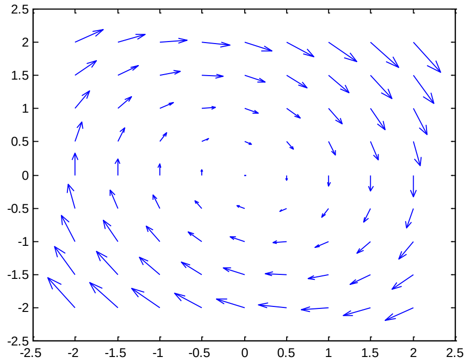
Linear state feedback



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[JML-CEE2019] Caps. 4 e 5, pp. 143 - 238

Plan

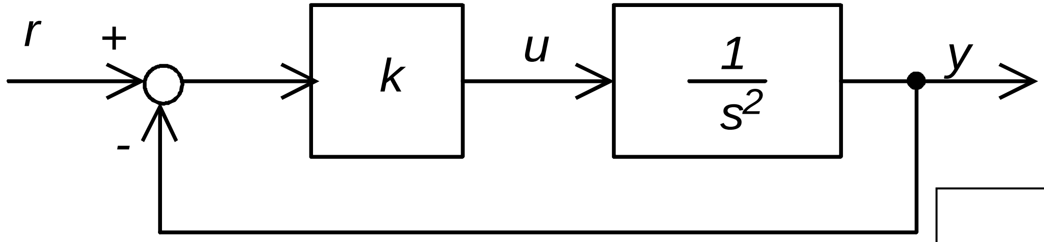
1. Motivation
2. Controlability and Observability
3. Linear state feedback (regulation)
4. Assymptotic observers
5. Separation theorem
6. Reference tracking and integral effect

1.Motivation to linear state feedback

Objective:

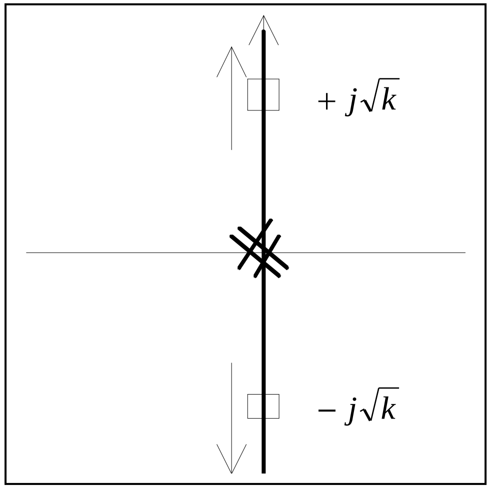
Motivate controller design based on linear state feedback. Present structural issues related to the concepts of controllability and observability.

Example: Control of the magnetic suspension



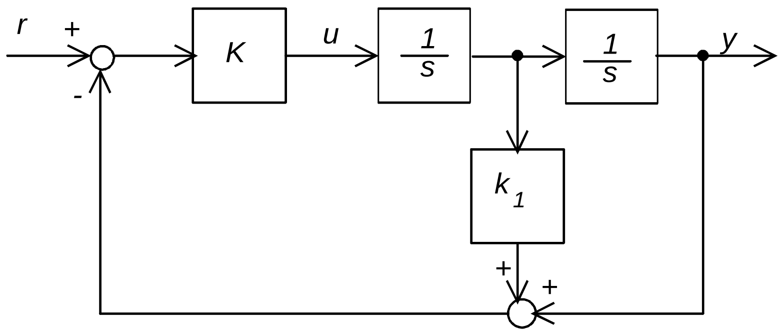
In closed loop:

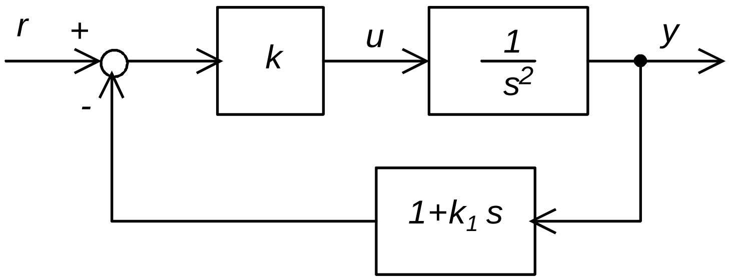
$$Y(s) = \frac{k}{s^2 + k}$$



With proportional control the closed-loop system is always marginally stable.

Velocity feedback





$$Y(s) = \frac{K}{s^2 + K(1 + k_1 s)} R(s)$$

Closed-loop characteristic equation:

$$s^2 + Kk_1 s + K = 0$$

By adjusting the coefficients we can place the closed-loop poles arbitrarily.

For instance, if we want to place the poles at $-1 \pm j$, the characteristic polynomial is

$$(s+1)^2 + 1 = s^2 + 2s + 2$$

Compare with the polynomial obtained by velocity feedback

$$s^2 + Kk_1 s + K$$

Equating the coefficients of both polynomials we get the algebraic system of equations in the gains

$$\begin{cases} Kk_1 = 2 \\ K = 2 \end{cases}$$

Important question: *Does this system of equation always have a solution?*

Conclusion

State feedback improves design flexibility because it yields a systematic procedure to force closed-loop poles

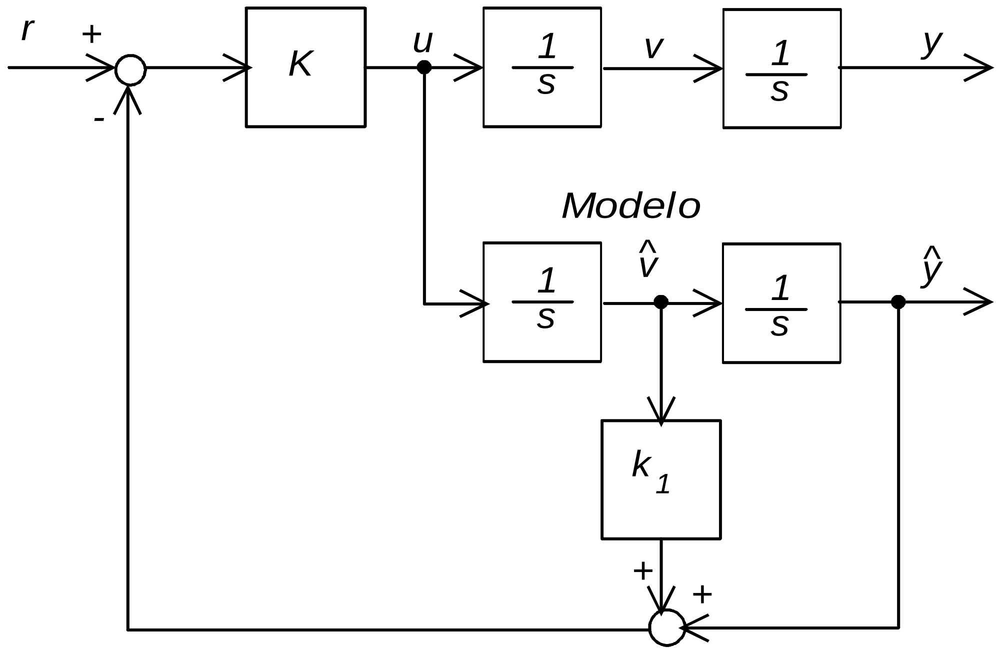
Two important questions

- Access to the state measure. The state is not always available for direct measure, for instance due to technological limiotations, or because it is not made of physical variables; We need state estimators.
- Existence of solution of the equations.

This issue will lead us to the concept of controlability

State estimation

When the state is not available, one possibility is to estimate it and feed back its estimates. Open-loop estimator:

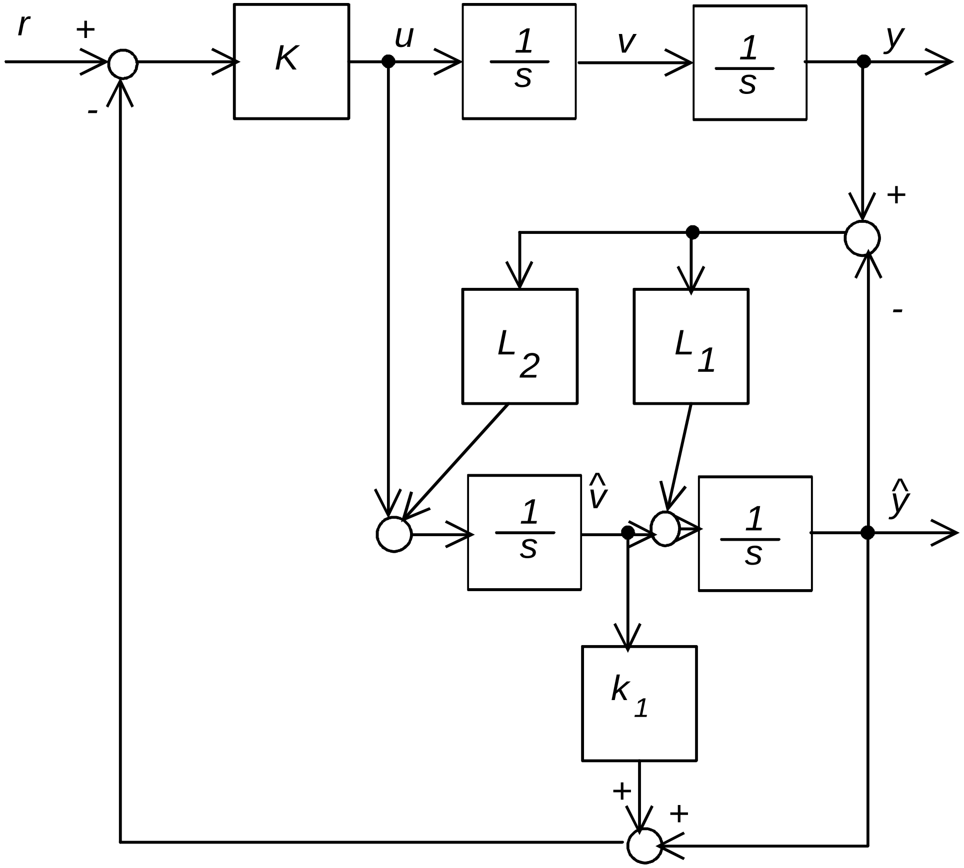


This is not a good solution.

It leads to an open-loop controller.

Disturbances and uncertainty are not attenuated.

Solution with an asymptotic observer



Relevant questions:

- Is it possible to find L_1 and L_2 such that the estimation error goes to zero?

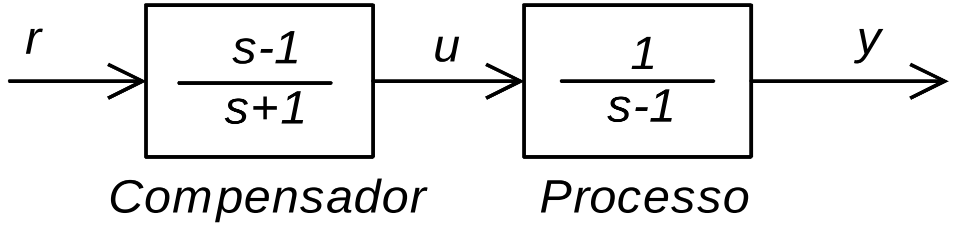
  → *OBSERVABILITY*

- What is the impact of model uncertainty?

  → *Limitations on* L_1 *and* L_2

Another example: Open-loop compensator of an unstable plant

Kailath, *Linear Systems*, (p. 31, chap. 2)

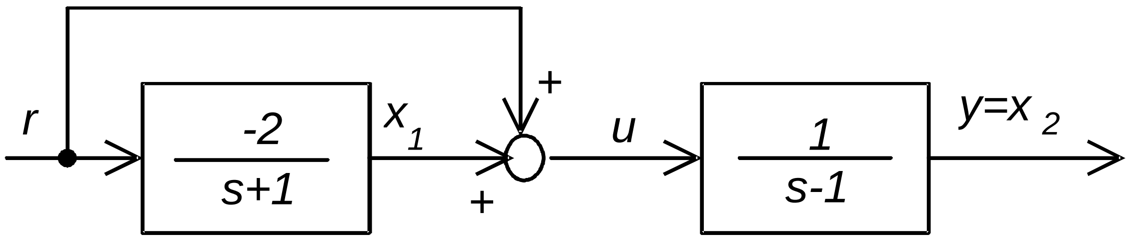


Is this system stable, even when the pole-zero cancelation is mathematically exact?

Start by building a state model. Observe that

$$\frac{s-1}{s+1} = 1 + \frac{-2}{s+1}$$

Block diagram with state variables x_1 e x_2:



State model

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 - 2r \\ \dfrac{dx_2}{dt} = x_2 + (x_1 + r) \end{cases}$$

The solution of these equations is ($*$ means "convolution"):

$$x_1(t) = e^{-t}x_{10} - 2e^{-t} * r(t)$$

$$y(t) = e^{t}x_{20} + \frac{1}{2}(e^{t} - e^{-t})x_{10} + e^{-t} * r(t)$$

Conclusion

Even when the cancelation is exact, the response does not go to infinity only if $x_{10} = x_{20} = 0$.

It is common to say that the series controller does not stabilize the system because the pole-zero cancellation is not exact. It is remarked that even in this case the controlled system is not stable.

The need for an internal description of systems

This example illustrates the importance of an internal description of the plant that clarifies the issues related to pole-zero cancellation.

In this respect, the concepts of controllability and observability are also crucial

2.Controlability and observability

Objectivo:

Introduce the concepts of observability, controllability, reconstructibility and reachability. Controllability and observability criteria. Controllability, observability and pole-zero cancellation in a transfer function.

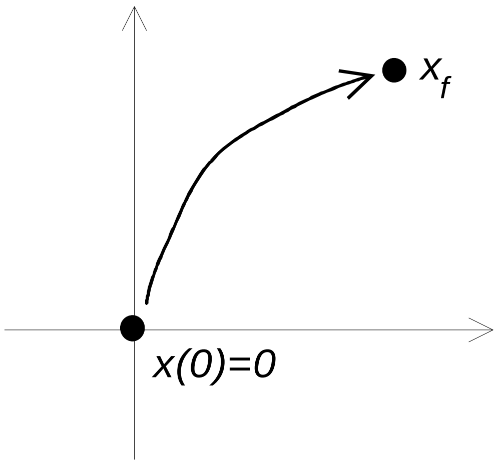
[JML-CEE2019] cap 4, excepto 159 – 162 e 167 (gramianos)

Controllability (definition – continuous systems)

The continuous state realization

$$\dot{x}(t) = Ax(t) + bu(t)$$

Is said to be *completely controllable or reachable* (atingível) if, given an initial state at the origin $x(0) = 0$, and any x_f, there exists a finite time t_f and an input function $u(t)$, $0 \le t \le t_f$ such that $x(t_f) = x_f$.



