DM DEPARTAMENTO DE MATEMÁTICA TÉCNICO LISBOA

Probability Theory

LMAC, MMA

1st. Semester – 2021/2022
2022/02/12 — 08:00
Exam 1

Duration: **120** minutes

- Please justify all your answers.
- This exam has TWO PAGES and TWELVE QUESTIONS. The total of points is 20.0.

Chap. 1 — Probability spaces — 3.5 points

1. A ($A \subset \Omega$) is called a *co-finite set* if A^c is finite. Let $\mathscr{A}$ consist of all the finite and co-finite subsets of Ω. **(1.5)**

Admit that Ω is finite. Show that, in this case, $\mathscr{A}$ is a $\sigma-$algebra on Ω.

- **Requested proof**

  We ought to mention that a minimal set of postulates for a non-empty class of subsets $\mathscr{A}$ of Ω to be a $\sigma-$algebra on Ω is:

  (i) $\Omega \in \mathscr{A}$;

  (ii) $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$;

  (iii) $A_1, A_2, \dots \in \mathscr{A} \Rightarrow \cup_{i=1}^{+\infty} A_i \in \mathscr{A}$.

  Hence, we have to prove that all these 3 postulates are true for the class of all the finite and co-finite subsets of Ω.

  (i) Ω is finite and $\Omega^c = \emptyset$ is also finite, we conclude that Ω is *co-finite*, hence $\Omega \in \mathscr{A}$.

  (ii) We know that $A \in \mathscr{A}$ iff it is finite or co-finite and since Ω is finite, then $A^c = \Omega \backslash A$ is also finite, thus $A^c \in \mathscr{A}$.

  (iii) Note that there are $2^{\#\Omega}$ subsets of the finite sample space Ω. Moreover, $\cup_{j=1}^{2^{\#\Omega}} B_j [= \Omega]$ is also finite. Consequently, if we consider $A_1, A_2, \dots \in \mathscr{A}$ then $\cup_{i=1}^{+\infty} A_i \subseteq \cup_{j=1}^{2^{\#\Omega}} B_j$, thus $\cup_{i=1}^{+\infty} A_i$ is also finite and therefore it belongs to $\mathscr{A}$. ✓

2. The Borel-Cantelli lemma reads as follows. **(2.0)**

(i) If $\sum_{n=1}^{+\infty} P(A_n) < +\infty$ then $P(\limsup_{n\to+\infty} A_n) = 0$.

(ii) If $\sum_{n=1}^{+\infty} P(A_n) = +\infty$ and $A_1, A_2, \dots$ are (mutually) independent events then $P(\limsup_{n\to+\infty} A_n) = 1$.

Now, consider $\Omega = (0,1)$, $\mathscr{A} = \mathscr{B}((0,1))$, and P the Lebesgue measure. Show that the sequence of events $\{A_n = (0, 1/n) : n \in \mathbb{N}\}$ illustrates two facts.

1. In general, the converse of (i) is not true.
2. The (mutual) independence condition in (ii) is essential.

- **Events**

  $A_n = (0, 1/n), \qquad P(A_n) = 1/n, \qquad A_n \downarrow$

  For $i, j \in \mathbb{N}$ and $i < j$, we have $P(A_i \cap A_j) = P(A_j) = \frac{1}{j} \neq P(A_i) \times P(A_j) = \frac{1}{i} \times \frac{1}{j} = \frac{1}{ij}$. Therefore these events are not (pairwise) independent.

