the perturbed control v.

In particular, x and λ are computed by integrating the state and co-state equations with the optimal control u.

$$\delta\bar{J} = \int_0^T \left[H\big(\lambda(t), x(t), v(t)\big) - H\big(\lambda(t), x(t), u(t)\big) \right] dt$$

If u is optimal, at any instant ***t***:

$$H\big(\lambda(t), x(t), v\big) \leq H\big(\lambda(t), x(t), u(t)\big)$$

$$\forall v \in U$$

This statement must be proved.

The proof is possible because if happens for an arbitrary control perturbation.

$$\delta \bar{J} = \int_0^T \left[H\big(\lambda(t), x(t), v(t)\big) - H\big(\lambda(t), x(t), u(t)\big) \right] dt$$

Assume that there is an instant t_1 and a function φ such that

$$H\big(\lambda(t_1), x(t_1), \varphi(t_1)\big) > H\big(\lambda(t_1), x(t_1), u(t_1)\big)$$

Since H is a continuous function, there exists an interval $[t_1 - \sigma, t_1 + \sigma]$ in which this property holds. Let $v(t) = u(t)$ except in this interval in which we select $v(t) = \varphi(t)$. For this perturbed function

$$\delta \bar{J} = \int_{t_1-\sigma}^{t_1+\sigma} \left[H\big(\lambda(t), x(t), v(t)\big) - H\big(\lambda(t), x(t), u(t)\big) \right] dt > 0$$

Where the inequality results from the integrand function being always positive. This fact contradicts the assumption that u is the optimal control.

Problems with equality constraints on the terminal state

Let x be the state of a plant with input u defined by

$$\dot{x} = f(x, u) \qquad x(0) = x_0 \qquad t \in [0, T] \quad u(t) \in U$$

T *given*

Find the function u, defined in the time interval $[0, T]$ that maximizes

$$J(u) = \Psi(x(T)) + \int_0^T L(x, u)\, dt$$

Subject to the equality constraints in the terminal state

$$x_i(T) = \bar{x}_i \qquad i = 1, 2, \dots, r \leq n$$

Maximum Principle (Equality constraints on the terminal state)

Alomng the optimal trajectory for *x*, *u* and λ the following necessary conditions for the maximization of *J* are verified

$$\dot{x} = f(x, u) \qquad x(0) = x_0 \qquad t \in [0, T] \;\; u(t) \in U$$

$$x_i(T) = \bar{x}_i \qquad i = 1, 2, \ldots, r \le n$$

$$-\dot{\lambda}'(t) = \lambda'(t) f_x\big(x(t), u(t)\big) + L_x\big(x(t), u(t)\big)$$

$$\lambda_i'(T) = \Psi_x\big(x(T)\big)_i \qquad i = r+1, r+2, \ldots, n$$

For each *t*, the Hamiltonian $H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$ is maximum for the optimal value of $u(t)$.

Free terminal time problems

In addition to the conditions of the Maximum Principle, the following condition must hold:

$$H\big(\lambda(T), x(T), u(T)\big) = 0$$

The Linear Quadratic Problem

Dynamics:

$$\dot{x}(t) = Ax(t) + bu(t)$$

$$x(0) = x_0 \qquad u(t) \in R^m$$

Cost functional:

T fixo

$$J = \frac{1}{2}\int_0^T \left[x'(t)Qx(t) + u'Ru\right]dt \qquad Q = Q' \geq 0 \qquad R = R' > 0$$

Since we want to minimize J the Lagrangian is

$$L(x,u) = -\frac{1}{2}(x'Qx + u'Ru)$$

Adjoint equation

$$-\dot{\lambda}' = \lambda' f_x + L_x$$

$-\dot{\lambda}'(t) = \lambda'(t)A - x'(t)Q$ subject to the terminal condition $\lambda(T) = 0$

Hamiltonian

$$H(\lambda, x, u) = \lambda' f(x,u) + L(x,u)$$

$$H(\lambda, x, u) = \lambda'(t)Ax(t) + \lambda'(t)bu(t) - \frac{1}{2}x'(t)Qx(t) - \frac{1}{2}u'(t)Ru(t)$$

Minimum condition on the Hamiltoniana

The Hamiltonian

$$H(\lambda, x, u) = \lambda'(t)Ax(t) + \lambda'(t)bu(t) - \frac{1}{2}x'(t)Qx(t) - \frac{1}{2}u'(t)Ru(t)$$

Is a quadratic function. A necessary condition of minimum is therefore

$$\frac{\partial H}{\partial u} = 0$$

or

$$\lambda'(t)b - u'(t)R = 0$$

Thus, the optimal control verifies

$$u(t) = R^{-1}b'\lambda(t)$$

Thus, the optimal trajectory verifies

$$u_{opt}(t)$$

$$\dot{x}(t) = Ax(t) + bR^{-1}b'\lambda(t)$$

$$\dot{\lambda}(t) = Qx(t) - A'\lambda(t)$$

Subject to

$$x(0) = x_0 \qquad \lambda(T) = 0$$

This is a problem in which the unknowns (x and λ) are specified at two points (0 and T). It is said to be a *Two point boundary value problem*.

How to solve it?

State and co-state equations with optimal control

$$\dot{x} = Ax + bR^{-1}b'\lambda$$

$$\dot{\lambda} = Qx - A'\lambda$$

Assume that there is a matrix $P(t)$ such that

$$\lambda = -Px$$

Under this assumption, the state and co-state equations can be written as

$$\dot{x} = \left[A - bR^{-1}b'P\right]x$$

$$\dot{\lambda} = \left[Q + A'P\right]x$$

Let's try to get an equation for $P(t)$. We have

$$\lambda = -Px$$

Differentiate

$$\dot{\lambda} = -\dot{P}x - P\dot{x}$$

Use the state and co-state equations

$$\left(Q + A'P\right)x = -\dot{P}x - P\left(A - bR^{-1}b'P\right)x$$

Factorize x

$$\left[\dot{P} + PA + A'P - PbR^{-1}b'P + Q\right]x = 0$$

$$\left[\dot{P} + PA + A'P - PbR^{-1}b'P + Q\right]x = 0$$

In order that this identity holds for all x, the term between brackets must vanish.

In this way, we arrive at the **Riccati differential equation**:

$$-\dot{P} = PA + A'P - PbR^{-1}b'P + Q$$

$$P(T) = 0 \qquad \text{(why?)}$$

Linear Quadratic (LQ) Problem

Given a system with linear dynamics

$$\dot{x}(t) = Ax(t) + bu(t) \qquad x(0) = x_0 \qquad u(t) \in R^m$$

The control that minimizes the quadratic cost over an infinite horizon

$$J = \frac{1}{2}\int_0^T \left[x'(t)Qx(t) + u'Ru\right]dt \qquad Q = Q' \geq 0 \qquad R = R' > 0$$

Is given by the state feedback with time varying gain:

$$u(t) = -K(t)x(t) \qquad K(t) = R^{-1}B'P(t)$$

Where $P(t)$ is a symmetric positive definite matrix that satisfies the Riccati differential equation

$$-\dot{P} = PA + A'P - PbR^{-1}b'P + Q \qquad P(T) = 0$$

Example (LQ Control of a 1st order system)