Nota sobre o conceito de controlabilidade

Para sistemas **contínuos** a definição de controlabilidade é **equivalente** a impôr que de qualquer estado se atinja a origem num intervalo de tempo finito por escolha conveniente da entrada. É esta a definição dada em [Rugh]. A definição dada no acetato anterior é normalmente referida como **atingibilidade**. Para sistemas contínuos as duas definições são equivalentes mas para sistemas discretos não.

Referências:

- Rugh (1996). *Linear System Theory.*
- Kailath (1980). *Linear Systems.*

Controllability criteria (continuous systems)

The continuous system

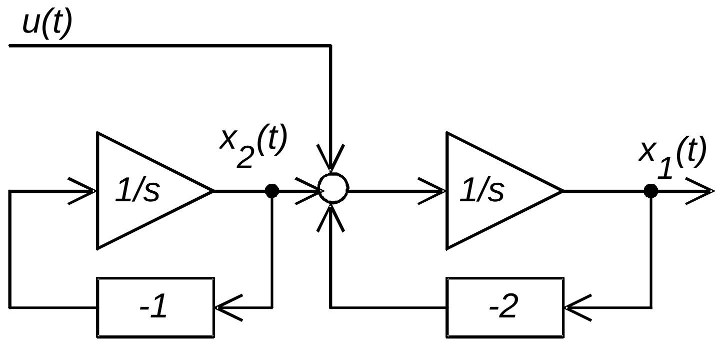
$$\dot{x}(t) = Ax(t) + bu(t)$$

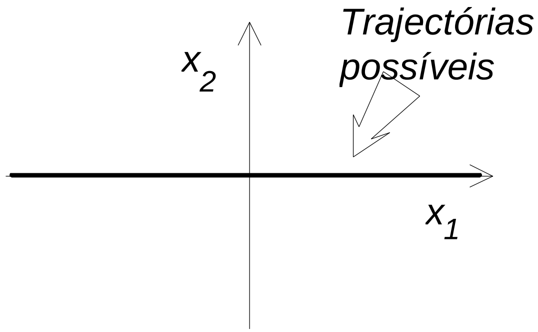
Is completely controllable iff the matrix

$$C[A, B] = [b \mid Ab \mid A^2 b \mid \cdots \mid A^{n-1} b]$$

called controllability matrix, has rank (*característica*) $n = \dim \, x$.

Example of a system not completely controllable





From the block diagram we conclude that values of x_2 different from zero cannot be zero.

$$\begin{cases} \dfrac{dx_1}{dt} = -2x_1 + x_2 + u \\ \dfrac{dx_2}{dt} = -x_2 \end{cases} \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

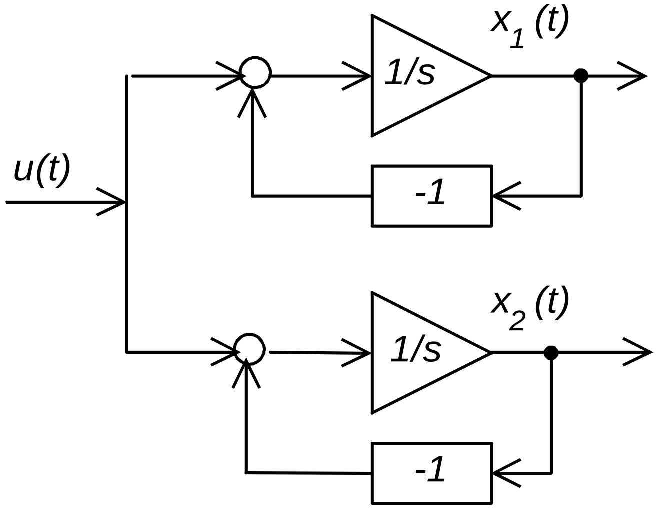
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

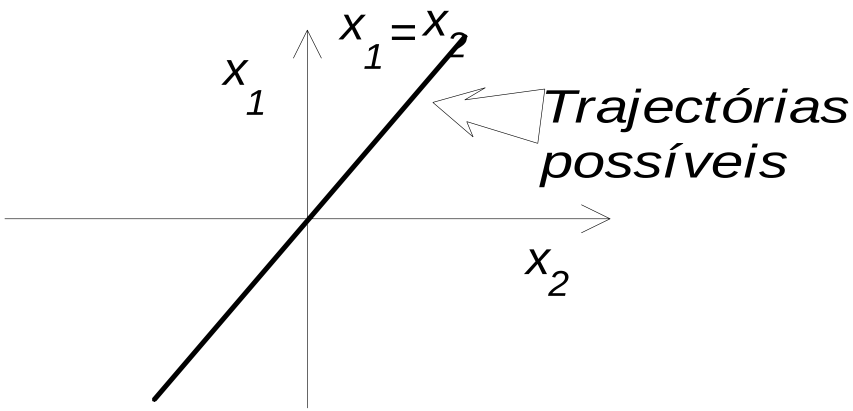
$$C(A,b) = \left[b \mid Ab \right] = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$car\ C(A,b) = 1 < n = 2$$

Hence this space realization is not controllable. We can only reach points in a subspace of dimension $car\ C(A,b) = 1$ of the tate space (that has dimension 2).

Another example of a system that is not completely controllable





Remark that the eigenvalues are equal

$$\begin{cases} \dfrac{dx_1}{dt} = -x_1 + u \\ \dfrac{dx_2}{dt} = -x_2 + u \end{cases}$$

With zero initial conditions is $x_1(t) = x_2(t) = \int_0^t e^{-(t-\tau)} u(\tau) d\tau$

We can only reach points along the straight line $x_1 = x_2$

Apply the criteria of controllebility

$$A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C(A,b) = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

Since $car\ C(A,b) = 1 < n = 2$ the realization is not controllable.

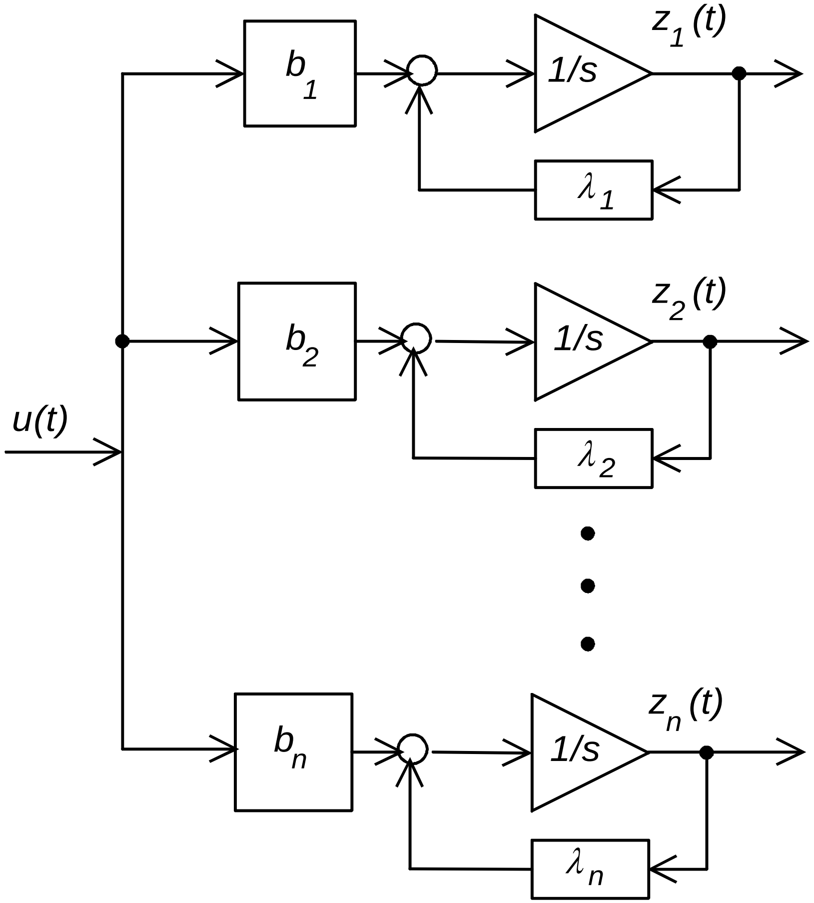
Interpretation in terms of diagonal systems (continuous time)

$$\dot{z}(t) = \Lambda z(t) + bu(t) \qquad \Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

$$C(\Lambda, b) = \begin{bmatrix} b_1 & \lambda_1 b_1 & \lambda_1^2 b_1 & \cdots & \lambda_1^{n-1} b_1 \\ b_2 & \lambda_2 b_2 & \lambda_2^2 b_2 & \cdots & \lambda_2^{n-1} b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b_n & \lambda_n b_n & \lambda_n^2 b_n & \cdots & \lambda_n^{n-1} b_n \end{bmatrix}$$

For this state realization to be controllable it must be $b_i \neq 0 \quad \forall_i$ (none of the lines vanishes) and $\lambda_i \neq \lambda_j \quad \forall_{i \neq j}$ (no proportional lines).

Interpretation of the controllability condition for diagonal systems



If any b_i is zero, the input does not affect the corresponding state that stays at the origin.

If there equal eigenvalues, the corresponding states will be proportional.

Definition of controllability (discrete systems)

The state realization of order n

$$x(k+1) = Ax(k) + bu(k)$$

Is said to be completely controllable (or reachable) iff for an initial condition $x(0) = 0$ and any x_f there exists N finite and a sequence of control inputs

$$u(0), u(1), \ldots, u(N-1)$$

such that

$$x(N) = x_f$$

Criteria of controllability (discrete systems)

The discrete system

$$x(k+1) = Ax(k) + bu(k)$$

Is completely controllable iif the matrix

$$C(A,b) = \left[b \mid Ab \mid A^2 b \mid \cdots \mid A^{n-1} b \right]$$

Called controllability matrix, has rank $n = \dim\ x$.

This result is proved in the following slides.

Cailey-Hamilton theorem

Given a square matrix A with characteristic polynomial

$$a(s) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_n = \det(sI - A)$$

The matrix verifies the equation

$$A^n + a_1 A^{n-1} + a_2 A^{n-2} + \ldots + a_n I = 0$$

With an abuse of language we say that a matrix verifies its characteristic equation.

Reference: Strang (1980). *Linear Algebra and its Applications*. Academic Press.

Lemma

$\forall_{N \geq n}$ we have

$$rank\left[b \mid Ab \mid \cdots \mid A^{n-1}b \mid \cdots \mid A^{N-1}b\right] = rank\left[b \mid Ab \mid \cdots \mid A^{n-1}b\right]$$

The proof of this lemma is a consequence of the Cailey-Hamilton theorem (try to do it before looking at the next slide).

Proof of the Lemma

Let $a(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n$ be the characteristic polynomial of A.

From Cailey-Hamilton theorem

$$A^n + a_1 A^{n-1} + \ldots + a_n I = 0$$

Multiply in the right by b

$$A^n b + a_1 A^{n-1} b + \ldots + a_n b = 0$$

This means that $A^n b$ is a linear combination of $A^{n-1}b, \ldots, b$. The proof that $A^{n+i}b \quad i \geq n$ is also a linear combination of the same vector is made by induction.

Proof of the controllability criteria (discrete time)

From the formula of variation of constants with zero initial conditions, the sate at time $N \geq n$ is given by

$$x(N) = A^{N-1}bu(0) + A^{N-2}bu(1) + \ldots + bu(N-1)$$

The points of the state space that can be reached are thus the linear combination of

$$b,\ Ab,\ \ldots,\ A^{N-2}b,\ A^{N-1}b$$

By the Lemma, the subspace generated by these vectors is the same as the subspace generated by

$$b,\ Ab,\ \ldots,\ A^{n-2}b,\ A^{n-1}b$$

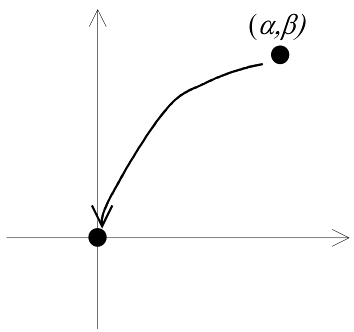
Problem

Consider the discrete system

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(k)$$

a) Show that the system is not controllable, i. e. that it is not always possible to transfer the origin to an arbitrarily specified state;

b) Show that starting from a generic state $x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$, with α, β given, there exists a control law that brings the state to the origin in 1 step, *i.e.*, exists $u(0)$ function of α, β such that $x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.



a) The state realization is not controllable. Indeed:

$$C(A,b) = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad rank\ C(A,b) = 1 < \dim x = 2$$

According to the criteria of controllability the system is not controllable and there are states that cannot be achieved staring from the origin..

b) Since

$$x(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad x(1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$u(0)$ must verify the system of equations

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0)$$

That has the solution

$$u(0) = -\alpha - \beta$$

Controllability and controllability to the origin

As the previous problem shows, in discrete systems, the concepts of controllability to the origin (being able to attain the origin from any state) and controllability (being able to attain any state from the origin, also called reachability) are not equivalent.

In continuous systems both concepts are equivalent.

The criteria refers to controllability (reachability) and not to controllability to the origin.

Stabilizablibility

A state realization

$$\dot{x} = Ax + bu$$

is stabilizable if there is a state feedback $u(t) = -Fx(t)$ such that the closed loop is asymptotically stable, which equivalent to state that there is a vector F such that all the eigenvalues of $A - bF$ have negative real part.

All the non-controllable modes of a stabilizable system are asymptotically stable.

Similar definitions hold for discrete time systems.