- **Checking fact 1.**

  Note that

$$
\begin{aligned}
A_n \downarrow \quad &\Rightarrow \quad \limsup_{n\to+\infty} A_n = \bigcap_{n=1}^{+\infty} A_n = \emptyset \\
P(\limsup_{n\to+\infty} A_n) \quad &= \quad P(\emptyset) = 0,
\end{aligned}
$$

hence, (i) checks. However, even though $P(\limsup_{n\to+\infty} A_n) = 0$, we have

$$\sum_{n=1}^{+\infty} P(A_n) \quad = \quad \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty, \qquad (*)$$

thus, the converse of (i) is not true. ✓

- **Checking fact 2.**

  Looking at these dependent events, at (*), and at (ii), we can conclude that the condition $\sum_{n=1}^{+\infty} P(A_n) = +\infty$ does not imply that $P(\limsup_{n\to+\infty} A_n) = 1$ and consequently the independence of $A_1, A_2, \ldots$ is absolutely essential in (ii).

Chap. 2 — Random variables	3.5 points

3. Let X and Y be two r.v. and prove that XY is also a r.v. **(2.0)**

Hint: Prove that X^2 is a r.v.; rewrite XY, for example, in terms of a difference between the square of a sum and two squares; take for granted that the sum and difference of two r.v. are also r.v.

- **R.v.**

  Let $(\Omega, \mathscr{F})$ and $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ be two measurable spaces. Then, $X : \Omega \to \mathbb{R}$ and

  $$X^{-1}(B) \quad = \quad \{\omega \in \Omega : X(\omega) \in B\} \in \mathscr{F}, \quad \forall B \in \mathscr{B}(\mathbb{R}).$$

- **Auxiliary result**

  [A function $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable iff $g^{-1}(B) = \{x \in \mathbb{R} : g(x) \in B\} \in \mathscr{B}(\mathbb{R}), \forall B \in \mathscr{B}(\mathbb{R})$. Moreover,] if

  $$g^{-1}((-\infty, z]) \quad = \quad \{x \in \mathbb{R} : g(x) \le z\} \in \mathscr{B}(\mathbb{R}), \quad \forall z \in \mathbb{R},$$

  then $g : \mathbb{R} \to \mathbb{R}$ is Borel measurable.

  Now, let us consider $g(X) = X^2$.

  - for $z < 0$,
    $$g^{-1}((-\infty, z]) \quad = \quad \{x \in \mathbb{R} : g(x) = x^2 \le z\} = \emptyset \in \mathscr{B}(\mathbb{R});$$
  - for $z \ge 0$,
    $$\begin{aligned} g^{-1}((-\infty, z]) \quad &= \quad \left\{x \in \mathbb{R} : g(x) = x^2 \le z\right\} = \left\{x \in \mathbb{R} : -\sqrt{z} \le x \le \sqrt{z}\right\} \\ &= \quad (-\infty, \sqrt{z}] \backslash (-\infty, -\sqrt{z}) = (-\infty, \sqrt{z}] \cap (-\infty, -\sqrt{z})^c \in \mathscr{B}(\mathbb{R}). \end{aligned}$$

  As a result, $g(X) = X^2$ is a Borel measurable function and therefore a r.v.

- **Requested proof**

  Since we just proved that X^2 is a r.v. and we can take for granted that the sum and difference of two r.v. are r.v., we conclude that Y^2, $(X+Y)$, and $(X+Y)^2$ are also r.v., and so is

  $$XY = \frac{-X^2 - Y^2 + (X+Y)^2}{2}. \qquad ✓$$

  □

4. Let: X, Y, and Z be r.v. such that X and Y are identically distributed; $g : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function. **(1.5)**

Show that $g(X)$ and $g(Y)$ are identically distributed and give a simple example to show that XZ and YZ can have different distributions.

Hint: Consider X a discrete r.v., with values in $\mathbb{R}_X = \mathbb{Z}\backslash\{0\}$ and a symmetric p.f.