Consider the 1st order, open loop unstable system

$$\dot{x}(t) = x(t) + u(t) \qquad x(0) = 1$$

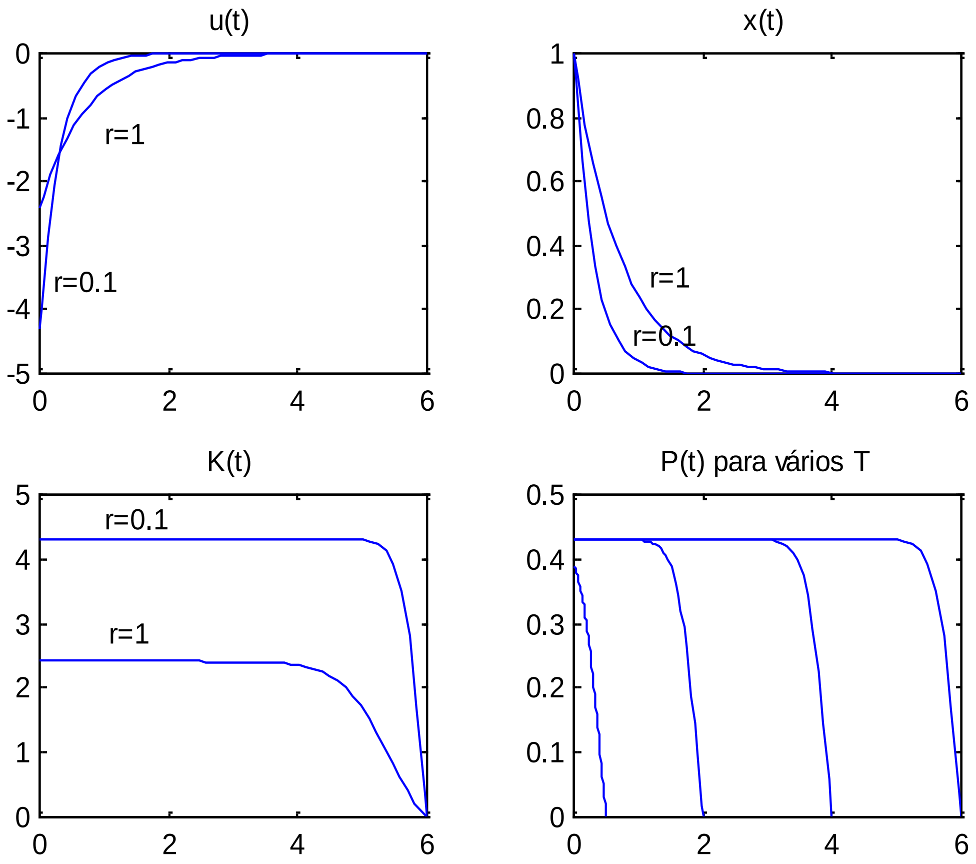
Find the control law that minimizes

$$J(u) = \frac{1}{2}\int_0^T \left[x^2(t) + ru^2(t)\right]dt \qquad T > 0, \quad r > 0$$

The solution is given by

$$\dot{p}(t) = -2p(t) + \frac{1}{r}p^2(t) - 1 \qquad p(T) = 0$$

$$u(t) = -K(t)x(t) \qquad K(t) = \frac{1}{r}p(t)$$



u(t)
r=1
r=0.1
x(t)
r=1
r=0.1
K(t)
r=0.1
r=1
P(t) para vários T

When the weight in the control action, r, decreases:

- The closed-loop becomes faster
- The controller gain increases

Increasing the horizon, T, the solution of the Riccati equation is initially a constant and there is a transient close to the end of T.

This suggests that, when $T \to \infty$ the solution of the Riccati equation becomes connsatnt for all times and the optimal control is a constant feedback of the state.

The previous example suggests the consideration of the problem that consists in minimizing a cost over an infinite horizon

$$J_{LQ\infty} = \int_0^\infty \left[x'(t)Qx(t) + u'(t)Ru(t)\right]dt$$

The solution is given by the constant state feedback ciontrol law

$$u(t) = -Kx(t) \qquad K = R^{-1}B'P$$

where P is the solution of the **algebraic Riccati equation**, given by

$$PA + A'P - PbR^{-1}b'P + Q = 0$$

If the system

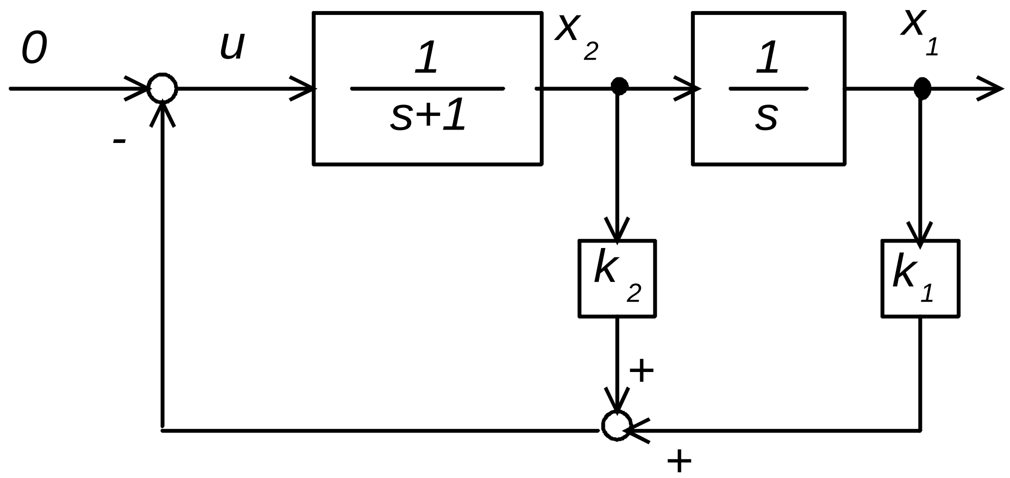
$$\dot{x}(t) = Ax(t) + bu(t)$$

Is stabilizable, *i. e.*, if there is a vector if gains F such that the closed-loop system

$$\dot{x}(t) = \left(A - bF\right)x(t)$$

Is stable, then the solution of the algebraic Riccati equation is positive semidefinite (at least) and corresponds to the limit of the solution of the Riccati differential equation when T increases.

Problem: Given the system defined by the block diagram



find the values of k_1 and k_2 that minimize

$$J = \int_0^{\infty} \left[x' Q x(t) + u' R u(t) \right] dt \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} \qquad R = 1$$

State model of the open-loop system

$$X_1(s) = \frac{1}{s} X_2(s) \quad \text{and hence} \quad \dot{x}_1(t) = x(2)$$

$$X_2(s) = \frac{1}{s+1} U(s) \quad \text{or} \quad sX_2(s) = -X_2(s) + U(s) \quad \text{and hence} \quad \dot{x}_2(t) = -x_2(t) + u(t)$$

The open-loop state model is thus

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

In this case, the algebraic Riccati equation

$$PA + A'P - PBR^{-1}C'P + Q = 0$$

becomes

$$\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} - \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{1}{1}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & p_{11} - p_{12} \\ 0 & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} - p_{12} & p_{12} - p_{22} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12}p_{22} \\ p_{12}p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & p_{11} - p_{12} \\ 0 & p_{12} - p_{22} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} - p_{12} & p_{12} - p_{22} \end{bmatrix} - \begin{bmatrix} p_{12}^2 & p_{12} p_{22} \\ p_{12} p_{22} & p_{22}^2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Equating the entries of the matrices in both members yields:

$$p_{12}^2 = 1$$

$$p_{11} - p_{12} - p_{12} p_{22} = 0$$

$$2\left(p_{12} - p_{22}\right) - p_{22}^2 + 0.1 = 0$$

The equation $p_{12}^2 = 1$ is verified by $p_{12} = \pm 1$. However, only the positive root leads to a positive definite matrix P. Therefore, $p_{12} = 1$.

$$p_{11} - p_{12} - p_{12}p_{22} = 0$$

$$2\left(p_{12} - p_{22}\right) - p_{22}^2 + 0.1 = 0$$

Being $p_{12} = 1$, these equations become

$$p_{11} - p_{22} = 1$$

$$p_{22}^2 + 2p_{22} - 1.9 = 0$$

The 2nd equation has roots $-1 \pm \sqrt{2.9}$. Again, only the positive root leads to a positive definite P. Thus:

$$P = \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix}$$

$$P = \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix}$$

The vector of optimal gains is given by

$$K = R^{-1} B' P$$

$$K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1.7 & 1 \\ 1 & 0.7 \end{bmatrix} = \begin{bmatrix} 1 & 0.7 \end{bmatrix}$$

The optimal LQ control law is therefore

$$u(t) = -\left(x_1 + 0.76 x_2 \right)$$

This computation may also be performed with MATLAB (Control Systems Toolbox) using the function *lqr* (continuous time) or *dlqr* (discrete time).