The effect of a linear state transformation on controllability

Given the state model

$$\dot{x} = Ax + bu$$

Consider the state model in the transformed coordinates $x = Tz$.

a) If T is invertible, show that if (A,b) is controllable, then the state realization in the new coordinates is also controllable. Write the controllability matrix in the new coordinates, C_z.

b) Relate C_z and $C(A,b)$.

c) Express T in terms of C_z and $C(A,b)$

$$x = Tz \qquad \dot{x} = T\dot{z} \qquad \dot{z} = T^{-1}(Ax + bu) = T^{-1}ATz + T^{-1}bu$$

$$\dot{z} = T^{-1}ATz + T^{-1}bu$$

$$C_z = \begin{bmatrix} T^{-1}b & (T^{-1}AT)T^{-1}b & (T^{-1}AT)(T^{-1}AT)T^{-1}b & \cdots & \end{bmatrix}$$

$$C_z = T^{-1}\begin{bmatrix} b & Ab & A^2b & \cdots & A^{n-1}b \end{bmatrix}$$

$$C_z = T^{-1}C(A,b)$$

$$\boxed{T = C(A,b)C_z^{-1}}$$

Conclusion

Given two controllable state realizations with the same dimension,

$$\dot{x} = A_x x + b_x u \quad \text{e} \quad \dot{z} = A_z z + b_z u$$

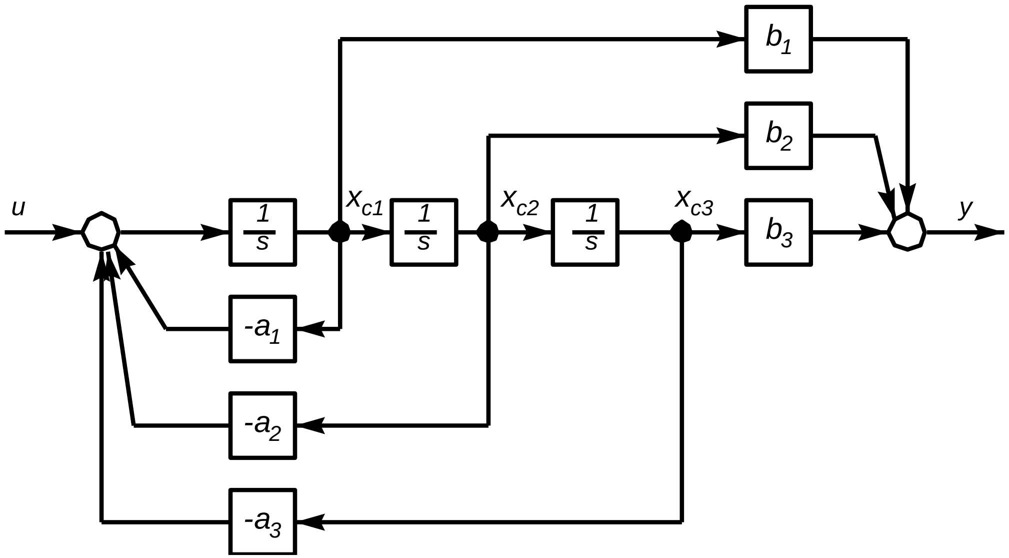
They are similar, and the similarity transformation T that maps one in the other

$$x = Tz$$

Is given by

$$\boxed{T = C(A_x, b_x) C^{-1}(A_z, b_z)}$$

Example – Controller canonical form



$$G(s)=\frac{b_1 s^2+b_2 s+b_3}{s^3+a_1 s+a_2 s+a_3}$$

$$\dot{x}_c = A_c x_c + b_c u \qquad y = c_c x_c$$

$$A_c = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad b_c = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad c_c = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$$

Controllability matrix:

$$C_c = \begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$$

Transformation that leads to the controller canonical form:

$$x = Tx_c$$

$$T = C_x \begin{bmatrix} 1 & a_1 & \cdots & a_{n-1} \\ & 1 & \ddots & \vdots \\ & \underline{0} & \ddots & a_1 \\ & & & 1 \end{bmatrix}$$

This transform will be used later in relation to controller design.

Definition of observability (continuous systems)

The continuous system

$$\dot{x}(t) = Ax(t) \qquad y(t) = Cx(t)$$

Is said to be completely observable if exists finite t_1, $0 < t_1 < \infty$ such that the knowledge of the output $y(t)$ for $0 \le t \le t_1$ is sufficient to compute the initial state $x(0)$.

Observability criterion

The state realization

$$\dot{x}(t) = Ax(t) \qquad y(t) = Cx(t)$$

Is completelly observable iff the observability matrix

$$O(A,C) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Has rank $n = \dim\ x$.

o

Interpretation in terms of diagonal systems

Given the diagonal system

$$\dot{z}(t) = \Lambda z(t) \qquad \Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$$

$$y(t) = Cz(t)$$

a) Show that it is observable iff

$$\lambda_i \neq \lambda_j \qquad \forall i \neq j \quad 1, \ldots, n$$

$$c_i \neq 0 \qquad \forall i \quad 1, \ldots, n$$

b) Give an interpretation of these conditions with a block diagram.

Joint controllability and observability

Consider the diagonal systems

$$\dot{x}(t) = \Lambda x(t) + bu(t) \qquad \Lambda = diag\{\lambda_1, \ldots, \lambda_n\}$$

$$y(t) = Cx(t)$$

Transfer function

$$H(s) = C(sI - A)^{-1}b = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} \frac{1}{s-\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{1}{s-\lambda_n} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i}$$

Transfer function

$$H(s) = \sum_{i=1}^{n} \frac{c_i b_i}{s - \lambda_i}$$

The sum has less than n terms if there is i such that

- $b_i = 0$ $\Rightarrow$ loss of controllability;
- $c_i = 0$ $\Rightarrow$ loss of observability;
- λ_i repeated $\Rightarrow$ loss of controllability and observability

Conclusion: There are pole/zero cancellations if the state realization is not controllable or not observable. This fact is general.

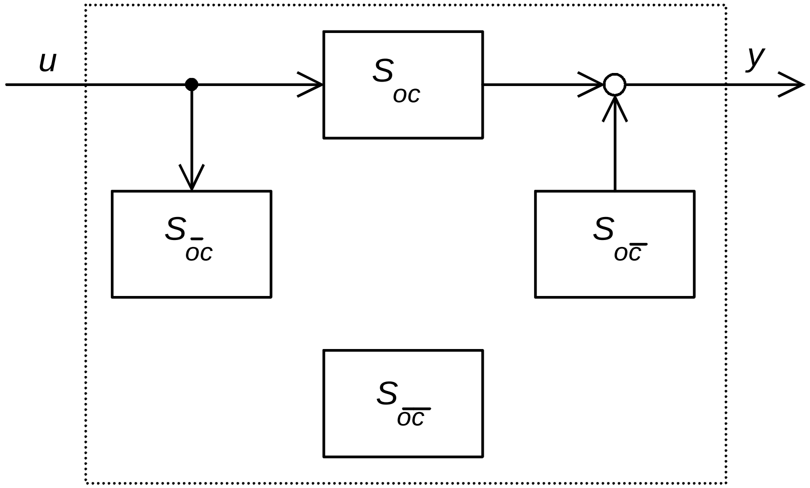
Kalman decomposition

In general, it is possible to transform the state realization to

$$\dot{x}(t) = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 \\ 0 & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & 0 & A_{44} \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ 0 \\ b_3 \\ 0 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} c_1 & c_2 & 0 & 0 \end{bmatrix} x(t)$$

$$x = \begin{bmatrix} x'_{oc} & x'_{o\bar{c}} & x'_{\bar{o}c} & x'_{\bar{o}\bar{c}} \end{bmatrix}'$$



The transfer function depends only on the observable and controllable part:

$$G(s) = c_1 (sI - A_{11})^{-1} b_1$$

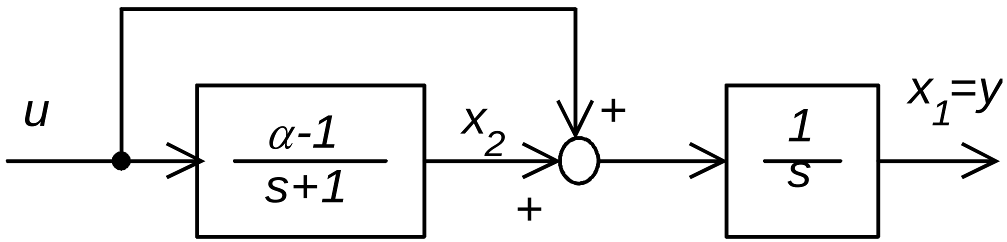
Reconstructibilty and Detectability

Two important concepts related to observability are:

- If, from output observations it is possible to reconstruct the most recent value of the state, the state realization is said to be reconstructible.
- If the part of the state that is not observable is asymptotically stable, the state realization is said to be detectable.

Problem (Interpretation of controllability and observability)

Consider the system



Build a state model using the state variables indicated.

Say for what values of α is the state realization:

a) Controlable.

b) Observable

Give an interpretation in terms of a transfer function.

Solution

$$sX_1 = X_2 + U \qquad X_2 = \frac{\alpha - 1}{s+1} U \quad \rightarrow \quad \begin{cases} \dfrac{dx_1}{dt} = x_2 + u \\ \dfrac{dx_2}{dt} = -x_2 + (\alpha - 1)u \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ \alpha - 1 \end{bmatrix} \qquad c = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$C(A,b) = \begin{bmatrix} 1 & \alpha - 1 \\ \alpha - 1 & 1 - \alpha \end{bmatrix}$$

$$\det C(A,b) = 1 - \alpha - (\alpha - 1)^2 = \alpha(1 - \alpha)$$

Observe that $$1+\frac{\alpha-1}{s+1}=\frac{s+\alpha}{s+1}$$

$$\det C(A,b)=1-\alpha-(\alpha-1)^2=\alpha(1-\alpha)$$

Hence, there is loss of controlability for:

a) $\alpha=0 \quad \rightarrow \quad x_2$ is not affected by u and there is a zero that cancels the pole at the origin;

b) $\alpha=1 \quad \rightarrow$ There is a zero at *–1* that cancels a pole.

$$O(A,c)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$ The state realization is always observable.

Problem (Interpretation of controllability and observability)

Consider

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \beta \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & \gamma \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

For what values of α, β and γ is the system:

a) Controlable?

b) Observable?

Give an interpretation in terms of the transfer function.

$C(A,b)=\begin{bmatrix}0 & \beta \\ \beta & \beta\end{bmatrix}$ is controllable whenever $\beta \neq 0$.

In terms of the transfer function:

$H(s)=C(sI-A)^{-1}b$ $\quad (sI-A)^{-1}=\begin{bmatrix}s & -1 \\ \alpha & s-1\end{bmatrix}^{-1}=\dfrac{1}{\det(A)}\begin{bmatrix}s-1 & -\alpha \\ 1 & s\end{bmatrix}^{T}=\dfrac{1}{s(s-1)+\alpha}\begin{bmatrix}s-1 & 1 \\ -\alpha & s\end{bmatrix}$

$H(s)=\beta\dfrac{\gamma s+1}{s^2-s+\alpha}$ Hence, for $\beta=0$ the gain of the TF vanishes.

$O=\begin{bmatrix}1 & \gamma \\ -\alpha\gamma & 1+\gamma\end{bmatrix}$ Since the observability matrix is square, we can look at the determinant. When $\det(O(A,c))=0$ there is loss of observability.

$\det(O(A,c)) = 1 + \gamma + \alpha\gamma^2$ There is loss of observability if $1 + \gamma + \alpha\gamma^2 = 0$ or

$\alpha = -\frac{1+\gamma}{\gamma^2}$. In this situation (loss of observability) the poles are the roots of

$s^2 - s - \frac{1+\gamma}{\gamma^2} = 0$. This equation is satisfied by $s = -\frac{1}{\gamma}$ meaning that when there is loss of observability there is a pole/zero cancelation.

3.Linear state feedback

Objective:

Design a pole-placement regulator by state feedback.

Proof of the Bass-Gura formula.

[JML-CEE2019] cap. 5, pp. 181-225

Problem (Regulator design by state feedback)

Given a controllable and observable state realization

$$\dot{x}(t) = Ax(t) + bu(t) \qquad y = Cx(t)$$

With characteristic polynomial

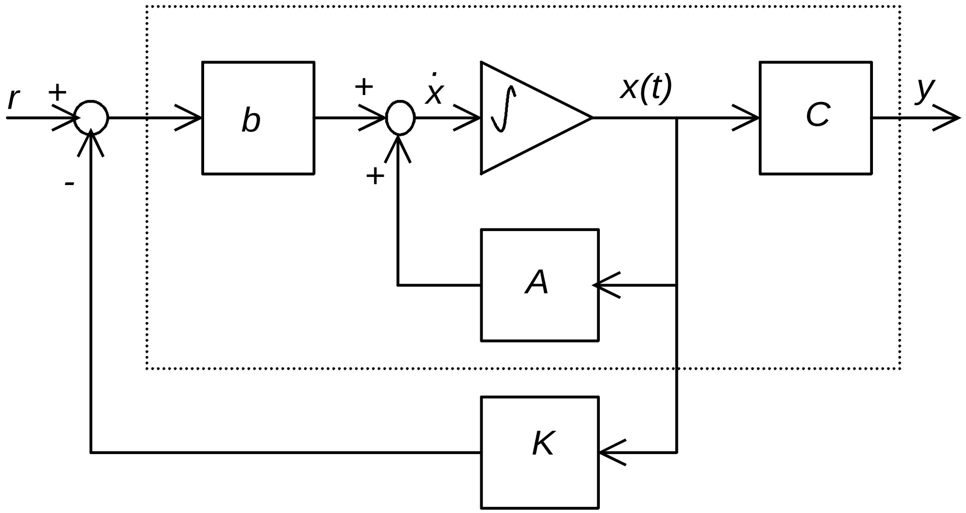
$$a(s) = \det(sI - A) = s^n + a_1 s^{n-1} + \ldots + a_n$$

Admissible control law:

$$u(t) = r - Kx(t)$$

Start by considering $r = 0$ *(regulation problem).*

Compute the vector of controller gains K such that the closed-loop characteristic polynomial is $\alpha(s) = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$



$$\dot{x}(t) = Ax(t) + bu(t) \qquad u(t) = r - Kx(t)$$

Closed-loop system:

$$\dot{x}(t) = (A - bK)x(t) + br(t)$$

Closed-loop characteristic polynomial:

$$a_k(s) = \det(sI - A + bK)$$

Can be adjusted by choice of K

Objective:

Compute K such that

$$a_K(s) = \alpha(s)$$

Characteristic polynomial of the closed-loop system. Depends on K

Specified characteristic polynomial

Method of unknown coefficients

Solve the equation with respect to K

$$\det(sI - A + bK) = \alpha(s)$$

$= a_K(s)$

Characteristic polynomial as a function of K

Specified characteristic polynomial. Depends on the specifications for the closed-loop.

$$s^n + a_K^1 s^{n-1} + \ldots + a_K^n = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

$$s^n + a_K^1 s^{n-1} + \ldots + a_K^n = s^n + \alpha_1 s^{n-1} + \ldots + \alpha_n$$

Equate the coefficients of the monomials with the same degree to obtain the system of linear algebraic equations verified by K:

$$\begin{cases} a_K^1 = \alpha_1 \\ \quad \vdots \\ a_K^n = \alpha_n \end{cases}$$

When does this system as a solution $\forall\ \alpha_1, \ldots, \alpha_n$?

