Equality constraint in the terminal state

Maximize $J(u) = \Psi(x(T) + \int_0^T L(x,u)dt$

Subject to $\dot{x} = f(x,u)$, $x(0) = x_0$, $x(T) = x_T$, $u(t) \in U$

Pontryagin's Principle (necessary conditions for optimality)

$$\dot{x} = f(x,u),\ x(0) = x_0,\ x(T) = x_T,\ u(t) \in U$$

$$-\dot{\lambda}' = \lambda' f_x + L_x$$

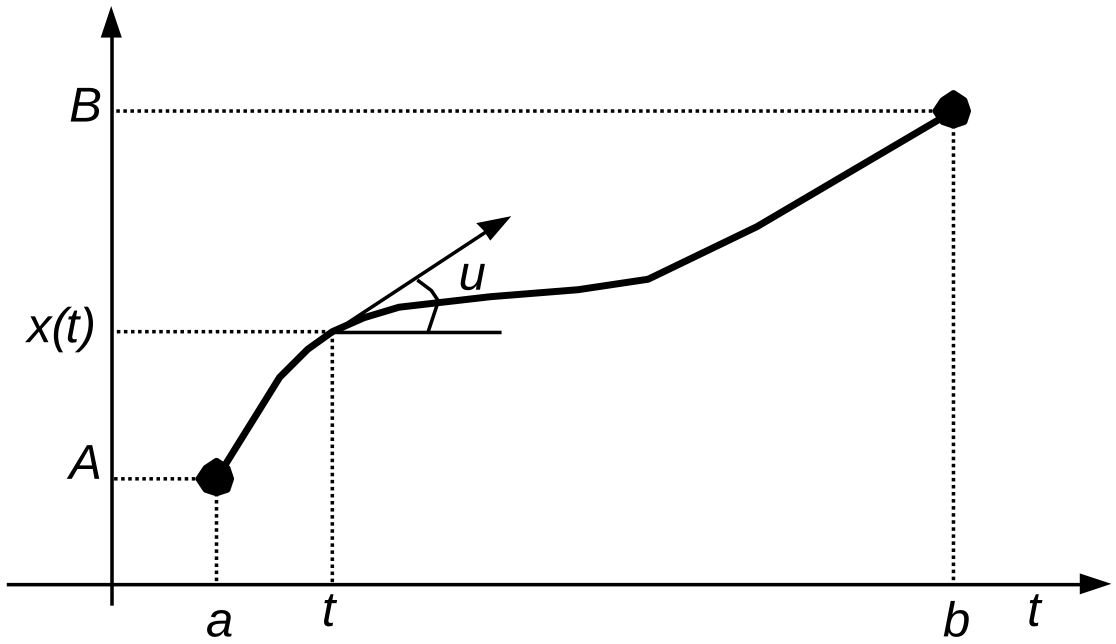
There are no constraints on $\lambda(T)$ because the terminal state is specified

$$H(\lambda, x, u) := \lambda' f(x,u) + L(x,u)$$

At each $t \in [0,T]$ the Hamiltonian H is maximal with respect to u

[JML-CEE] chap. 12

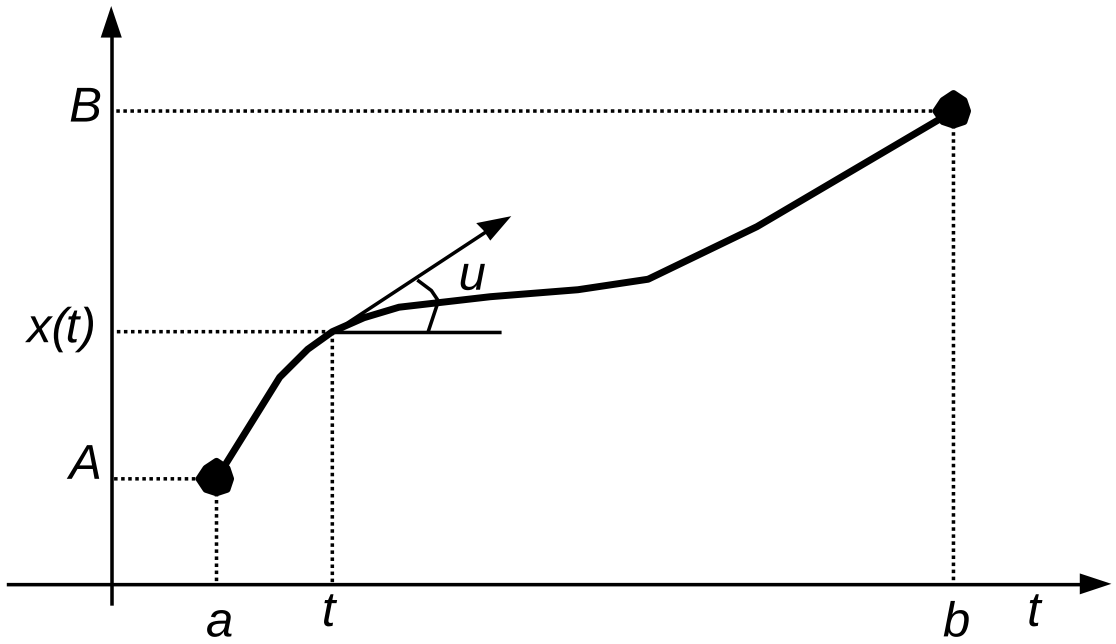
Ex. 1 – Shortest path between 2 points [JML-CEE] p.468-470