Output quadratic regulation with infinite horizon

Model:

$$\dot{x}(t) = Ax(t) + bu(t) \qquad\qquad y(t) = Cx(t)$$

Cost functional

$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt$$

Since

$$y^2(t) = x'(t)C'Cx(t)$$

This problem reduces to the previous one by selecting Q as

$$Q = C'C$$

The solution of the problem that consists of minimizing

$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt$$

where the system is modelled by

$$\dot{x}(t) = Ax(t) + bu(t) \qquad\qquad y(t) = Cx(t)$$

Is given by

$$u(t) = -Kx(t) \qquad K = R^{-1}B'P$$

where P is the unique positive definite solution of the algebraic Riccati equation

$$PA + A'P - \frac{1}{\rho} Pbb'P + C'C = 0$$

In relation to this controil law, we have the following theorem:

If the pair (A, B) is stabilizable, and the pair (A, C) is observable, the positive definite solution of the algebraic Rioccati equation exists and is unique, and the closed loop system is asymptotically stable.

The pair (*A*,*C*) is observable if

$$car\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \qquad n = \dim(x)$$

Definition

A matrix P is positive definite if

$$x'Px > 0 \qquad \forall x \neq 0$$

Is said to be positive semidefinite if

$$x'Px \geq 0 \qquad \forall x \neq 0$$

Problem: *What is the place of the closed-loop poles that corresponds to minimize* J_∞ *(for SISO systems)?*

Answer [Chang/Letov]: The poles of the optimal closed-loop system (with $T=\infty$) are the n stable roots of the degree $\boldsymbol{2n}$ polynomial $\Delta(s)$

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

where

$$b(s) = C\ adj(sI - A)B$$

Open-loop zeros

$$a(s) = \det(sI - A)$$

Open-loop poles

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

If $s = s_1$ is a root of $\Delta(s)$, then:

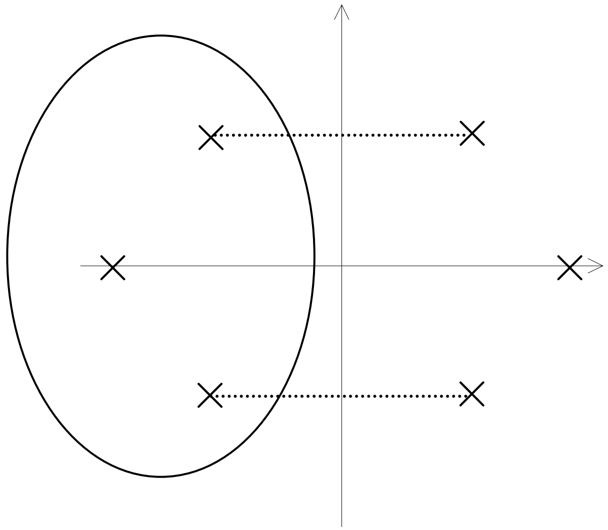
$$\Delta(s_1) = a(s_1)a(-s_1) + \frac{1}{\rho} b(s_1)b(-s_1) = 0$$

Hence, for $s = -s_1$:

$$\Delta(-s_1) = a(-s_1)a(s_1) + \frac{1}{\rho} b(-s_1)b(s_1) = 0$$

Meaning that if $s = s_1$ is a root of $\Delta(s)$, then $s = -s_1$ is also a root.

The roots of $\Delta(s)$ are symmetric with respect to the imaginary axis.



We can **always** select n stable poles

Since the poles of the controlled system are given by the roots of $\Delta(s)$ on the left-hand plane, then the system controlled with the LQ law with an infinite horizon is asymptotically stable.

Solution of the LQ ($T = \infty$) problem by pole placement

The solution of the infinite horizon LQ problem may be done as follows:

1. Compute the polynomial

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

2. Compute the $n = \partial a(s)$ roots of $\Delta(s)$ on the left half-plane.

3. Compute the vector of controller gains such that the closed loop system has the poles coincident with the roots found in step 2.

Example

Given the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

find the state feedback control law that minimizes

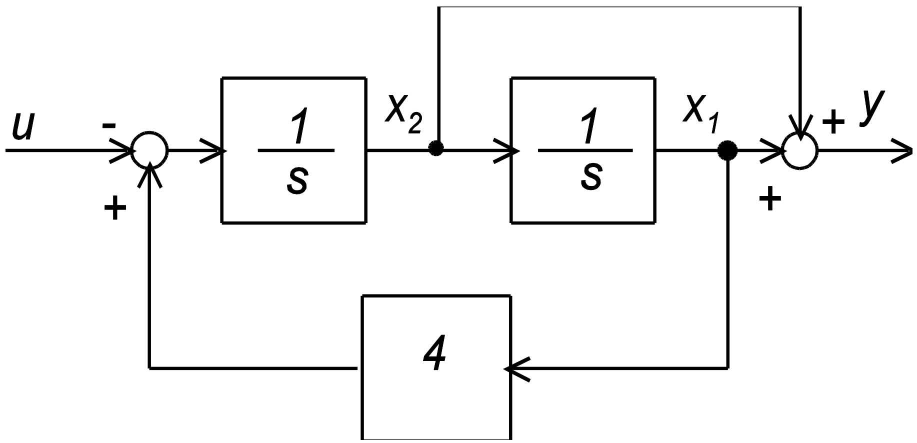
$$J_\infty = \int_0^\infty \left[y^2(t) + \rho u^2(t) \right] dt \qquad \rho = 10$$

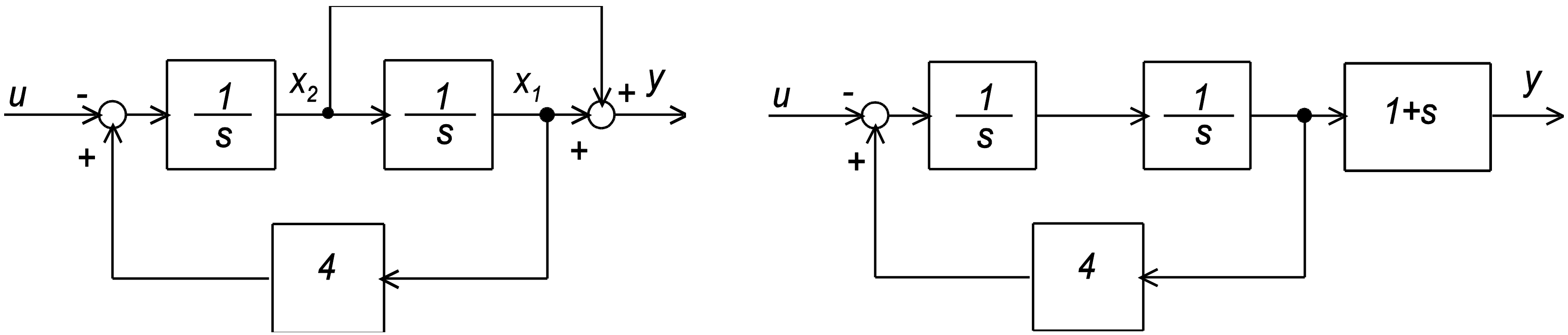
State equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = 4x_1 - u$$