There is always a solution if (A,b) is controllable. (Proved next)

Bass-Gura formula

The closed-loop poles can be arbitrarily placed iff (A,b) is controllable.

Compute the gains by

$$K = (\alpha - a) M^{-T} C^{-1}$$

where $C = C(A,b) = \left[b \mid Ab \mid A^2 b \mid \cdots \mid A^{n-1} b \right]$ is the controllability matrix of (A,b), and

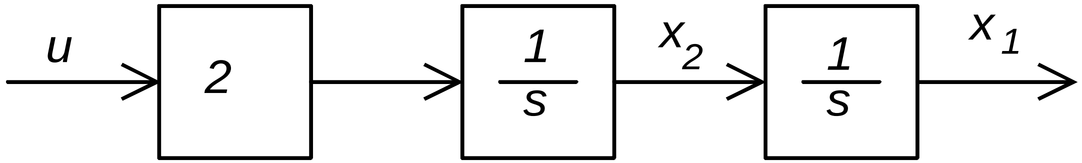
$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{bmatrix}$$

$$a = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}$$

Exercise (Bass-Gura formula)

Consider the system:



a) Obtain a state realization with state variables x_1 and x_1.

b) Using Bass-Gura formula compute the gains of the controller that place the closed-loop poles at $-1 \pm j$.

c) Solve the same problem by the method of unknown multipliers.

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = 2u \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} u$$

$$C(A,b) = \begin{bmatrix} b & Ab \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \qquad C^{-1}(A,b) = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

Characteristic polynomial of A:

$$a(s) = \det(sI - A) = \begin{vmatrix} s & -1 \\ 0 & s \end{vmatrix} = s^2$$ (as expected!)

Recall the notation:

$$a(s) = s^2 + a_1 s + a_2$$ from which $a_1 = 0 \qquad a_2 = 0 \qquad a = \begin{bmatrix} 0 & 0 \end{bmatrix}$

$$M = \begin{bmatrix} 1 & 0 \\ a_1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad M^{-T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Desired characteristic polynomial:

$$\alpha(s) = s^2 + 2s + 2 \qquad \text{(poles at } -1 \pm j\text{)}$$

Application of Bass-Gura formula

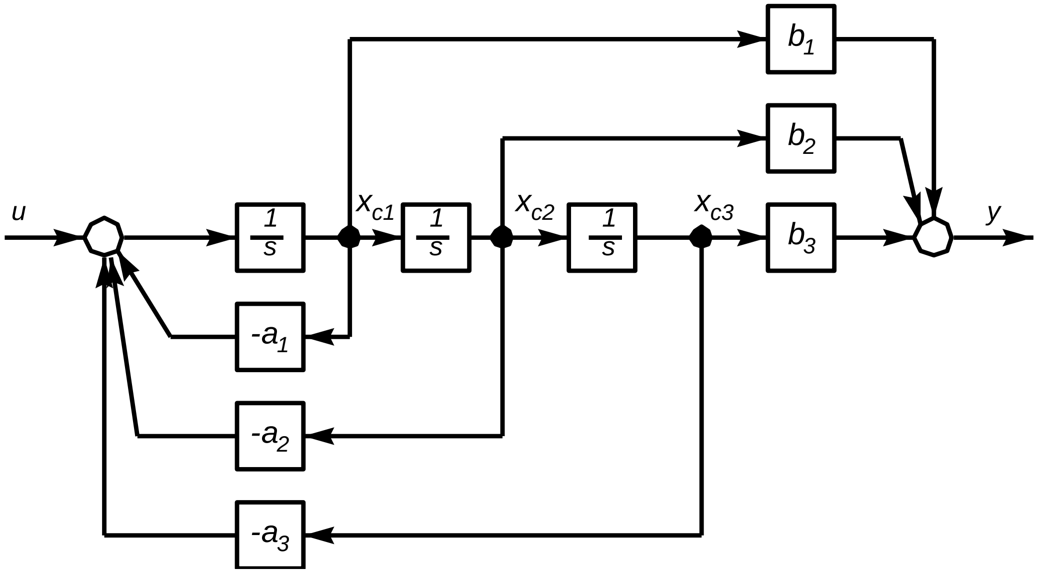
$$K = (\alpha - a)M^{-T}C^{-1}(A,b) = \left(\begin{bmatrix} 2 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Problem (that leads to the proof of Bass-Gura formula)

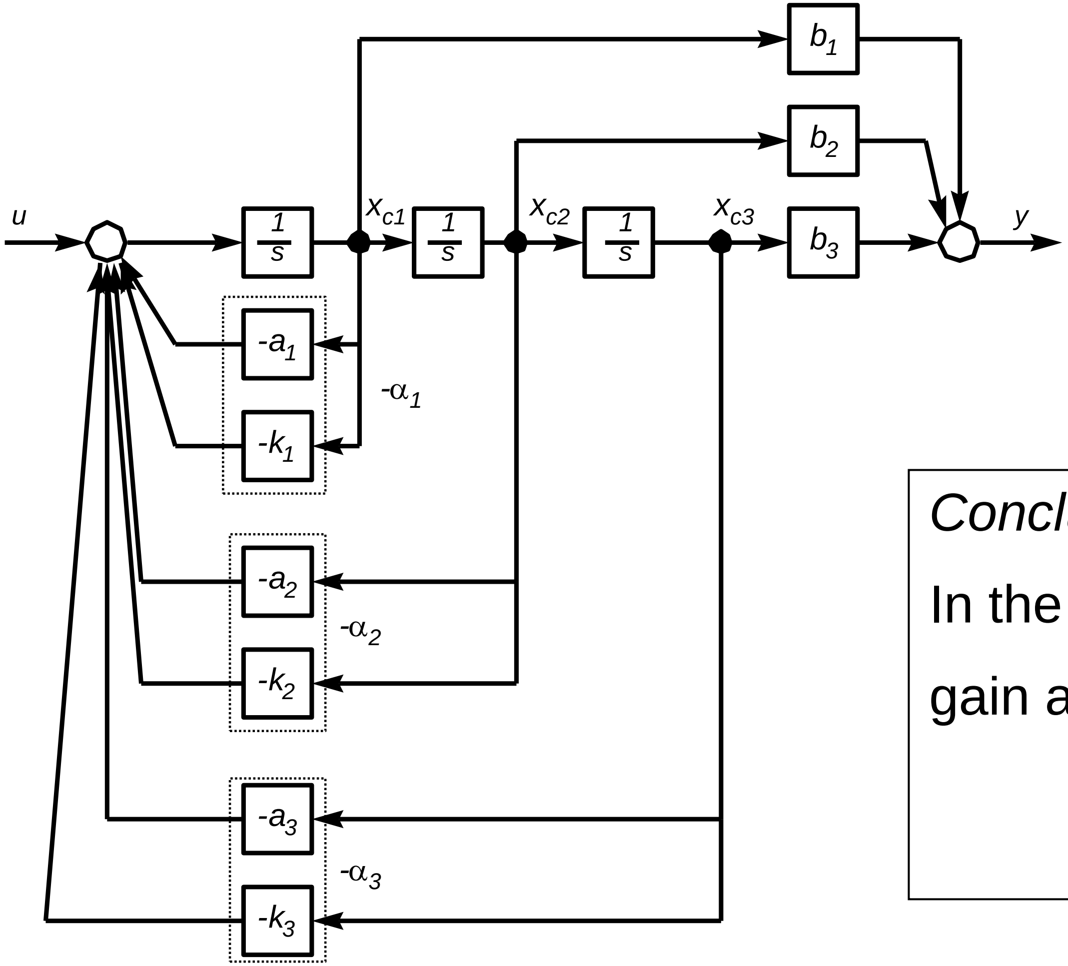
Is there any state realization such that the gain computation is trivial?

Sugestion: Consider the controller canonical form



$$G(s) = \frac{b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s + a_2 s + a_3}$$

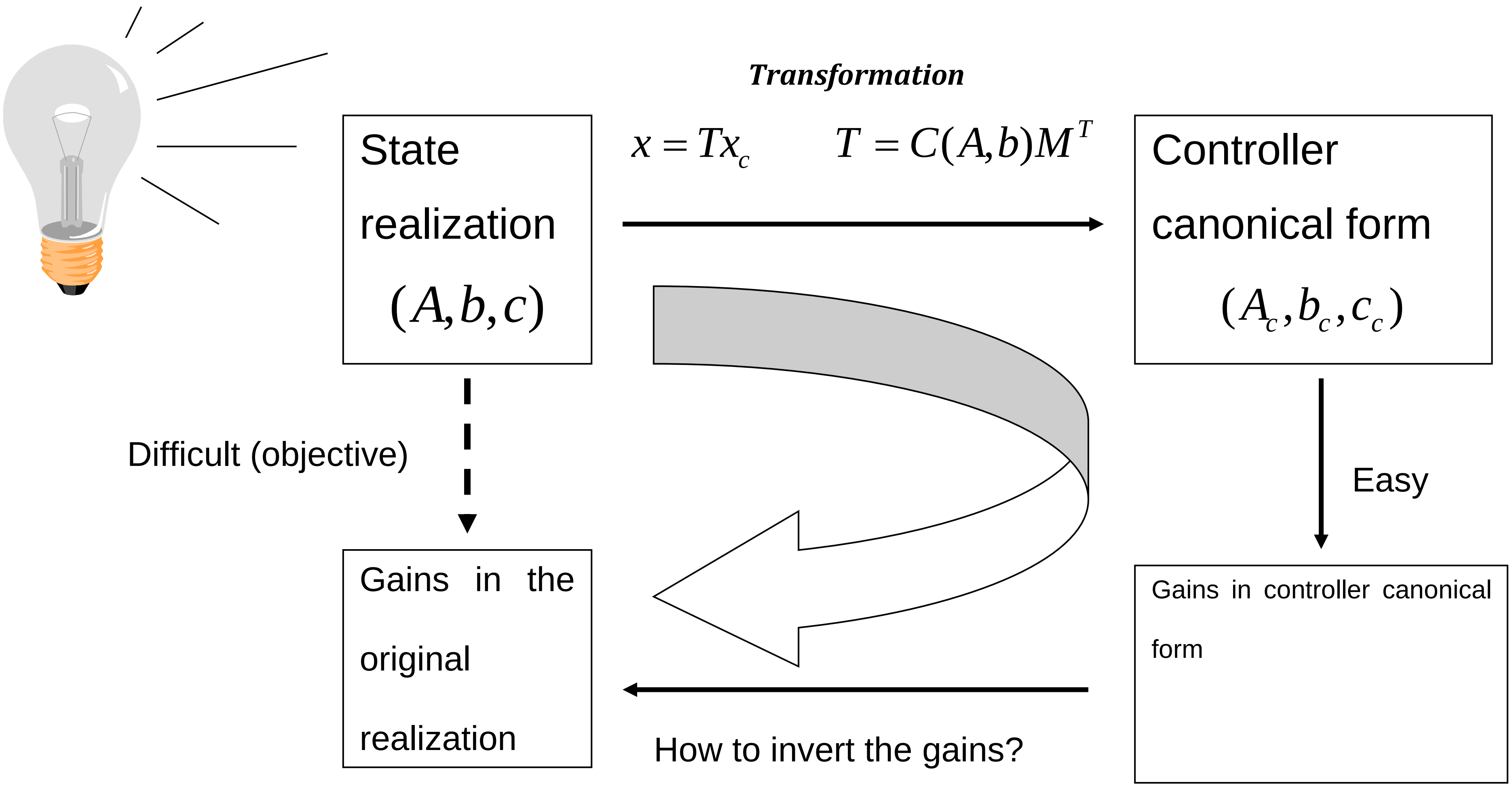
Superimposed to this block diagram draw the blocks of the state feedback.



Conclusion:

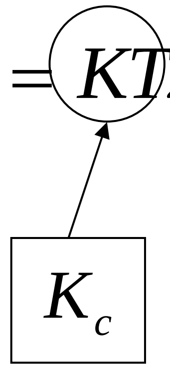
In the controller canonical form the gain are simply computed by

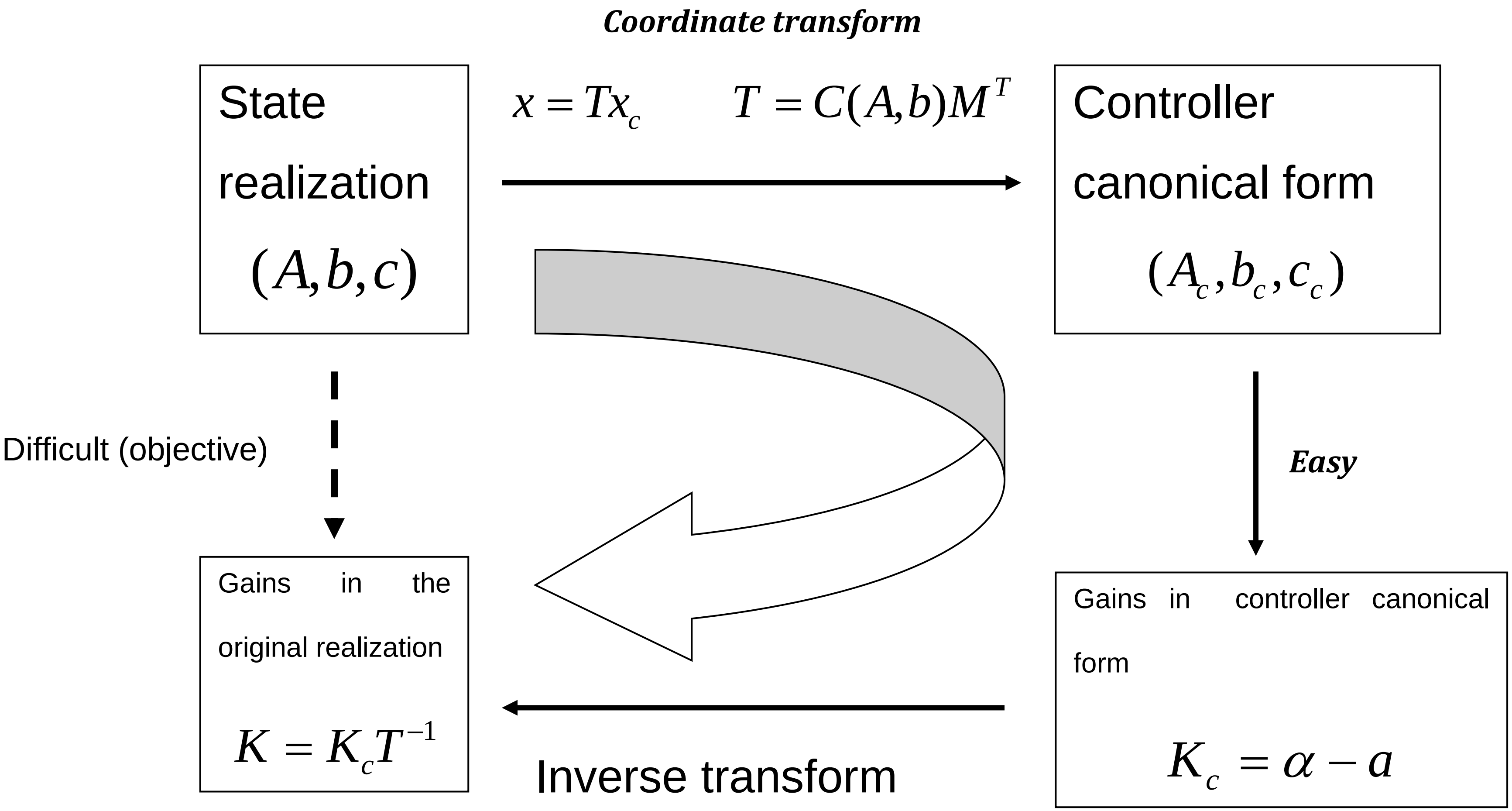
$$K_c = \alpha - a$$


Transformation
$x = Tx_c$
$T = C(A,b)M^T$
State realization
(A,b,c)
Controller canonical form
(A_c,b_c,c_c)
Difficult (objective)
Easy
Gains in the original realization
Gains in controller canonical form
How to invert the gains?