- **R.v.**

  $X \sim Y$,

  $g(X), g(Y)$, where $g: \mathbb{R} \to \mathbb{R}$ is a Borel measurable function

- **Requested proof**

  Since g is a Borel measurable function, we can add that $g(X)$ and $g(Y)$ are both r.v. Furthermore,

$$\begin{aligned} F_{g(Y)}(z) &= P[g(Y) \le z] = P[Y \in \{x \in \mathbb{R} : g(x) \le z\}] \overset{X \sim Y}{=} P[X \in \{x \in \mathbb{R} : g(x) \le z\}] = P[g(X) \le z] \\ &= F_{g(X)}(z), \quad z \in \mathbb{R}, \end{aligned}$$

  we conclude that $g(X)$ and $g(Y)$ are identically distributed r.v. ✓

- **Requested example**

  X a discrete r.v., with values in $\mathbb{R}_X = \mathbb{Z} \backslash \{0\}$ and a symmetric p.f., i.e., $P(X = x) = P(X = -x)$, for $x \in \mathbb{Z} \backslash \{0\}$

  $Y = -X, \qquad Z = Y$

  $XZ = -X^2, \quad R_{XZ} = \mathbb{Z}^-, \qquad YZ = (-X)^2 = X^2, \quad R_{YZ} = \mathbb{Z}^+$

  It suffices to note that the r.v. XZ and YZ have different ranges to conclude that they cannot possibly have the same distribution.

- **[Note**

  The trivial case of X such that $P(X = 0) = 1$ is of no interest.]

Chap. 3 — Independence	4.5 points

5. Let $0 < \epsilon \le \frac{1}{16}$ and A, B, and C be events such that: **(1.0)**

- $P(A \cap B \cap C) = P(A \cap B \cap C^c) = \frac{1}{8}$;
- $P(A \cap B^c \cap C) = P(A^c \cap B \cap C) = \frac{1}{8} - \epsilon$;
- $P(A \cap B^c \cap C^c) = P(A^c \cap B \cap C^c) = \frac{1}{8} + \epsilon$;
- $P(A^c \cap B^c \cap C) = \frac{1}{8} + 2\epsilon$;
- $P(A^c \cap B^c \cap C^c) = \frac{1}{8} - 2\epsilon$.

Show that the events A, B, and C are not mutually independent.

- **Events and probabilities**

$$\begin{aligned} P(A) &= P(A \cap B \cap C) + P(A \cap B \cap C^c) + P(A \cap B^c \cap C) + P(A \cap B^c \cap C^c) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} - \epsilon + \frac{1}{8} + \epsilon = \frac{1}{2} \\ P(B) &= P(A \cap B \cap C) + P(A \cap B \cap C^c) + P(A^c \cap B \cap C) + P(A^c \cap B \cap C^c) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} - \epsilon + \frac{1}{8} + \epsilon = \frac{1}{2} \\ P(C) &= P(A \cap B \cap C) + P(A \cap B^c \cap C) + P(A^c \cap B \cap C) + P(A^c \cap B^c \cap C) = \frac{1}{8} + \frac{1}{8} - \epsilon + \frac{1}{8} - \epsilon + \frac{1}{8} + 2\epsilon \\ &= \frac{1}{2} \end{aligned}$$

- **Requested proof**

  Note that:

$$\begin{aligned} P(A \cap B) &= P(A \cap B \cap C) + P(A \cap B \cap C^c) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4} \equiv P(A) \times P(B); \\ P(A \cap C) &= P(A \cap B \cap C) + P(A \cap B^c \cap C) = \frac{1}{8} + \frac{1}{8} - \epsilon = \frac{1}{4} - \epsilon \ne P(A) \times P(C). \end{aligned}$$

  Hence, events A and C are not pairwise independent and therefore events A, B, and C cannot be mutually independent. ✓

6. Let X and Y be two independent r.v. with common p.d.f. $f(x) = x^{-2} \times I_{[1,+\infty)}(x)$. **(2.0)**

Derive (directly) the c.d.f. of $Z = \sqrt{XY}$ and describe a method to generate pseudorandom numbers from the distribution of Z.

Note: There is no explicit expression for the quantile function of Z.