Equivalent block diagram





$$Y = \frac{-\frac{1}{s^2}}{1-\frac{4}{s^2}}(1+s)U \qquad Y = -\frac{1+s}{s^2-4}U$$

$$b(s) = -(1+s)$$

$$a(s) = s^2 - 4$$

The optimal poles are the stable roots of

$$\Delta(s) = a(s)a(-s) + \frac{1}{\rho} b(s)b(-s)$$

$$a(s) = s^2 - 4 \qquad b(s) = -(1+s)$$

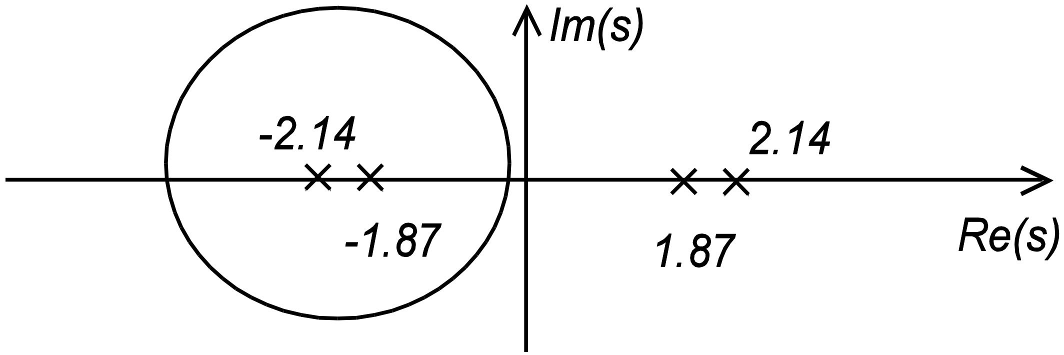
$$\Delta(s) = (s^2 - 4)^2 + \frac{1}{\rho}(1+s)(1-s) \qquad = 1 - s^2$$

Change of variable $\quad z = s^2$

$$(z-4)^2 + \frac{1}{\rho}(1-z) = 0$$

$$z^2 - 8.1z + 16.1 = 0 \qquad z_1 = 4.6 \qquad z_2 = 3.5$$

$$s_1 = 2.14 \qquad s_2 = -2.14 \qquad s_3 = 1.87 \qquad s_4 = -1.87$$



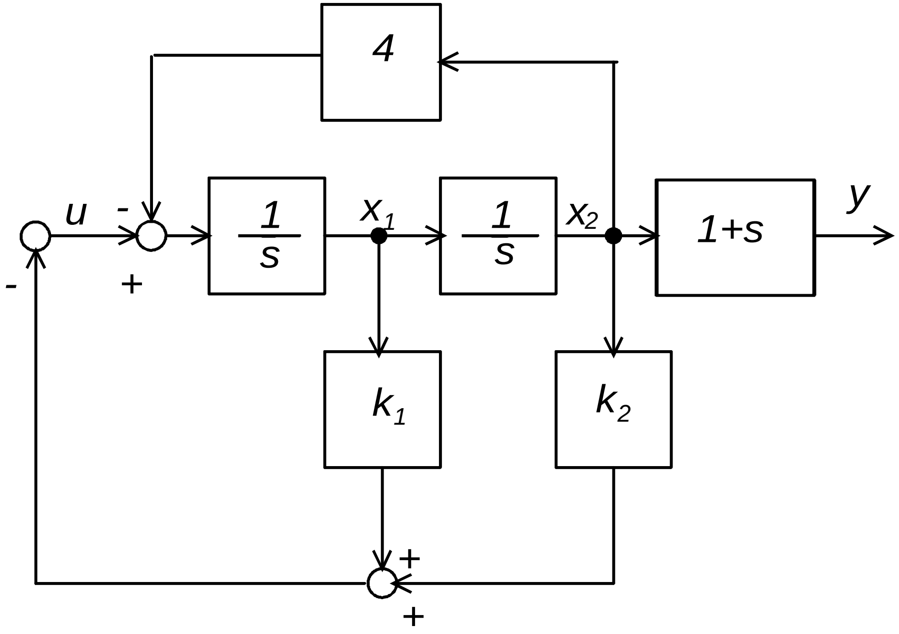
Estes são os pólos do sistema em cadeia fechada com o controlador óptimo

The optimal gain vector is computed such that the closed-loop poles are –2.14 and –1.87

The desired closed-loop polynomial is thus

$$\alpha(s) = (s + 2.14)(s + 1.87) = s^2 + 4.01s + 4$$

Block diagram of the closed-loop system with generic state feedback:



Closed-loop characteristic equation

$$1-\frac{1}{s^2}\left(4+k_1 s+k_2\right)=0$$

Closed-loop characteristic polynomial

$$\alpha_K(s)=s^2+k_1 s+k_2+4$$

Compare with the desired characteristic polynomial

$$\alpha(s)=s^2+4.01s+4$$

The optimal gains are

$$k_1^{opt}=4{,}01 \quad k_2^{opt}=0$$

Root square locus

The optimal closed-loop poles are the stable roots of

$$a(s)a(-s) + \frac{1}{\rho} b(s)b(-s) = 0$$

This equation may be written as

$$\frac{1}{\rho} \cdot \frac{b(s)b(-s)}{a(s)a(-s)} = -1$$

What happens to the roots of this equation when ρ varies?

$$a(s)a(-s)+\frac{1}{\rho}b(s)b(-s)=0$$

For ρ very big, the equation becomes approximatively

$$a(s)a(-s)=0$$

Thus, for ρ very big, the optimal poles are either the open loop poles if they are stable, or their symmetric if they are not.

$$a(s)a(-s)+\frac{1}{\rho}b(s)b(-s)=0$$

What happens for ρ very little?