$$J = \int_a^b \sqrt{1 + (\dot{x}(t))^2}$$

What is the shape of the function $x(t)$ such that the length of the line connecting points (a, A) and (b, B) is minimum?

Transform the geometrical problem into an optimal control problem



$$J = \int_a^b \sqrt{1 + \left(\dot{x}(t)\right)^2}$$

$$\dot{x} = u$$

$$J(u) = \int_a^b \sqrt{1 + u^2}dt$$

minimize $J(u) = \int_a^b \sqrt{1 + u^2}dt$

Subject to $\dot{x} = u$, $x(a) = A$, $x(b) = B$

maximize $J(u) = -\int_a^b \sqrt{1+u^2}dt$

Subject to $\dot{x} = u$, $x(a) = A$, $x(b) = B$

$$L(x,u) = -\sqrt{1+u^2}, \quad L_x = 0, \quad f_x = 0$$

$$-\dot{\lambda}' = \lambda' f_x + L_x \quad \dot{\lambda} = 0 \quad \rightarrow \quad \lambda = Const.$$

$$H(\lambda, x, u) = \lambda' f(x,u) + L(x,u) = \lambda u - \sqrt{1+u^2}$$

Optimality condition: $\frac{\partial H}{\partial u} = \lambda - \frac{u}{\sqrt{1+u^2}} = 0$ Since $\lambda = C$, then u is also constant

The solution is a straight line (constant slope).

To find the value of the constant, solve the state equation and impose the terminal condition.

Solution of the state equation with u constant:

$$\dot{x} = u,\ x(a) = A,\ x(b) = B$$

Integrate both sides and use the fundamental theorem of calculus:

$$x(t) = x(a) + \int_a^t u(\sigma)d\sigma$$

Since u is constant:

$$x(t) = x(a) + u(t - a)$$

Apply the terminal condition

$$B = A + u(b - a) \quad \rightarrow \quad u = \frac{B-A}{b-a}$$

The geometrical interpretation is immediate.

Ex. 2 – Mobile robot with final state specified [CEE-JML] p. 471-474

Robot moves along a straight line only.

Model (double integrator):

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = u$$

Find the optimal control that drives the robot from the initial state

$$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

to the terminal state

$$x(1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Minimizing $\qquad J(u) = \frac{1}{2}\int_0^1 u^2(t)dt$

Como o campo de vetores é

$$f = \begin{bmatrix} x_2 \\ u \end{bmatrix},$$

a respetiva matriz jacobiana é

$$f_x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Além disso, a lagrangiana é

$$L(x,u) = -\frac{1}{2}u^2,$$

em que o sinal "menos" é devido a pretender-se minimizar o funcional, e

$$L_x = \begin{bmatrix} 0 & 0 \end{bmatrix},$$

pelo que a equação adjunta se escreve

$$\begin{bmatrix} -\dot{\lambda}_1 & -\dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

ou seja

$$\dot{\lambda}_1 = 0,$$

$$\dot{\lambda}_2 = -\lambda_1.$$

Não há condições fronteira a impôr ao coestado uma vez que o estado terminal é completamente especificado. Assim, é

$$\lambda_1 = C_1,$$

em que C_1 é uma constante e, pela equação para λ_2,

$$\lambda_2 = C_2 - C_1 t,$$

em que C_2 é uma constante.