- **Random vector and range**

  $$(X,Y), \quad X \perp\!\!\!\perp Y, \quad X \sim Y, \quad f_X(x) = f_Y(x) = f(x) = \begin{cases} x^{-2}, & x \geq 1 \\ 0, & \text{otherwise}, \end{cases} \qquad \mathbb{R}_{X,Y} = [1,+\infty)^2$$

- **Transformation of (X,Y) and its range**

  $$Z = g(X,Y) = \sqrt{XY}, \qquad \mathbb{R}_Z = g(\mathbb{R}_{X,Y}) = [1,+\infty)$$

- **C.d.f. of Z**

  $$\begin{aligned}
  F_Z(z) &= P\left(\sqrt{XY} \leq z\right) \\
  &= \int\int_{\{(x,y)\in[1,+\infty)^2:\, \sqrt{xy}\leq z\}} f_{X,Y}(x,y)\, dy\, dx \\
  &\overset{X \overset{i.i.d.}{\sim} Y}{=} \int\int_{\left\{(x,y)\in[1,+\infty)^2:\, y\leq \frac{z^2}{x}\right\}} f(x) \times f(y)\, dy\, dx \\
  &= \int_1^{z^2} \int_1^{\frac{z^2}{x}} \frac{1}{x^2} \times \frac{1}{y^2}\, dy\, dx \\
  &= \int_1^{z^2} \frac{1}{x^2} \times \left(-\frac{1}{y}\Big|_1^{\frac{z^2}{x}} \right) dx \\
  &= \int_1^{z^2} \frac{1}{x^2} \times \left(1 - \frac{x}{z^2}\right) dx \\
  &= \left(-\frac{1}{x} - \frac{\ln(x)}{z^2}\right)\Big|_1^{z^2} \\
  &= \frac{z^2 - 2\ln(z) - 1}{z^2}, \quad z \geq 1.
  \end{aligned}$$

- **Generation of a pseudorandom number from Z**

  There is no explicit expression for the quantile function of Z but there is one for the common quantile function of X and Y. Indeed:

  $$F(x) = P(X \leq x) = \begin{cases} 0, & x < 1 \\ \int_1^x \frac{1}{t^2}\, dt = -\frac{1}{t}\Big|_1^x = 1 - \frac{1}{x}, & x \geq 1; \end{cases}$$

  $$F(x) = u \Leftrightarrow 1 - \frac{1}{x} = u \Leftrightarrow F^{-1}(u) = \frac{1}{1-u}, \quad 0 < u < 1.$$

  Furthermore, by resorting to the quantile transformation, we know that if $U \sim \text{uniform}(0,1)$ then $F^{-1}(U) \equiv \frac{1}{1-U} \sim X \sim Y$.

  Consequently, to generate a pseudorandom number from Z, z, we have to:

  - independently generate two pseudorandom numbers, u_1 and u_2, from the uniform$(0,1)$ distribution;
  - assign $x = \frac{1}{1-u}$, $y = \frac{1}{1-u_2}$, and, finally, $z = \sqrt{xy}$.

7. Admit that jobs arrive to a workstation according to a non-homogeneous Poisson process with intensity function $\lambda(t) = 1 + e^{-t}$, $t \geq 0$ (time in hours). **(1.5)**

Suppose two jobs arrived during the first hour. What is the probability that both jobs arrived during the first 20 minutes?