Relation of the gains in the original state x and the state in controller canonical form, x_c :

$$u = Kx = KTx_c$$




Coordinate transform
State realization
(A,b,c)
$x = Tx_c$
$T = C(A,b)M^T$
Controller canonical form
(A_c, b_c, c_c)
Difficult (objective)
Easy
Gains in the original realization
$K = K_c T^{-1}$
Inverse transform
Gains in controller canonical form
$K_c = \alpha - a$

Ackerman's formula

Does not require the explicit knowledge of the open loop characteristic polynoimial

$$K = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} C^{-1}(A,b)\alpha(A)$$

Last line of the inverse of the controllability matrix

Desired chracteristic polynomial computed for $s = A$

MATLAB functions: *acker, place*

Example (double integrator)

Consider again the problem of the double integrator (slide 62)

The last line of the controllability matrix is $\begin{bmatrix} 1/2 & 0 \end{bmatrix}$.

$$\alpha(A) = A^2 + 2A + 2I = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

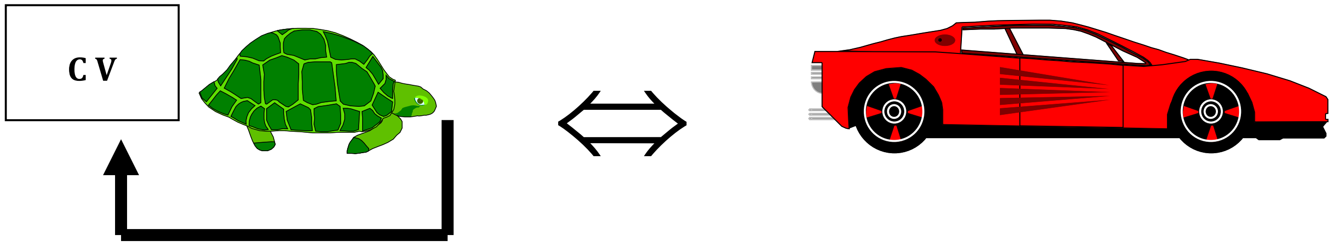
Gains computed using Ackerman's formula:

$$K = \begin{bmatrix} 1/2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The same as the ones computed with Bass-Gura formula.

Question (practical and of upmost importance!)

Can we transform a FIAT PUNTO into a Ferrari by velocity feedback?



Optimal Linear Quadratic design of the state feedback gains

Linear dynamics, quadratic cost → LQ control

$$\dot{x} = Ax + Bu$$

Select u so as to minimize (details to be studied later) the infinite horizon quadratic cost

$$J = \int_0^{\infty} (x^T Q_r x + R_r u^2) dt.$$

Consider the special case of u scalar.

Works also for multivariable plants with a minor modification.

$$\dot{x} = Ax + Bu \qquad J = \int_0^\infty (x^T Q_r x + R_r u^2)dt.$$

Solution

Find the unique positive definite matrix P that satisfies the Algebraic Riccati Equation (ARE)

$$A^T P + PA^T - PBR_r^{-1}B^T P + Q_r = 0,$$

This solution always exists if (A, B) is controllable.

Compute the state feedback gains K (a row vector) using

$$K = R_r^{-1}BP.$$

Can be computed with MATLAB function *lqr*

Apply to the plant the feedback control

$$u(t) = -Kx(t).$$

Understanding the cost functional

$$J = \int_0^\infty (x^T Q_r x + R_r u^2) dt.$$

Always yields an asymptotically stable closed loop for all $Q_r \succcurlyeq 0, R_r \succ 0$.

The cost entails a compromise between keeping the state low without much activity in control.

The diagonal entries of Q_r tell which entries of x are more important.

The higher the value of Q_{rii}, the lower the modulus of the entry x_{ii}

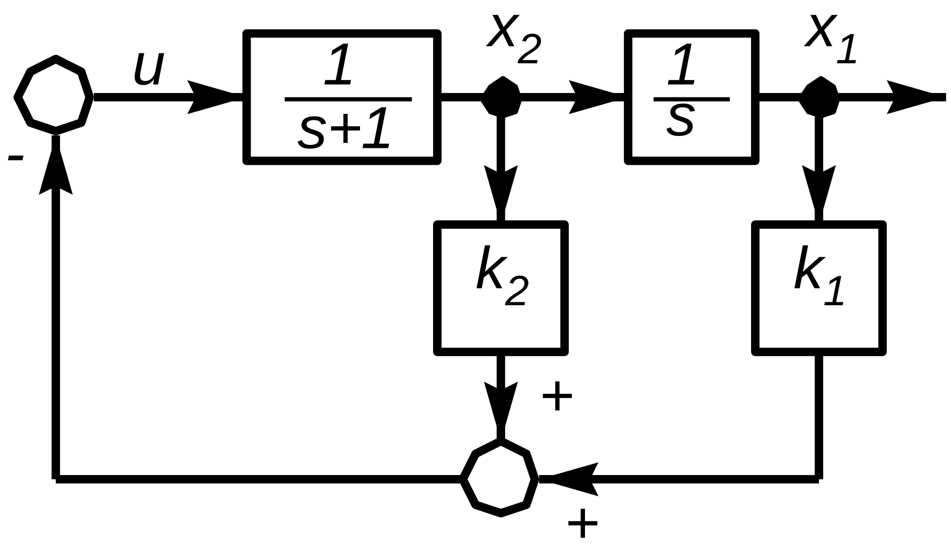
Increasing R_r decreases the modulus of u and allows x to have a higher value. The closed-loop system becomes slower (smaller bandwidth).

Example: Optimal control of a DC motor

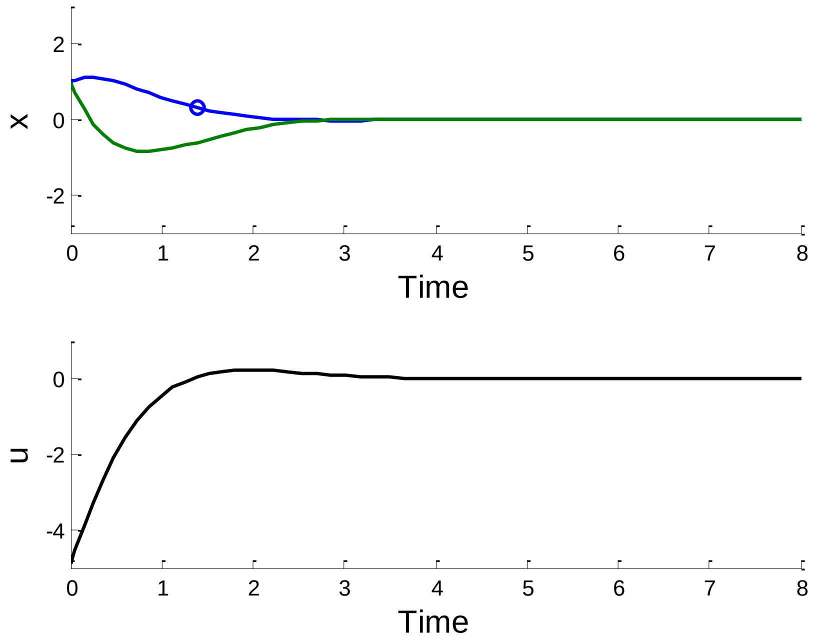
$$\frac{dx_1}{dt} = x_2$$

$$\frac{dx_2}{dt} = -x_2 + u$$

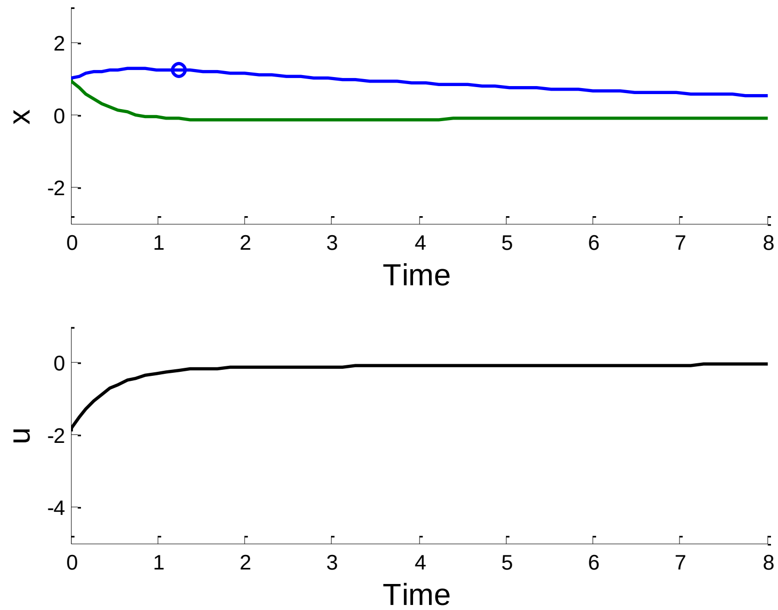
$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



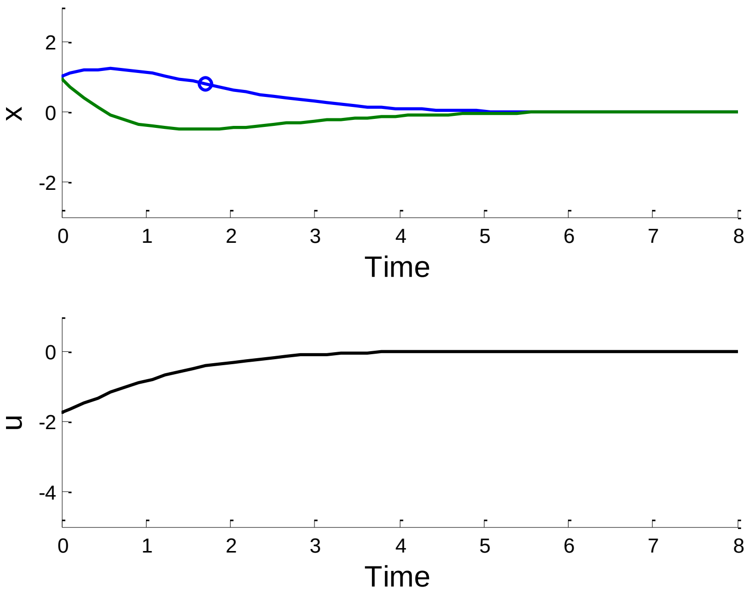
$$Q_r = \begin{bmatrix} 10 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 1$$



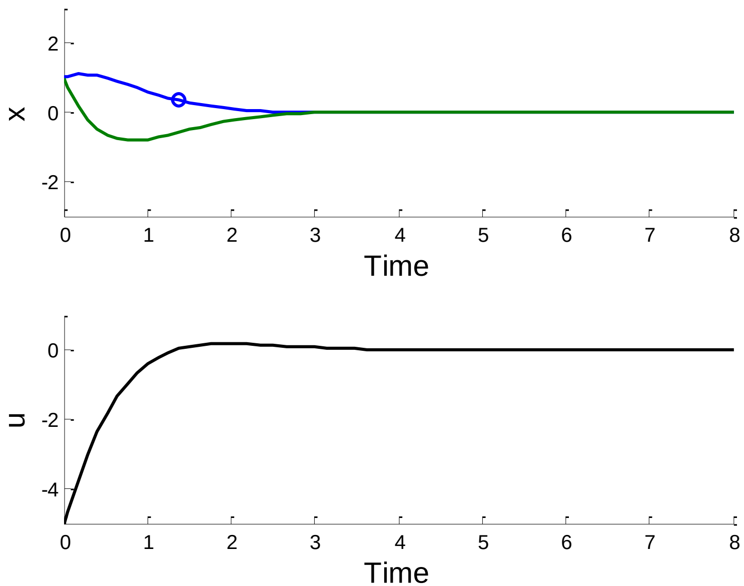
$$Q_r = \begin{bmatrix} 0{,}1 & 0 \\ 0 & 5 \end{bmatrix} \quad R_r = 1$$



$$Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 1$$



$$Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 0{,}1 \end{bmatrix} \quad R_r = 0.1$$



Decreasing R_r makes the closed-loop faster. However, this trend can be a danger if there are un-modelled high-frequency dynamics.

4.Asymptotic observers

Objective:

Design state estimators.