A hamiltoniana é

$$H(\lambda, x, u) = \lambda_1 x_2 + \lambda_2 u - \frac{1}{2}u^2$$

e a condição de máximo é

$$\frac{\partial H}{\partial u} = \lambda_2 - u = 0,$$

pelo que o controlo ótimo é

$$u^*(t) = \lambda_2(t) = C_2 - C_1 t.$$

Para calcular as constantes C_1 e C_2, resolvem-se as equações de estado com $u = u^*$ e impõem-se as condições inicial e final do estado. Da equação de estado para x_2,

$$\dot{x}_2 = C_2 - C_1 t.$$

Integrando ambos os membros

$$x_2(t) = x_2(0) + \int_0^t (C_2 - C_1\sigma)d\sigma.$$

Calculando o integral e usando a condição inicial, vem

$$x_2(t) = 1 + C_2 t - \frac{1}{2}C_1 t^2.$$

Integrando agora ambos os membros da equação diferencial de x_1,

$$x_1(t) = x_1(0) + \int_0^t x_2(\sigma)d\sigma$$

$$x_1(t) = 1 + \int_0^t \left[1 + C_2\sigma - \frac{1}{2}C_1\sigma^2\right] d\sigma,$$

ou seja,

$$x_1(t) = 1 + t + \frac{1}{2}C_2t^2 - \frac{1}{6}C_1t^3.$$

Para calcular C_1 e C_2 faça-se $t = 1$ (instante terminal) e usem-se as condições terminais para x_1 e x_2, obtendo-se

$$\begin{cases} 1 + \frac{1}{2}C_2 - \frac{1}{6}C_1 = 0 \\ 1 + C_2 - \frac{1}{2}C_1 = 0 \end{cases} .$$

A solução deste sistema de equações algébricas em C_1 e C_2 é

$$C_1 = -6, \quad C_2 = -4,$$

sendo, pois, o controlo ótimo

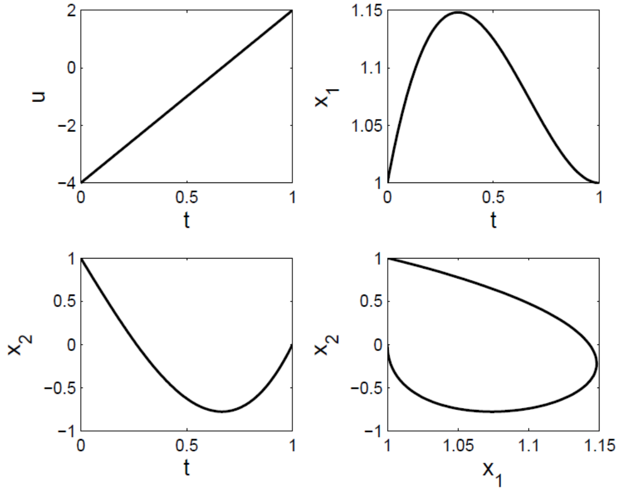
$$u^*(t) = 6t - 4.$$

As trajetórias ótimas para o estado obtêm-se a
ao sistema, sendo

$$x_1^*(t) = 1 + t - 2t^2 + t^3$$

e

$$x_2^*(t) = 1 - 4t + 3t^2.$$



Hamilton form of the state and adjoint equations

Remember: $H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$ $\qquad -\dot{\lambda}' = \lambda' f_x(x, u) + L_x(x, u)$

Then: $\frac{\partial H}{\partial \lambda} = f(x, u) = \dot{x}$ $\qquad \frac{\partial H}{\partial x} = \lambda' f_x(x, u) + L_x(x, u) = -\dot{\lambda}'$

Hence, we conclude the Hamilton form of the state and adjoint equations:

$$\dot{x} = \frac{\partial H}{\partial \lambda} \qquad \dot{\lambda}' = -\frac{\partial H}{\partial x}$$

When the maximum of the Hamiltonian is attained for a point that is interior tp the set of admissible control values, U, the optimal condition is

$$\frac{\partial H}{\partial u} = 0$$

Hamiltonian properties

For time invariant dynamics and Lagrangian, the Hamiltonian is constant in time.

Remember: $\dot{x} = \frac{\partial H}{\partial \lambda} \qquad \dot{\lambda}' = -\frac{\partial H}{\partial x} \qquad \frac{\partial H}{\partial u} = 0$

$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u)$$

Then:

$$\frac{dH}{dt} = \frac{\partial H}{\partial \lambda}\dot{\lambda} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial u} = 0$$

Meaning that the Hamiltonian is constant in time.

This property holds even if u is not smooth (the previous proof is no longer valid).