- **Stochastic process**

  $\{N(t): t > 0\} \sim NHPP(\lambda(t))$

  $N(t)$ = number of jobs arrived to the workstation until time t

- **Intensity and mean value functions**

$$\begin{aligned} \lambda(t) &= 1 + e^{-t}, \quad t \geq 0 \\ m(t) &= \int_0^t \lambda(s)\, ds = \int_0^t (1 + e^{-s})\, ds = t + 1 - e^{-t}, \quad t \geq 0 \end{aligned}$$

Requested probability

Since

$$(N(s) \mid N(t) = n) \sim \text{binomial}(n, m(s)/m(t)), \quad 0 < s < t,$$

$s = 1/3$, $t = 1$, $n = 2$, and

$$\frac{m(s)}{m(t)} = \frac{1/3 + 1 - e^{-1/3}}{1 + 1 - e^{-1}} \simeq 0.377914,$$

we get

$$P[N(1/3) = 2 \mid N(1) = 2] \simeq \binom{2}{2} \times (0.377914)^2 \times (1 - 0.377914)^{2-2} \simeq (0.377914)^2 \simeq 0.142819.$$

Chap. 4 — Expectation	3.5 points

8. Let X and Y be a two i.i.d. r.v. with standard normal distribution. Show that $E(\max\{X,Y\}) = \frac{1}{\sqrt{\pi}}$. **(2.0)**

- **R.v.**

  $X \overset{i.i.d.}{\sim} Y \sim \text{normal}(0,1), \qquad f_X(x) = f_Y(x) = \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad -\infty < x < +\infty$

- **Requested expected value**

$$\begin{aligned} E(\max\{X,Y\}) &\overset{X \perp\!\!\!\perp Y}{=} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \max\{x,y\} \times f_X(x) \times f_Y(y)\, dy\, dx \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \max\{x,y\} \times \phi(x) \times \phi(y)\, dy\, dx \\ &= \int_{-\infty}^{+\infty} \left[\int_y^{+\infty} x \times \phi(x)\, dx \right] \times \phi(y)\, dy + \int_{-\infty}^{+\infty} \left[\int_x^{+\infty} y \times \phi(y)\, dy \right] \times \phi(x)\, dx \\ &= 2 \times \int_{-\infty}^{+\infty} \left[\int_y^{+\infty} x \times \phi(x)\, dx \right] \times \phi(y)\, dy \\ &= 2 \times \int_{-\infty}^{+\infty} \left[\int_y^{+\infty} \frac{x\, e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\, dx \right] \times \phi(y)\, dy \\ &= 2 \times \int_{-\infty}^{+\infty} \left(-\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \Big|_y^{\infty} \right) \times \phi(y)\, dy \\ &= 2 \times \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \times \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy \\ &= \frac{1}{\pi} \sqrt{2\pi \left(1/\sqrt{2}\right)^2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi \left(1/\sqrt{2}\right)^2}} e^{-\frac{y^2}{2\left(1/\sqrt{2}\right)^2}}\, dy \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} f_{N\left(0,\left(1/\sqrt{2}\right)^2\right)}(y)\, dy \\ &= \frac{1}{\sqrt{\pi}}. \end{aligned}$$

9. Admit that $(X,Y) \sim \text{normal}_2(\underline{\mu}, \Sigma)$, where $\underline{\mu}$ and Σ are such that: $\mu_X = 55.57$, $\mu_Y = 75.86$; $\sigma^2_X = 7.6735$, $\sigma^2_Y = 35.5510$, $cov(X,Y) = 13.6531$. **(1.5)**

Compute $P(Y > X + 10)$.

- **Random vector** (X,Y)

  $$(X,Y) \sim \text{normal}_2(\underline{\mu}, \Sigma), \quad \text{where:} \quad \underline{\mu} = \begin{bmatrix} 55.57 \\ 75.86 \end{bmatrix}; \quad \Sigma = \begin{bmatrix} 7.6735 & 13.6531 \\ 13.6531 & 35.5510 \end{bmatrix}.$$