[JML-CEE2019] pp196 – 210

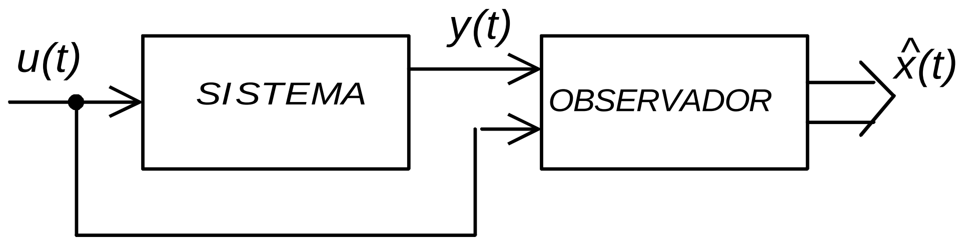
Problem: State estimation

Given a state realization $\{A, b, c\}$

$$\dot{x}(t) = Ax(t) + bu(t)$$

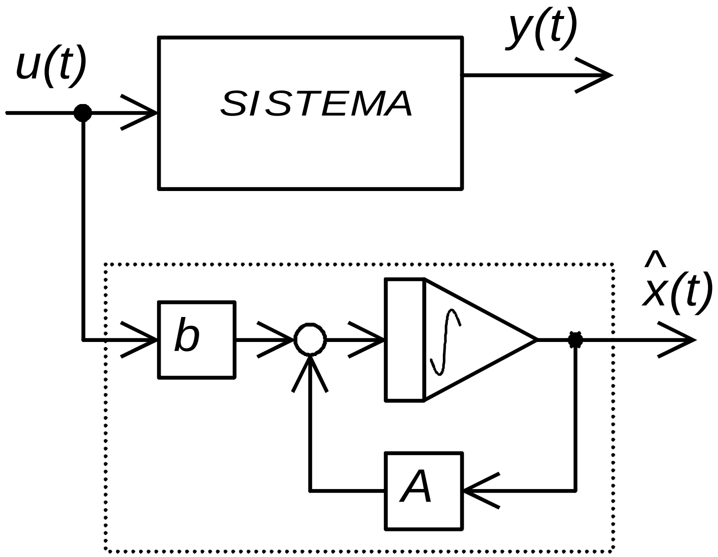
$$y(t) = cx(t)$$

Find an estimate $\hat{x}(t)$ of $x(t)$ built from observations of y and u. We want a recursive estimate, *i.e.* defined by an ODE which integration yields $\hat{x}(t)$.



1st Solution: Open-loop observer

A replica of the system, excited with the same input



Does it work?

What is the equation satisfied by the estimation error $\tilde{x} = x - \hat{x}$?

Estimation error in the open-loop observer

What is the equation satisfied by the estimation error $\tilde{x} = x - \hat{x}$?

Subtract the estimator equation from the system equation

$$\dot{x} \quad = Ax + bu$$

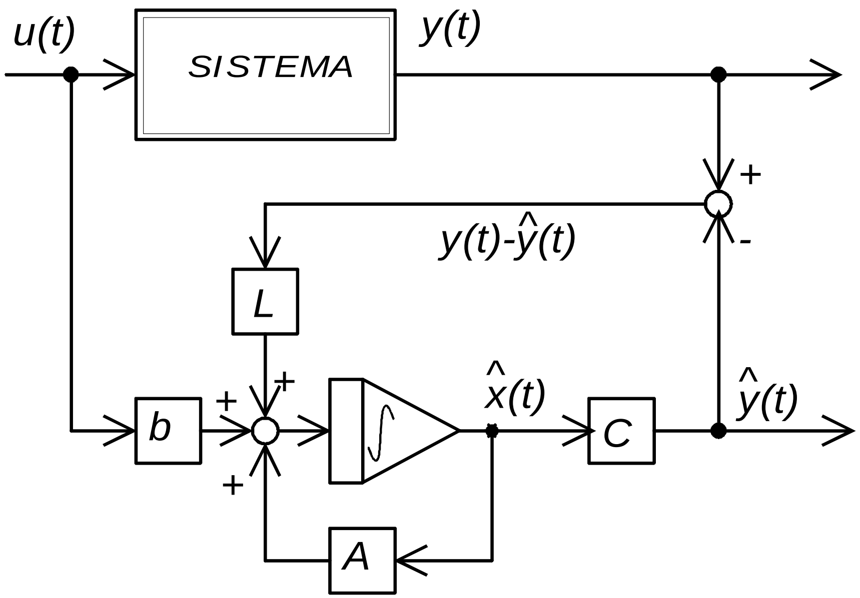
$$\dot{\hat{x}} \quad = A\hat{x} + bu$$

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + bu - bu$$

from which $\dot{\tilde{x}} = A\tilde{x}$

Conclusion: In the open-loop observer, the error goes to zero only for stable systems and with a rate equal to the eigenvalues of A.

2nd Solution: Closed-loop observer (asymptotic)



$$\dot{\hat{x}}(t) = A\hat{x}(t) + bu(t) + L\left[y(t) - C\hat{x}(t)\right]$$

Vector coluna com

dim L= dim x

When the estimate is correct, $y - c\hat{x}$ vanishes and the estimate satisfies the system dynamics.

What is now the equation satisfied by the error?

$$\begin{aligned} \dot{x} &= Ax + bu \\ \dot{\hat{x}} &= A\hat{x} + bu + L[y - c\hat{x}] \end{aligned}$$

$$\dot{x} - \dot{\hat{x}} = A(x - \hat{x}) + bu - bu - L[y - c\hat{x}]$$

↑
y=cx

Conclusion: For the closed-loop (asymptotic) observer, the estimation error satisfies

$$\dot{\tilde{x}}(t) = [A - Lc]\tilde{x}(t)$$

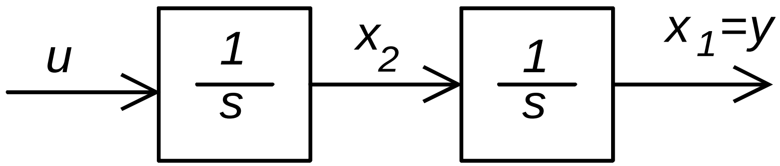
Error dynamics in the asymptotic observer

$$\dot{\tilde{x}}(t) = [A - Lc]\tilde{x}(t)$$

If (A, c) is observable, the eigenvalues of the error dynamics $A - Lc$ can be placed anywhere.

Exemplo: Observador para o integrador duplo

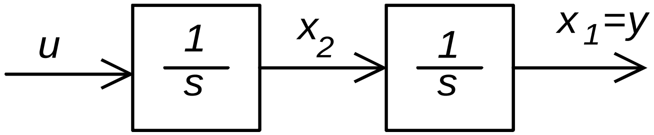
Consider the system (double integrator):



1. Draw a block diagram of an asymptotic observer

2. Compute the observer gains such that the error eigenvalues are at –1.

Sugestion: *Find A, b, c ; Write A-Lc and write its characteristic polynomial for generic L; Compute L using the method of unknown multipliers.*



$$\dot{x}_1 = x_2 \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$\dot{x}_2 = u$$

$$y = x_1 \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

$$A - LC = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} = \begin{bmatrix} -L_1 & 1 \\ -L_2 & 0 \end{bmatrix} \qquad \det(sI - A + Lc) = \begin{vmatrix} s + L_1 & -1 \\ L_2 & s \end{vmatrix} = s^2 + L_1 s + L_2$$

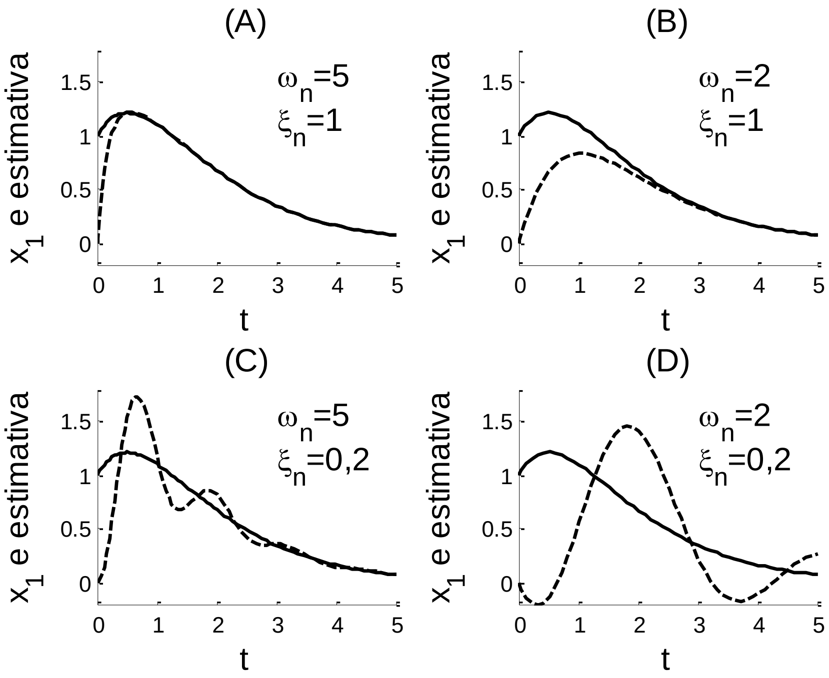
We want the eigenvalues of the error dynamics to be the roots of

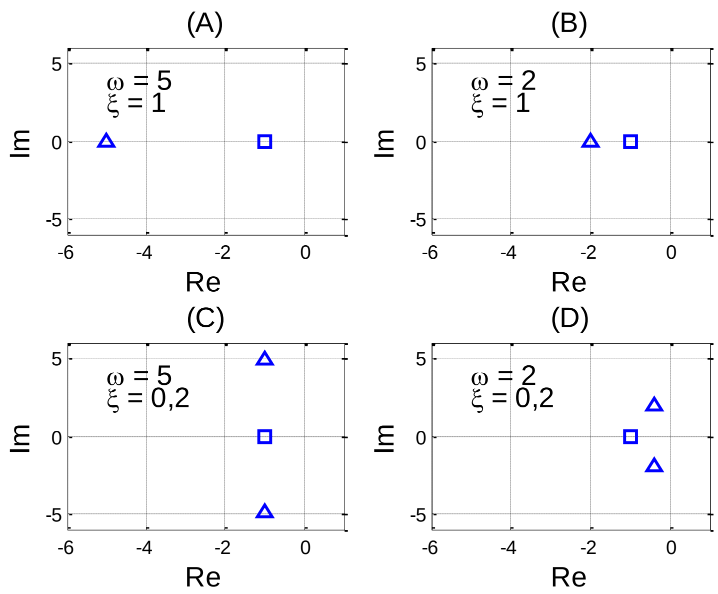
$$(s+1)^2 = s^2 + 2s + 1$$

Comparing the coefficients

$$L_1 = 2 \qquad L_2 = 1$$

Observer eigenvalues specification – Example with perfect matching





▫ Process poles (open-loop) △ observer poles

Choice of the error dynamics eigenvalues

The choice of the eigenvalues of the error dynamics $A-Lc$ results from the following trade-off:

- They cannot be too small to avoid the error tending to 0 slowly;
- They cannot be too big. Otherwise, the estimator may be “cheated” by high frequency model errors. If there is a feedback of the state estimates, the loop-gain must satisfy the robust stability condition.

Bass- Gura formula for the observer gains

Obtain a Bass-Gura type formula to compute the observer gains.

Sugestion: Write an ODE for the estimation error and do the transform that leads to the observer canonical form, where

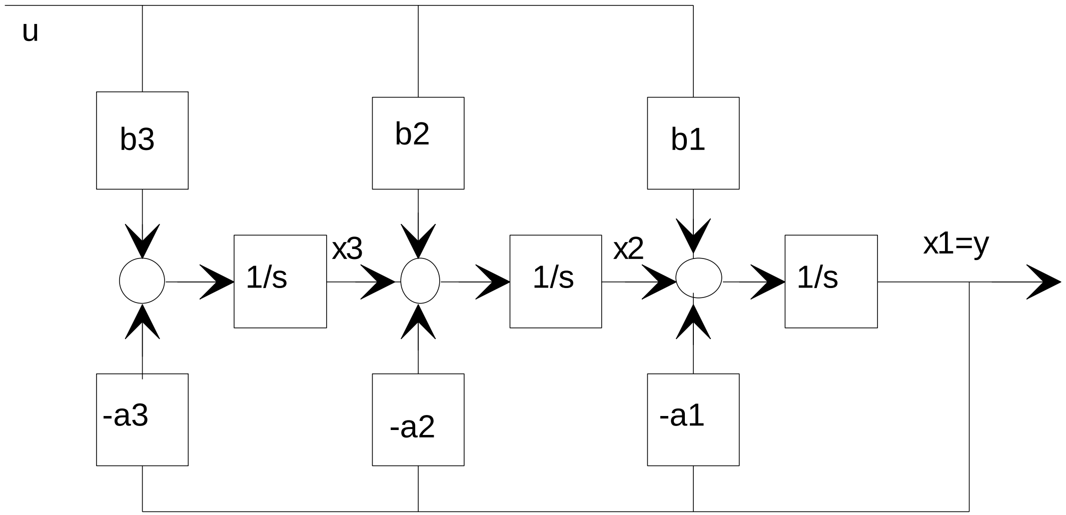
$$x_o = Tx \qquad T = MO(A,C) \qquad M = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ a_1 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{n-1} & \cdots & a_1 & 1 \end{bmatrix}$$

See the observer canonical form in the next slide.

Observer canonical form

Let the a_i be the coefficients of the characteristic polynomials in open-loop.

The observer canonical form is as shown below



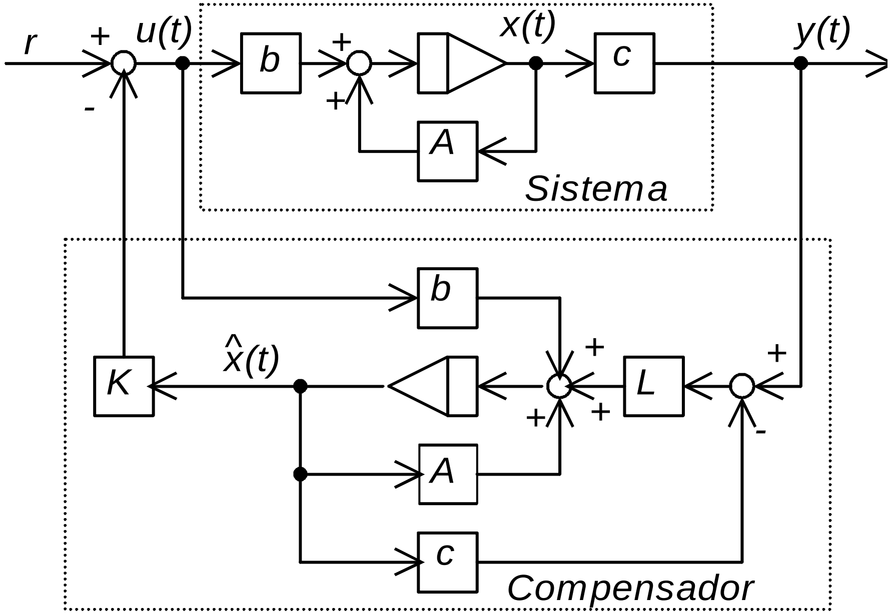
5.The separation theorem

Objective:

Show that, in a regulator obtained by feeding back the state estimate, the observer gains and the controller gains can be designed independently of each other.