Root square locus **- example**

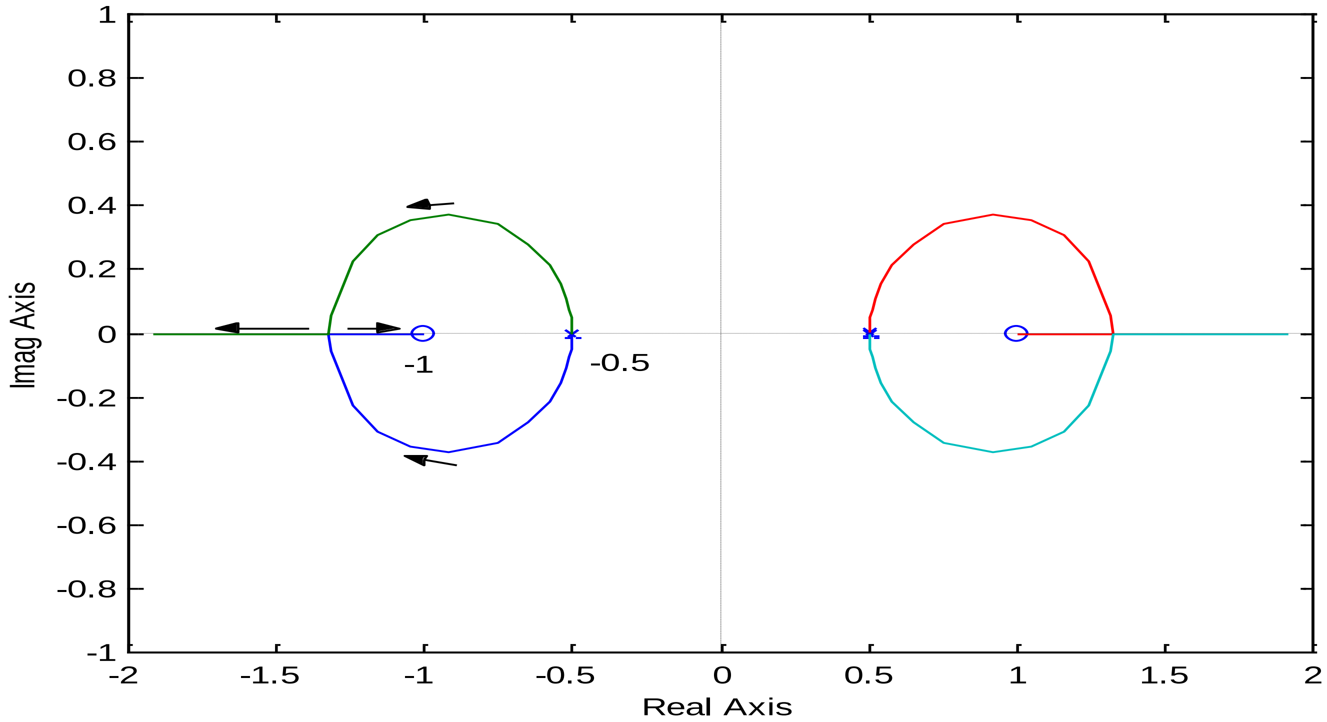
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0.25 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The corresponding transfer function is

$$\frac{b(s)}{a(s)} = \frac{s+1}{s^2 - 0.25}$$

The root square locus is



Relative stability of the LQ controller

$$\dot{x}(t) = Ax(t) + bu(t)$$

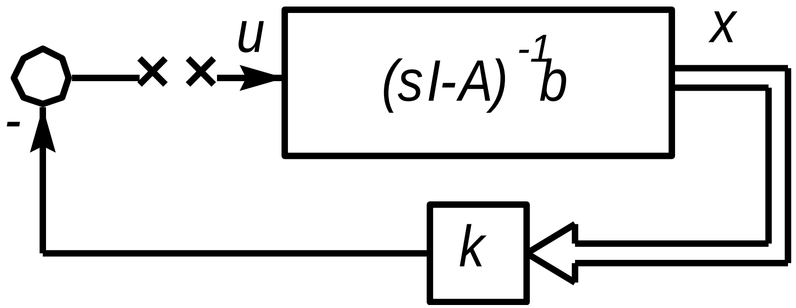
$$y(t) = Cx(t)$$

Laplace transform of the open loop state transition matrix:

$$\Phi(s) = \left(sI - A\right)^{-1}$$

Loop gain:

Intrrupt the loop and multiply all the gains.



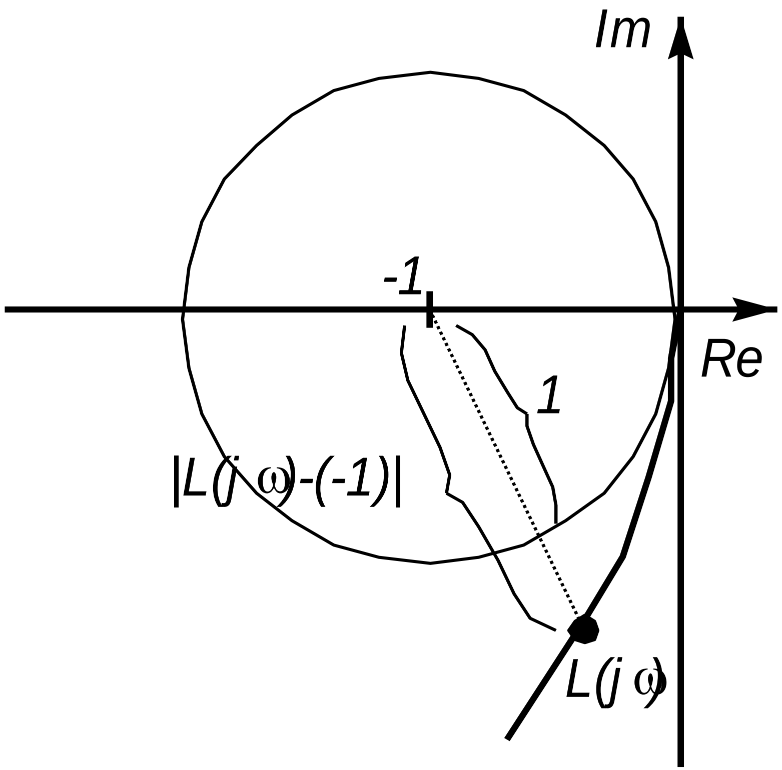
$$L(s) = k\Phi(s)b$$

Kalman inequality:

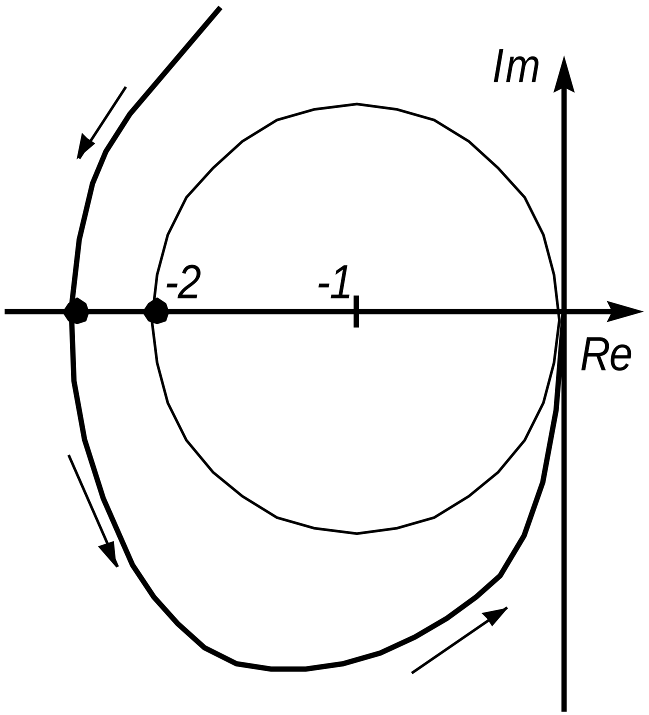
$$\left|1 + L(j\omega)\right| \geq 1$$

A consequence of the Kalman inequality:

$$\left|1+L(j\omega)\right|\geq 1 \quad \Leftrightarrow \quad \left|L(j\omega)-(-1)\right|\geq 1$$



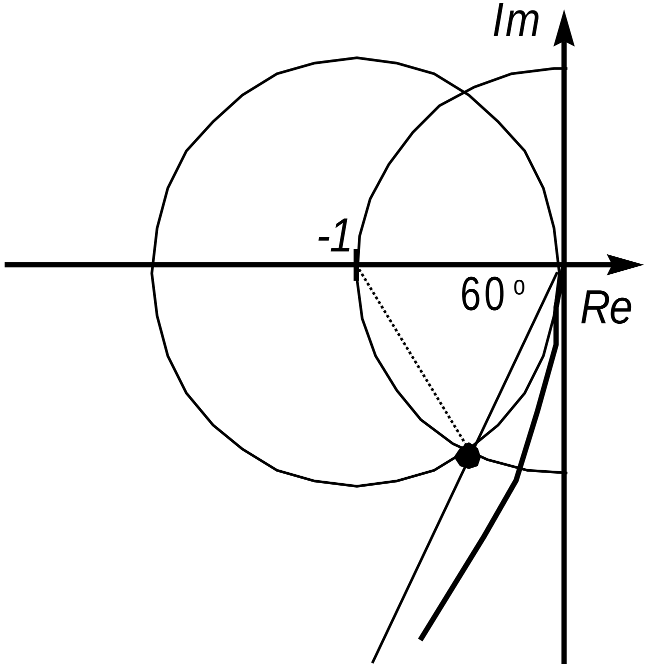
Conclusion: The Nyquist dyagram of $L(j\omega)$ never enters the circle of radius 1, with centre in –1.