- **Requested probability and auxiliary r.v.**

  $P(Y > X + 10) = P(W = Y - X > 10)$

  $$W = Y - X = \mathbf{C} \times \begin{bmatrix} X \\ Y \end{bmatrix} + \underline{b}, \quad \text{where:} \quad \mathbf{C} = [-1 \quad 1]; \quad \underline{b} = [0].$$

  $$W \overset{Th.\, 4.216}{\sim} \text{normal}(E(W), V(W)), \quad \text{where:}$$

  $$\begin{aligned} E(W) &= \mathbf{C}\underline{\mu} + \underline{b} - 55.57 + 75.86 + 0 = 20.29; \\ V(W) &= \mathbf{C}\Sigma\mathbf{C}^\top = [-1 \quad 1] \times \begin{bmatrix} 7.6735 & 13.6531 \\ 13.6531 & 35.5510 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= [-7.6735 + 13.6531 \quad -13.6531 + 35.5510] \times \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 15.9183. \end{aligned}$$

  Hence,

  $$\begin{aligned} P(W > 10) &= 1 - \Phi\left[\frac{10 - E(W)}{\sqrt{V(W)}}\right] \\ &= 1 - \Phi\left(\frac{10 - 20.29}{\sqrt{15.9183}}\right) \\ &\simeq 1 - \Phi(-2.58) \\ &= \Phi(2.58) \\ &\overset{tables}{=} 0.9951. \end{aligned}$$

Chap. 5 — Stochastic convergence concepts and classical limit theorems	5.0 points

10. Prove that complete convergence of sequences of r.v. is stronger than almost sure convergence. **(1.5)**

Hint: Without loss of generality, assume that $\{X_n : n \in \mathbb{N}\}$ is completely convergent to 0 ($X_n \overset{c}{\to} 0$) and use an alternative criterion when it comes to almost sure convergence of $\{X_n : n \in \mathbb{N}\}$ to zero ($X_n \overset{a.s.}{\to} 0$).

- **Sequence of r.v.**

  $\{X_n : n \in \mathbb{N}\}$

- **Requested proof**

  Without loss of generality, let us assume that $\{X_n : n \in \mathbb{N}\}$ is completely convergent to 0 ($X_n \overset{c}{\to} 0$), i.e., $\sum_{n=1}^{+\infty} P(|X_n| > \epsilon) < +\infty$, $\forall \epsilon > 0$. Equivalently,

  $$\lim_{n\to+\infty} \sum_{k=n}^{+\infty} P(|X_k| > \epsilon) = 0, \quad \forall \epsilon > 0. \qquad (\star)$$

  In order to relate this mode of convergence with a.s. convergence, recall that

  $$\begin{aligned} X_n \overset{a.s.}{\to} 0 \quad &\Leftrightarrow \quad P(\{\omega : \lim_{n\to+\infty} X_n(\omega) = 0\}) = 1 \\ &\Leftrightarrow \quad \lim_{n\to+\infty} P(\sup_{k\geq n} |X_k| > \epsilon) = 0, \quad \forall \epsilon > 0 \\ &\Leftrightarrow \quad \lim_{n\to+\infty} P(\cup_{k\geq n} \{|X_k| > \epsilon\}) = 0, \quad \forall \epsilon > 0. \end{aligned}$$

Since the probability P is semi-additive, we obtain, for every $\epsilon > 0$,

$$\lim_{n\to+\infty} P\left(\cup_{k\geq n}\{|X_k| > \epsilon\}\right) \quad \leq \quad \lim_{n\to+\infty} \sum_{k=n}^{+\infty} P(|X_k| > \epsilon) \overset{(\star)}{=} 0.$$

Hence, $X_n \overset{a.s.}{\to} 0$. ✓

11. Let $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. r.v. with common p.d.f. $f_X(x) = \frac{2x}{3\alpha^2} \times I_{[\alpha,2\alpha]}(x)$, where α is an unknown positive constant. **(2.0)**

After having derived the c.d.f. of $Y_n = \frac{X_{(n:n)}}{2}$, where $X_{(n:n)} = \max_{i=1,\ldots,n} X_i$, show that $Y_n \overset{d}{\to} \alpha$.