The Hamiltonian is constant along an optimal path – Ex. 1

Mobile robot with minimum energy [CEE-JML] p. 434-436

maximize $J(u) = x_1(T) - \frac{1}{2}\int_0^T u^2(t)dt$

subject to $\dot{x}_1 = x_2,\ \dot{x}_2 = u$

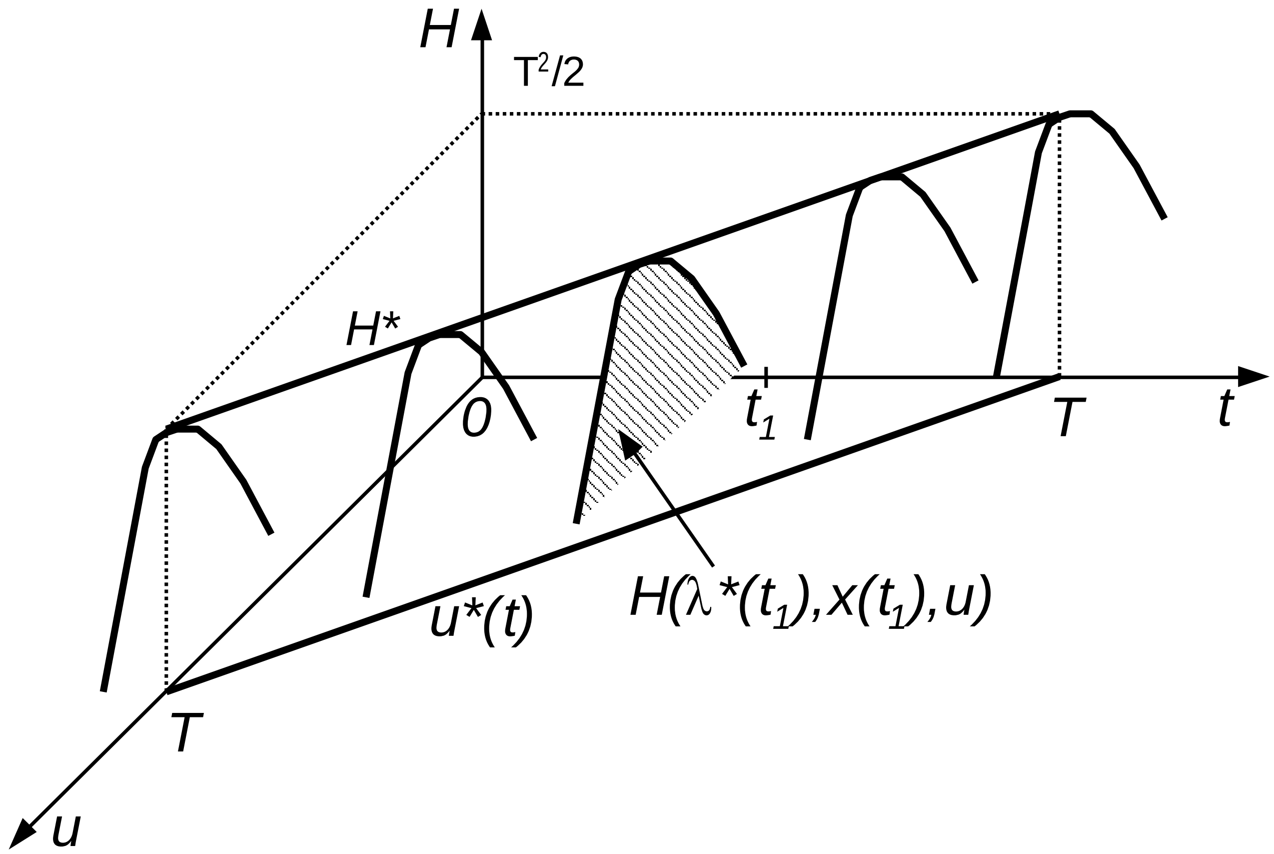
$$H(\lambda, x, u) = \lambda' f(x, u) + L(x, u) = \lambda_1 x_2 + \lambda_2 u - \frac{1}{2}u^2$$

$$\lambda_1(t) = 1, \qquad \lambda_2(t) = T - t, \qquad u^*(t) = T - t$$

Along an optimal trajectory $\dot{x}_1 = x_2,\ \dot{x}_2 = T - t$

$$x_2(t) = x_2(0) + \int_0^t (T - \sigma)d\sigma = Tt - \frac{1}{2}t^2$$

$$x_1(t) = x_1(0) + \int_0^t \left(T\sigma - \frac{\sigma^2}{2}\right)d\sigma = \frac{T}{2}t^2 - \frac{1}{6}t^3 \quad \rightarrow \quad H = \frac{1}{2}T^2$$