In the worst case, for open-loop unstable plants, the LQ controller allows a gain reduction of ½ before the loop gain crosses –1.

The gain margin is at least 0.5.

For open-loop stable plants the gain margin is ∞.



In the worst case, there can be a reduction of 60º before –1 is reached.

The phase margin of the LQ is at least 60º.

The Kalman-Bucy filter

Objective: Optimize the observer gains.

Process model:

White gaussian noise

$$\dot{x}(t) = Ax(t) + bu(t) + w(t)$$

$$y(t) = Cx(t) + v(t)$$

v and w are Gaussian and such that

$$E\left[w(t)w^T(t+\tau)\right] = Q_o\delta(\tau) \qquad E\left[v(t)v^T(t+\tau)\right] = R_o\delta(\tau)$$

The Kalman-Bucy filter propagates in a recursive way the state estimate $\hat{x}$ that is:

Unbiased (centrada):

$$E\left[x(t)-\hat{x}(t)\right]=0$$

Minimizes:

$$\int_0^{\infty}\left\|x(t)-\hat{x}(t)\right\|^2 dt$$

The estimation error has minimum energy.

Kalman-Bucy filter equations

The estimate $\hat{x}$ is obtained by solving the differential equation:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + L_o(y(t) - C\hat{x}(t))$$

Optimal gain vector (Kalman gain)

$$L_o = \Sigma C^T R_o^{-1}$$

The matrix Σ is the symmetric and positive semidefinte solution of the algebraic Riccati equation

$$A\Sigma + \Sigma A^T + Q_o - \Sigma C^T R_o^{-1} C\Sigma = 0$$

Linear Quadratic Gaussian Regulator (LQG)

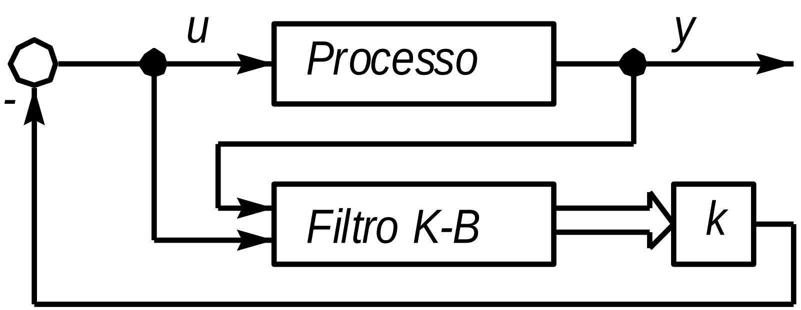
Combines:

The state estimation with a Kalman-Bucy filter

With

The feedback of the state estimate $\hat{x}$ with an optimal LQ,

Controller, designed assuming that there is acess to the state.



The Separation Theorem is valid for the LQG controller.

Rudolph Kalman nasceu em 1930, em Budapest na Hungria Emigrou para os U.S.A., onde estudou no MIT e, posteriormente, na Universidade de Colúmbia, onde fez o seu doutoramento. No início dos anos 60, o seu nome ficou ligado aos artigos que estabeleceram os fundamentos do Controlo LQ e LQG e à filtragem óptima linear com base no modelo de estado, que desenvolveu em conjunto com **Richard Bucy**.

Foi Kalman que "trouxe" para a comunidade do Controlo os métodos desenvolvidos por Lyapunov 70 anos antes e que os aplicou ao estudo da estabilidade de sistemas descritos por modelos de estado lineares.

LQG regulator equations

Estimator:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + L_o(y(t) - C\hat{x}(t))$$

$$L_o = \Sigma C^T R_o^{-1}$$

$$A\Sigma + \Sigma A^T + Q_o - \Sigma C^T R_o^{-1} C\Sigma = 0 \qquad \Sigma = \Sigma^T \geq 0$$

Controller

$$u(t) = -K\hat{x}(t) \qquad K = \frac{1}{\rho} B^T P \qquad A^T P + PA - \frac{1}{\rho} PBB^T P + Q = 0$$

Transfer function of the LQG regulator

$$G_{CLQG}(s) = K\left(sI - A + BK + L_o C\right)^{-1} L_o$$

It is like the RLVE controller transfer function.

The difference is the way in which the controller and estimation gains are computed.

Loop Transfer Recovery (LTR)

There is no warranty on the stability margins of LQG due to the inclusion of the Kalman-Bucy filter. These margins may be arbitrarily low, depending on the characteristics of the noise level.

Idea: Use the parameters that define the noise statistics, R_o and Q_o as design knobs to recover the loop-gain of the LQ.

This is LQG-LTR (LQG loop gain recovery).

Pode demonstrar-se que se:

1) $G(s)$ é de fase mínima;

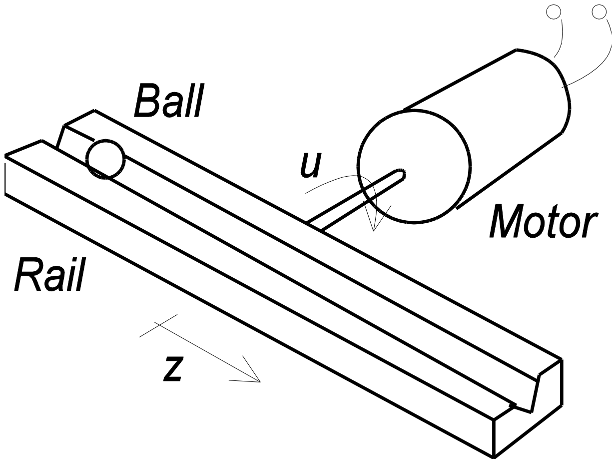
2) $R_0 = 1$ e $Q_0 = q^2 BB^T$

Então

$$\lim_{q \to \infty} L_{LQG}(s) = L_{LQ}(s)$$

Isto sugere que se projecte um filtro de Kalman-Bucy em que o parâmetro q é muito elevado.

Exemplo: Controlo de um integrador duplo



Este e outros sistemas podem ser modelados como um integrador duplo, tomando como variáveis de estado

$$x_1 = z \qquad x_2 = \dot{z}$$

Modelo do integrador duplo

Modelo de estado do integrador duplo:

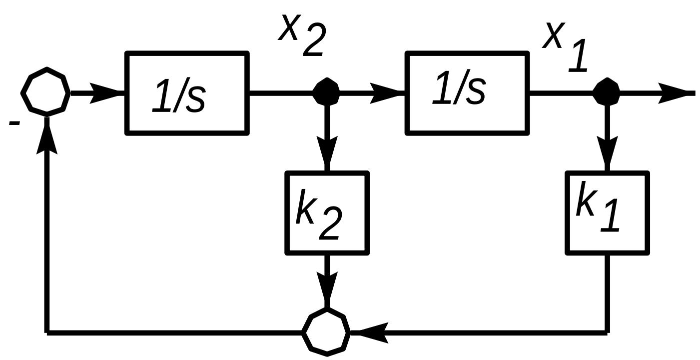
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Função de transferência do integrador duplo:

$$G(s) = \frac{1}{s^2}$$

Integrador duplo com regulador LQ



Pretende-se esciolher os ganhos k_1 e k_2 por forma a minimizar o custo quadrático de horizonte inifinito:

$$J_{LQ} = \frac{1}{2}\int_0^{\infty}\left[y^2(t) + \rho u^2(t)\right]dt$$

Assume-se que se tem acesso directo à medida de x_1 e x_2.