[JML-CEE2019] pp. 211 – 219

Regulator with state estimation feedback



System:

$$\dot{x}(t) = Ax(t) + bu(t) \qquad y(t) = cx(t)$$

Observerr:

$$\dot{\hat{x}}(t) = (A - Lc)\hat{x}(t) + Lcy(t) + bu(t)$$

Control law:

$$u(t) = -K\hat{x}(t)$$

The closed—loop system is of order *2n*.

Where are the eigenvalues of the closed-loop system placed?

Lema

Let A, C be square matrices. Then:

$$\begin{vmatrix} A & B \\ 0 & C \end{vmatrix} = |A| \cdot |C|$$

Proof:

We have $\begin{vmatrix} I & 0 \\ 0 & C \end{vmatrix} = |C|$ and also $\begin{vmatrix} A & B \\ 0 & I \end{vmatrix} = |A|$

Since $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} A & B \\ 0 & I \end{bmatrix}$ the result is concluded because the determinant of the product is the product of the determinants of the factors.

The eigenvalues of the closed-loop system

State model of the controlled system:

$$\dot{x}(t) = Ax(t) + bu(t) \qquad y(t) = cx(t)$$

$$\dot{\hat{x}}(t) = (A - Lc)\hat{x}(t) + Lcy(t) + bu(t)$$

$$u(t) = -K\hat{x}(t)$$

Eliminating u yields the autonomous system with order $2n$

$$\dot{x}(t) = Ax(t) - bK\hat{x}(t)$$

$$\dot{\hat{x}}(t) = Lcx(t) + (A - Lc - bK)\hat{x}(t)$$

$$\dot{x}(t) = Ax(t) - bK\hat{x}(t)$$

$$\dot{\hat{x}}(t) = Lcx(t) + (A - Lc - bK)\hat{x}(t)$$

It is better to work with $\tilde{x}(t) = x(t) - \hat{x}(t)$ This corresponds to make a linear transform of the state variables. Subtracting both equations

$$\dot{\tilde{x}}(t) = (A - Lc)\tilde{x}(t)$$

The whole system is thus described in an equivalent way by

$$\dot{x}(t) = (A - bK)x(t) + bK\tilde{x}(t)$$
$$\dot{\tilde{x}}(t) = (A - Lc)\tilde{x}(t)$$

State equations of the overall (controlled) system

$$\dot{x}(t) = (A - bK)x(t) + bK\tilde{x}(t)$$

$$\dot{\tilde{x}}(t) = (A - Lc)\tilde{x}(t)$$

IN matrix form

$$\begin{bmatrix} \dot{x} \\ \dot{\tilde{x}} \end{bmatrix} = \begin{bmatrix} A - bK & bK \\ 0 & A - Lc \end{bmatrix} \begin{bmatrix} x \\ \tilde{x} \end{bmatrix}$$

By the lemma, the characteristic polynomial of the overall system is

$$\begin{vmatrix} sI - A + bK & -bK \\ 0 & sI - A + Lc \end{vmatrix} = \underbrace{\det(sI - A + bK)}_{controller} . \underbrace{\det(sI - A + Lc)}_{observer}$$

Conclusion: The poles of the overall system are grouped in two sets:

- One set depends only on the controller gain K, as if we did a feedback of the state and not of its estimate.
- Another set depends only on the observer gain L, as if we do the estimate with a controller.

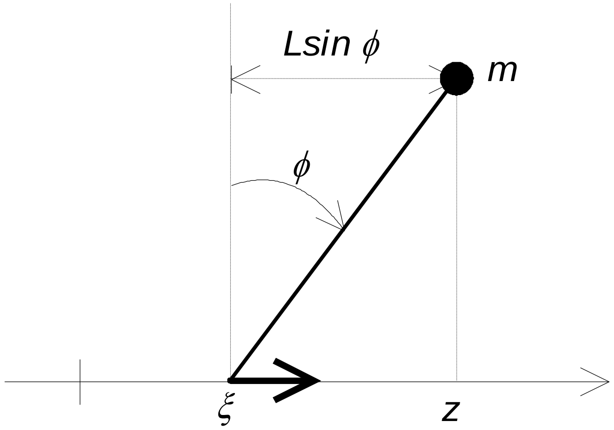
Separation theorem

The characteristic polynomial of the overall system (process, observer and estimated state feedback) is the product of the characteristic polynomials of $A-bK$ and $A-Lc$.

This theorem tells us the we can

- Design the vector of controller gains K as if we feedback the sate and not its estimate;
- Design the vector of observer gains L as if the observer acts on the system without control.

Example: Inverted pendulum



$$\begin{Bmatrix} m\ddot{z}(t) = mg\sin\phi(t) \\ z(t) = \xi(t) + L\sin\phi(t) \end{Bmatrix}$$

Linear model, valid for small angles:

$$\sin\phi \cong \phi$$

$$\begin{cases} \ddot{z}(t) = g\phi(t) \\ z(t) = \xi(t) + L\phi(t) \end{cases}$$

$$g\phi(t) = \ddot{\xi}(t) + L\ddot{\phi}(t)$$

$$g\phi(t) = \ddot{\xi}(t) + L\ddot{\phi}(t)$$

Define:

State variables: $x_1 = \phi \quad x_2 = \dot{\phi}$ Manipulated input: $u = \ddot{\xi} / L$

We obtain the state model:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ g/L & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Take $g / L = 9$.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u \qquad y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Controller design assuming access to the state:

$$A - bK = \begin{bmatrix} 0 & 1 \\ 9 & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 9 + k_1 & k_2 \end{bmatrix}$$

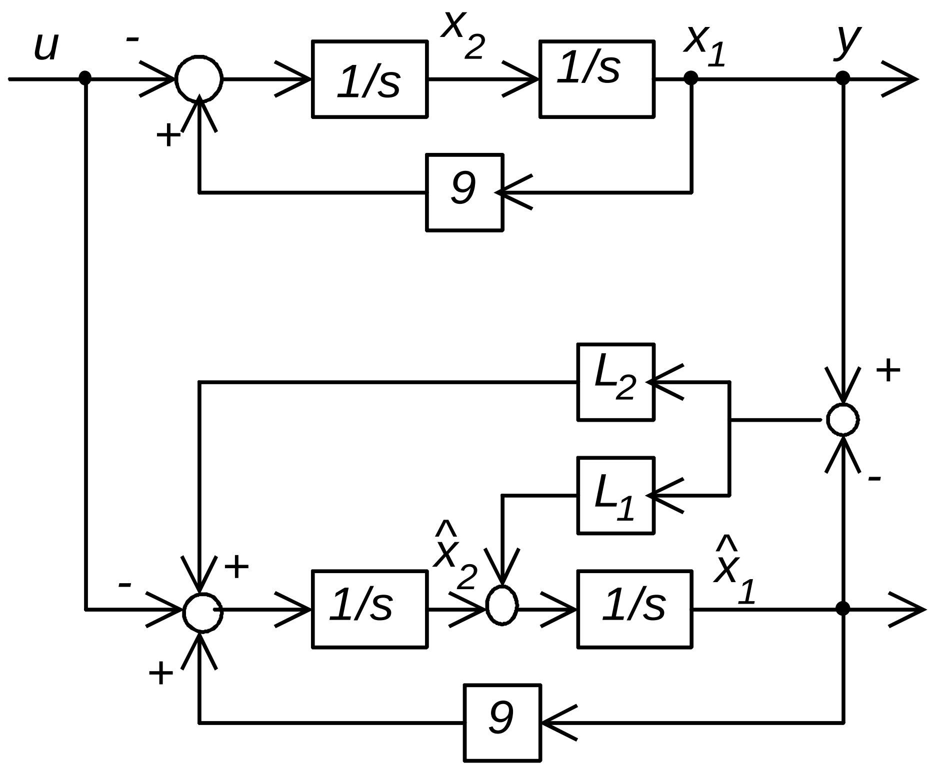
$$\det(sI - A + bK) = \begin{vmatrix} s & -1 \\ -9 - k_1 & s - k_2 \end{vmatrix} = s^2 - k_2 s - (9 + k_1)$$

Specified characteristic polynomial: $\alpha(s) = s^2 + 2s + 2$

Comparying the coefficiente sof both polynomials

$$k_1 = -11 \qquad k_2 = -2$$

The structure of the observer is a replica of the process with the state derivatives added by a term given by the output error amplified by L_1 e L_2:



To design the observer gains, impose the eigenvalues of $A-Lc$.

$$A-Lc=\begin{bmatrix}0 & 1\\ 9 & 0\end{bmatrix}-\begin{bmatrix}L_1\\ L_2\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix}=\begin{bmatrix}-L_1 & 1\\ 9-L_2 & 0\end{bmatrix}$$

$$\det(sI-A+Lc)=\begin{vmatrix}s+L_1 & -1\\ L_2-9 & s\end{vmatrix}=s^2+L_1 s+L_2-9$$

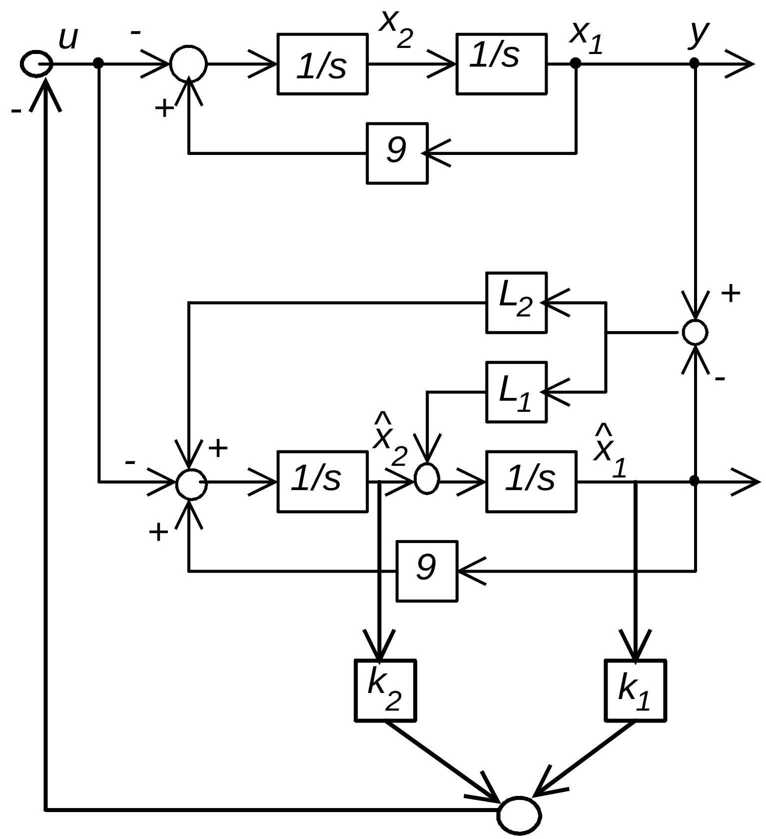
Specified error characteristic polynomial

$$\alpha_o(s)=(s+4)^2+4^2=s^2+8s+32$$

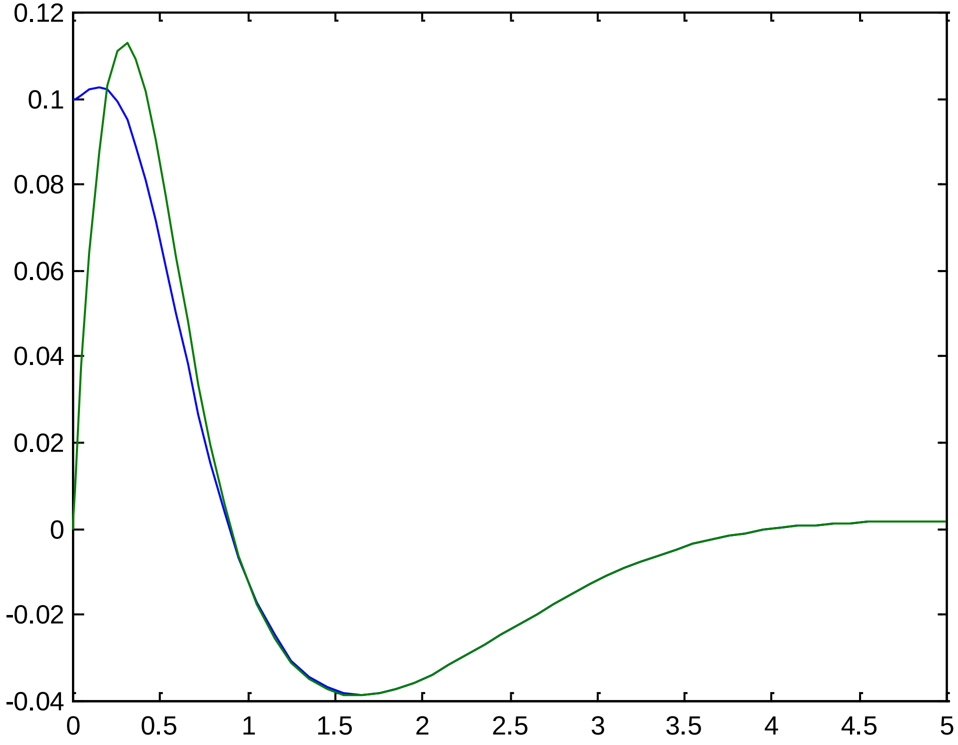
Observer gains obtained by equating the coefficients in both polynomials

$$L_1=8 \qquad L_2=41$$

The controller is build by feeding back the gains of the state estimates:



Response to the regulation of a non-zero initial condition



Compensator transfer function

Process model:

$$\dot{x} = Ax + bu$$

$$y = cx$$

$$G_p(s) = c(sI - A)^{-1}b$$

Observer/controller state model:

$$\dot{\hat{x}} = (A - Lc)\hat{x} + bu + Ly$$

$$u = -K\hat{x}$$

$$\dot{\hat{x}} = (A - Lc - bK)\hat{x} + Ly$$

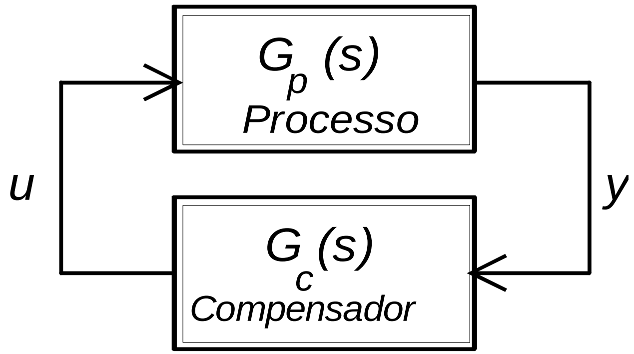
$$u = -K\hat{x}$$

$$\dot{\hat{x}} = (A - Lc - bK)\hat{x} + Ly$$

$$u = -K\hat{x}$$

The controller is described as having dynamics $A - Lc - bK$, input y ND OUTPUT u. The transfer function of the controller is thus

$$G_c(s) = -K(sI - A + Lc + bK)^{-1} L$$



Example: Controller transfer function

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad G_p(s) = \frac{1}{s^2}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Specified characteristic polynomial (desired poles):

$$\alpha(s) = s^2 + \sqrt{2}s + 1 \quad \Rightarrow \quad \text{Controller gains: } K = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}$$