H
T2/2
H*
0
t1
T
t
H(λ*(t1),x(t1),u)
u*(t)
T
u

The Hamiltonian is constant along an optimal path – Ex. 2

Mobile robot with minimum fuel [CEE-JML] p. 436-437

$$\text{maximize } J(u) = x_1(T) - \int_0^T u(t)dt$$

$$\text{subject to} \quad \dot{x}_1 = x_2, \;\; \dot{x}_2 = u \quad 0 \le u \le \bar{u}$$

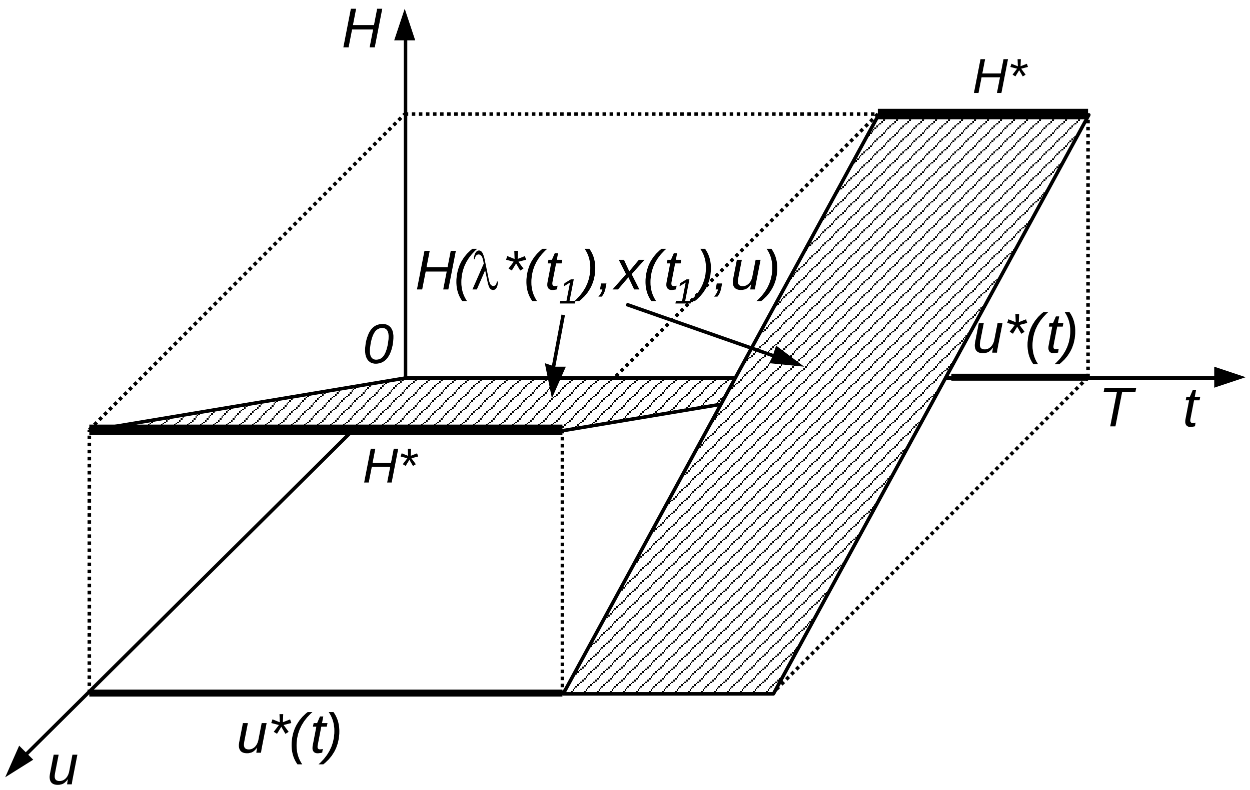
$$H(\lambda, x, u) = \lambda' f(x,u) + L(x,u) = \lambda_1 x_2 + (\lambda_2 - 1)u$$

In this example the optimal control is bang-bang.

Compute the Hamiltonian in each interval in which u is constant to get

$$H(\lambda, x, u) = \bar{u}(T-1)$$

The Hamiltonian is constant!


H
H*
H(λ*(t1),x(t1),u)
0
u*(t)
T
t
H*
u
u*(t)