Equação Algébrica de Riccati (ARE):

$$A'P + PA + Q - \frac{1}{\rho} PBB'P = 0$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad Q = C'C = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

A ARE fica:

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} + \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 & p_2 \\ p_2 & p_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} p_2^2 = 1 \\ p_1 = p_2 p_3 \\ 2p_2 - p_3^2 = 0 \end{cases} \quad \rightarrow \quad P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

Ganho óptimo:

$$K_{LQ} = \frac{1}{\rho} B' P$$

Como

$$B' = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}$$

Vem

$$K_{LQ} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

Com os ganhos óptimos, a dinâmica do sistema em cadeia fechada fica:

$$A - BK_{LQ} = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}$$

Equação característica da cadeia fechada:

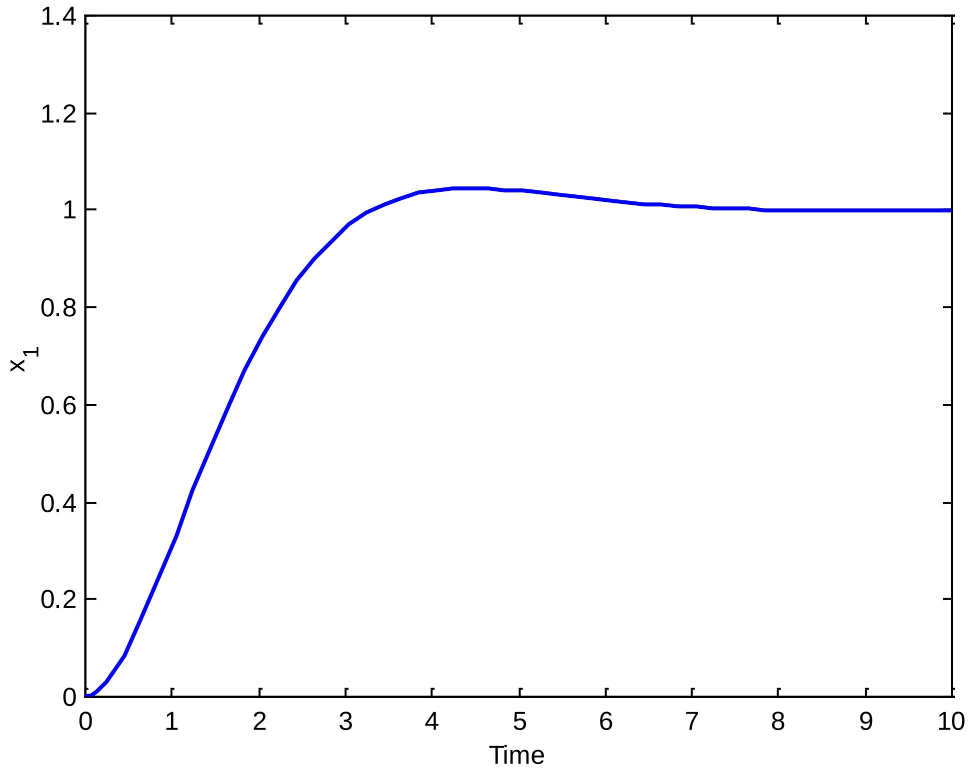
$$\det\left(sI - A + BK_{LQ}\right) = s^2 + \sqrt{2}s + 1 = 0$$

Pólos da cadeia fechada

$$s_{1,2} = \frac{\sqrt{2}}{2}(-1 \pm j)$$

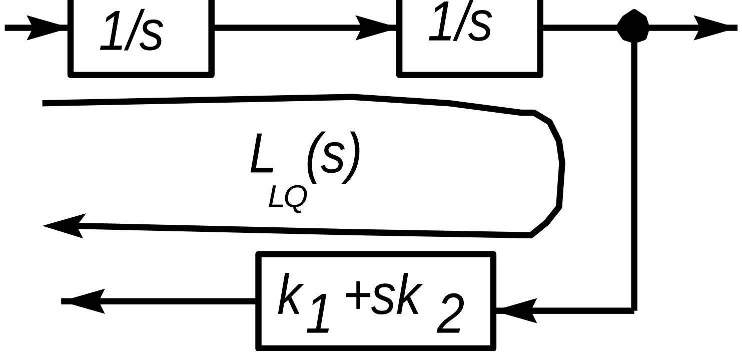
O sistema em cadeia fechada fica estável e com um coeficiente de amortecimento $\varsigma = 0.707$.

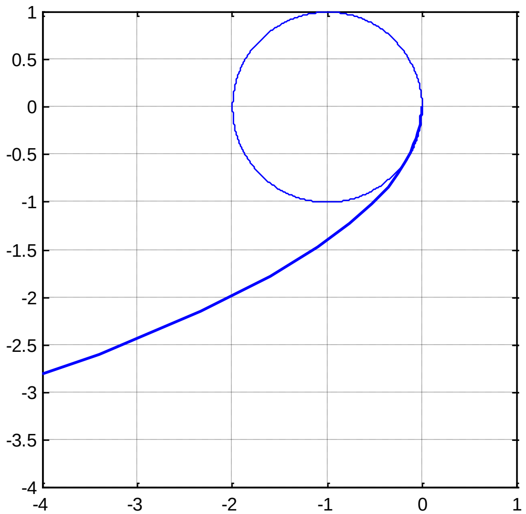
Resposta ao escalão do sistema com controlo LQ



Ganho de malha com controlo LQ:

$$L_{LQ}(s) = K\Phi(s)B = K\left(sI - A\right)^{-1}B \quad \rightarrow \quad L(s) = \frac{\sqrt{2}s + 1}{s^2}$$





Como esperado, o ganho de malha não entra no círculo de raio 1.

Modelo do integrador duplo com ruído:

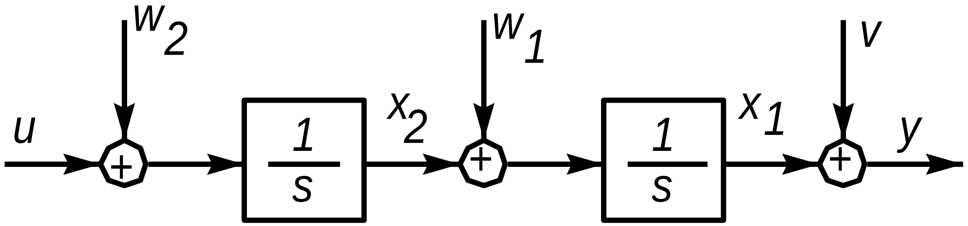
$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) + \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + v(t)$$

Os sinais v, w_1 e w_2 são sinais estocásticos mutuamente independentes, cujas características estatísticas são usadas para ajustar o ganho de malha:

$$E\left\{ \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \begin{bmatrix} w_1(t) & w_2(t) \end{bmatrix} \right\} = Q_o \qquad E\left[v^2(t)\right] = R_o$$