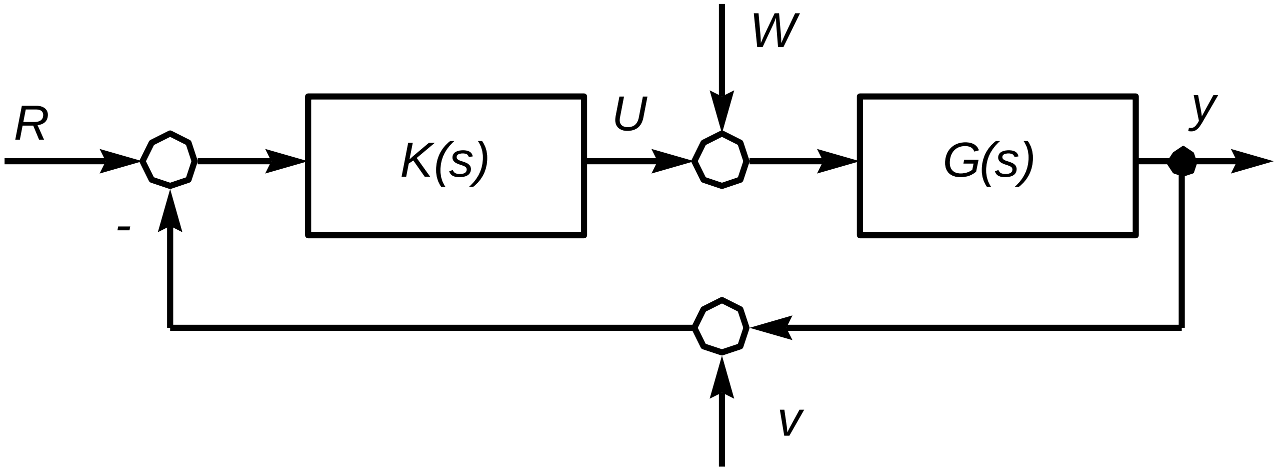
Observer error dynamics:

$$\alpha(s) = s^2 + \sqrt{2}s + 1 \quad \Rightarrow \quad \text{Observer gains } L = \begin{bmatrix} 5 \\ 25 \end{bmatrix}$$

Controller transfer function: $G_c(s) = \dfrac{-40.4(s+0.62)}{s + 3.21 \pm j4.77}$

The controller transfer function depends on the gains K and L.

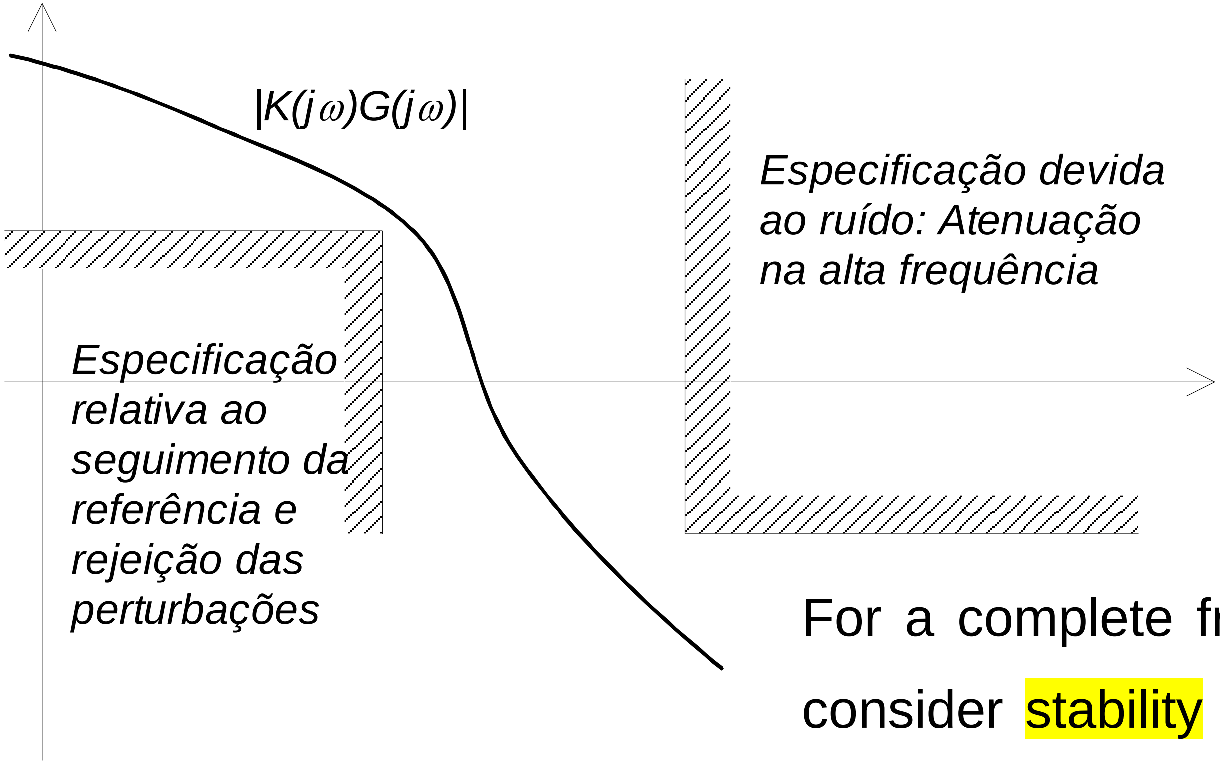
These gains can be envisaged as tuning knobs ("botões de ajuste") that allow to shape the loop gain such as to meet specifications.



$$Y_{cL} = \frac{KG}{1+KG}R + \frac{G}{1+KG}W - \frac{KG}{1+KG}V$$

KG is the loop gain

The controller $K(s)$ is designed such as to "shape" the loop gain:



For a complete framework we must also consider stability and uncertainty in plant model.

$$Y_{cL} = \frac{KG}{1+KG}R + \frac{G}{1+KG}W - \frac{KG}{1+KG}V$$

6.Reference tracking and integral gain

Objectivo:

Show how it is possible to change the basic regulator to track non-zero references, possibly including integral effect.

[JML-CEE2019] pp. 219 – 225

Servomechanism problem: Possibilities to insert a reference

Process model:

$$\dot{x} = Ax + bu \qquad y = cx$$

Controller:

$$\dot{\hat{x}} = (A - bK - Lc)\hat{x} + Ly + Mr$$

$$u = -K\hat{x} + Nr$$

There are several possibilities to select M (vector) and N (scalar).

a) Select M and N such that the error equation does not depend on r

$$\dot{x} - \dot{\hat{x}} = Ax + B(-K\hat{x} + Nr)$$
$$- (A - bK - Lc)\hat{x} + Ly + Mr$$

or

$$\dot{\tilde{x}} = (A - Lc)\tilde{x} + \underbrace{(bN - M)}_{=0} r$$

In order for this term to vanish, select

$$M = bN$$

With the choice $M = Nb$ we have:

$$\dot{\hat{x}} = (A - bK - Lc)\hat{x} + Ly + Nbr$$

or

$$\dot{\hat{x}} = (A - Lc)\hat{x} - b\underbrace{(K\hat{x} + Nr)}_{=-u} + Ly$$

The controller equations are thus

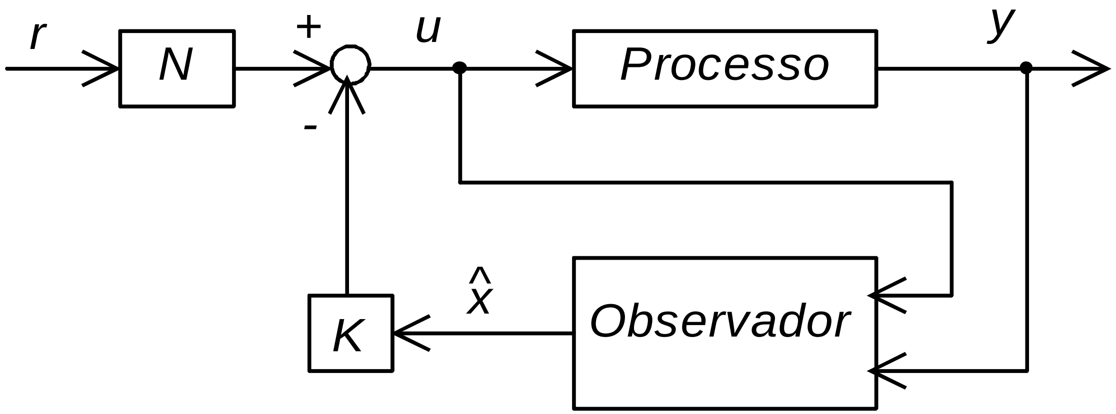
$$\dot{\hat{x}} = (A - Lc)\hat{x} + bu + Ly$$

$$u = -K\hat{x} + Nr$$

Controller structure such that the error does not depend on the reference

$$\dot{\hat{x}} = (A - Lc)\hat{x} + bu + Ly$$

$$u = -K\hat{x} + Nr$$



b) Select M and N such that the tracking error $e = r - y$ is used

$$\dot{\hat{x}} = (A - bK - Lc)\hat{x} + Ly + Mr$$

$$u = -K\hat{x} + Nr$$

Select

$$N = 0 \qquad M = -L$$

The controller is defined by the equations

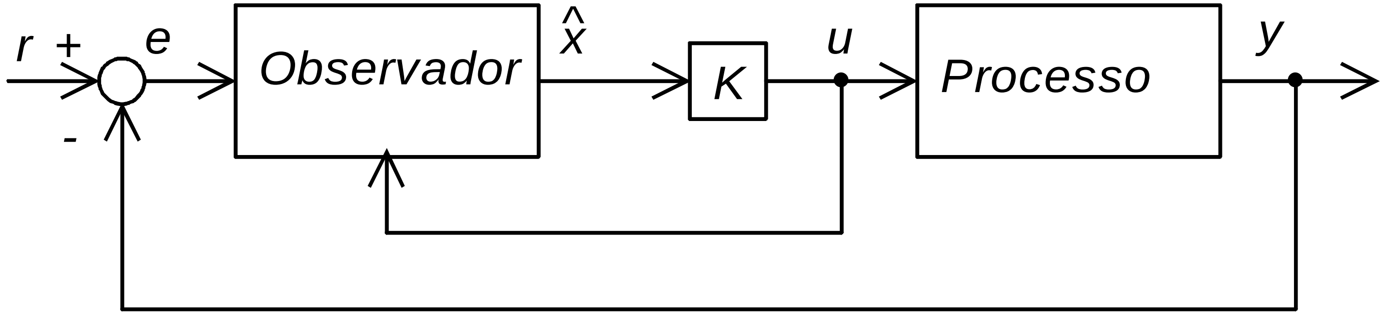
$$\dot{\hat{x}} = (A - bK - Lc)\hat{x} - Le$$

$$u = -K\hat{x}$$

Controller structure that uses the tracking error

$$\dot{\hat{x}} = (A - bK - Lc)\hat{x} - Le$$

$$u = -K\hat{x}$$



Inclusion of the integral effect

Augment the state x with the state x_I of the integral of the tracking error.

Differentiating $x_I(t) = \int_0^t (y(\tau) - r)d\tau$ for $r = const.$ we get $\dot{x}_I(t) = Cx(t)$

The ensemble process+integrator is described by the augmented state model

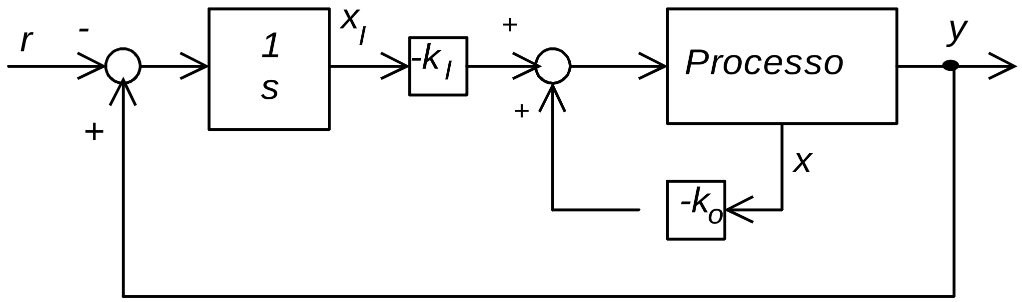
$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r$$

We can apply to this augmented system the pole placement methods studied before.

$$u = -\begin{bmatrix} K_0 & k_I \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

Yields the controller structure



The augmented state model

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & 0 \\ c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} b \\ 0 \end{bmatrix} u$$

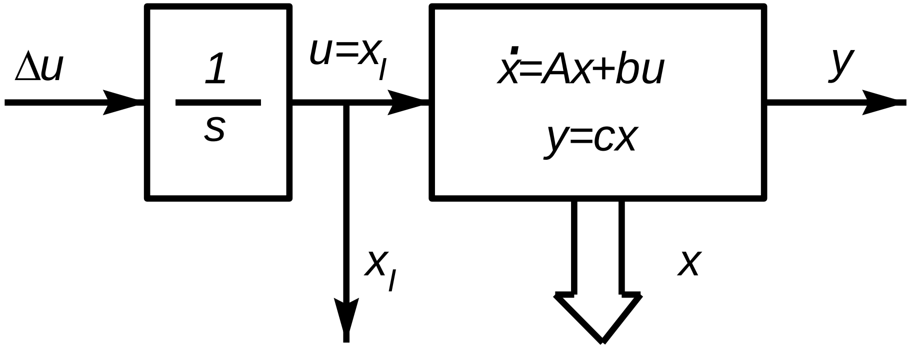
Is controllable but not observable.

We can estimate x with an observer and use the direct measure of x_I (since it is generated by the control computer).

The separation theorem is still valid.

Another way of forcing the integral effect: Integrator in series

Include an integrator at the input, in series with the system



Augmented state model (controllable and observable):

$$\begin{bmatrix} \dot{x} \\ \dot{x}_I \end{bmatrix} = \begin{bmatrix} A & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Delta u \qquad y = \begin{bmatrix} c & 0 \end{bmatrix} \begin{bmatrix} x \\ x_I \end{bmatrix}$$

Design the controller for the augmented model.