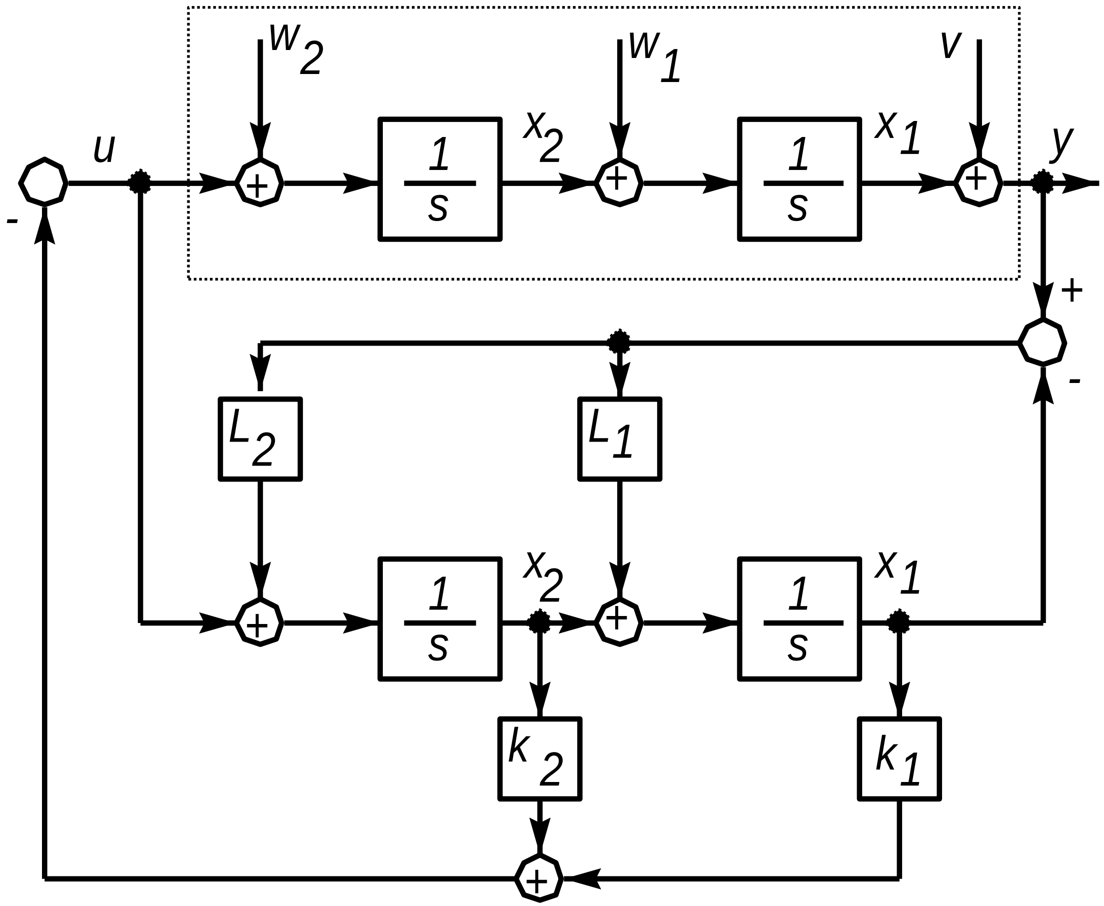
Diagrama de blocos do integrador duplo com ruído:



Vamos assumir

$$Q_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_o = 1$$

Integrador duplo com controlador LQG



k_1 , k_2 projectados tal como no regulador LQ.

L_1 , L_2 projectados de acordo com o dimensionamento do filtro de Kalman-Bucy

Cálculo dos ganhos do filtro de Kalman-Bucy

Equação de Riccati para o filtro:

$$A\Sigma + \Sigma A^T + Q_o - \Sigma C^T R_o^{-1} C\Sigma = 0$$

Assumindo $\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix}$ e usando o método dos coeficientes indeterminados obtém-se a solução definida positiva:

$$\Sigma = \begin{bmatrix} \sqrt{3} & 1 \\ 1 & \sqrt{3} \end{bmatrix}$$

Ganhos óptimos do filtro:

$$L = \Sigma C^T R_o^{-1} = \begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$$

Função de transferência do compensador LQG:

$$G_{CLQG} = K_{LQ}(sI - A + BK_{LQ} + LC)^{-1} L$$

$$G_{CLQG} = \frac{3.14(s + 0.31)}{s + 1.57 \pm j1.4}$$

Pólos da cadeia fechada do integrador duplo com LQG:

$$\frac{\sqrt{2}}{2}\left(-1 \pm j\right) \qquad \frac{-\sqrt{3} \pm j}{2}$$

Pólos do sistema

controlado com LQ,

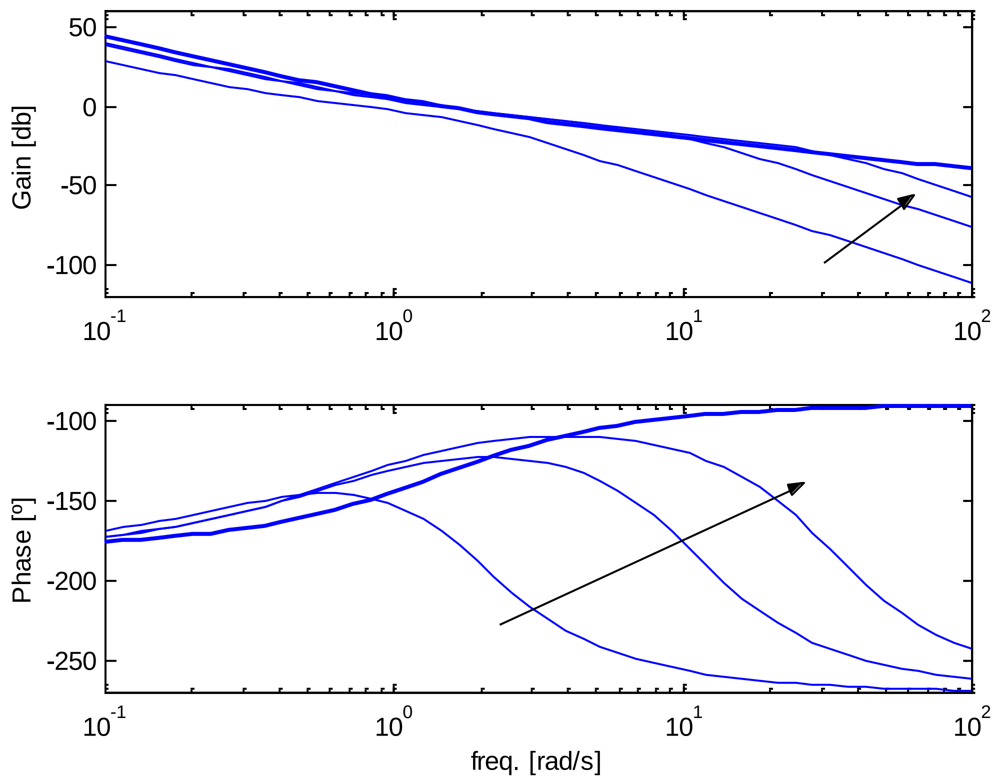
supondo acesso ao estado

Pólos do filtro

Comparação dos reguladores LQ e LQG

1. O LQ tem maiores margens de estabilidade.
2. Nas baixa frequência o ganho de malha do LQ é maior do que o do LQG. Isto implica que o LQ tem melhores propriedades de seguimento do que o LQG.
3. A frequência de corte é maior no LQ do que no LQG
   a. O LQ é mais susceptível ao ruído
   b. O LQ é mais rápido a responder
4. Na alta frequência, a inclinação da curva de ganho é –20dB/déc no LQ e –60db/déc no LQG. O LQ é mais susceptível ao ruído do que o LQG, mas tem melhor estabilidade relativa.

Recuperação do ganho de malha no integrador duplo com controlo LQG



$$q = 1, 100, 1000$$