- **Sequence of r.v.**

  $\{X_n : n \in \mathbb{N}\}$

  $X_n \overset{i.i.d.}{\sim} X, \quad n \in \mathbb{N}$

  $f_X(x) = \begin{cases} \frac{2x}{3\alpha^2}, & \alpha \leq x \leq 2\alpha \quad (\alpha > 0) \\ 0, & \text{otherwise} \end{cases}$

- **Another sequence of r.v.**

  $\{Y_n : n \in \mathbb{N}\}$

  $Y_n = \frac{X_{(n:n)}}{2}$

- **Requested c.d.f.**

  For $y \in \left[\frac{\alpha}{2}, \alpha\right]$, we have

$$\begin{aligned} F_{Y_n}(y) &= P\left[Y_n = \frac{X_{(n:n)}}{2} \leq y\right] = P\left[X_{(n:n)} \leq 2y\right] = \left[F_X(2y)\right]^n = \left(\int_\alpha^{2y} \frac{2x}{3\alpha^2}\,dx\right)^n = \left(\frac{x^2}{3\alpha^2}\Big|_\alpha^{2y}\right)^n \\ &= \left(\frac{4y^2}{3\alpha^2} - \frac{1}{3}\right)^n. \end{aligned}$$

  Moreover,

$$F_{Y_n}(y) = \begin{cases} 0, & y \leq \frac{\alpha}{2} \\ \left(\frac{4y^2}{3\alpha^2} - \frac{1}{3}\right)^n, & \frac{\alpha}{2} < y < \alpha \\ 1, & y \geq \alpha. \end{cases}$$

- **Requested proof**

  Since $\frac{4y^2}{3\alpha^2} - \frac{1}{3} \in (0,1)$, when $y \in \left(\frac{\alpha}{2}, \alpha\right)$, we have

$$\lim_{n\to+\infty} F_{Y_n}(y) = \begin{cases} 0, & y \leq \frac{\alpha}{2} \\ 0, & \frac{\alpha}{2} < y < \alpha \\ 1, & y \geq \alpha \end{cases} = \begin{cases} 0, & y < \alpha \\ 1, & y \geq \alpha, \end{cases}$$

  which coincides with c.d.f. of a degenerate r.v. at α, $F_\alpha(y) = I_{[\alpha,+\infty)}(y)$, for all $x \in \mathbb{R}$, thus, for all points at which $F_\alpha(y)$ is continuous. Hence, $Y_n \overset{d}{\to} \alpha$. ✓

12. Let: $\{X_n : n \in \mathbb{N}\}$ be a sequence of i.i.d. r.v. to $X \sim \text{beta}(\alpha, 1)$, where α is an unknown positive constant; $U_n = -\frac{1}{n}\sum_{i=1}^n \ln(X_i)$. **(1.5)**

Prove that $U_n \overset{P}{\to} \alpha^{-1}$.

Note: $Z_n = 2n\alpha \times U_n \sim \chi^2_{(2n)}$.

- **Sequence of r.v.**

  $\{X_n : n \in \mathbb{N}\}$

  $X_n \overset{i.i.d.}{\sim} X \sim \text{beta}(\alpha, 1), \quad n \in \mathbb{N} \quad (\alpha > 0)$

- **Another sequence of r.v.**

  $\{U_n : n \in \mathbb{N}\}$

  $U_n = -\frac{1}{n}\sum_{i=1}^{n} \ln(X_i)$

- **Rewriting U_n in terms of Z_n**

  $Z_n = 2n\alpha \times U_n \sim \chi^2_{(2n)}$

  $U_n = \frac{Z}{2n\alpha}$

- **Expected value and variance of U_n**

$$
\begin{aligned}
E(U_n) &= E\left(\frac{Z_n}{2n\alpha}\right) = \frac{1}{2n\alpha} \times E\left[\chi^2_{(2n)}\right] \overset{form}{=} \frac{1}{2n\alpha} \times 2n = \alpha^{-1} \\
V(U_n) &= V\left(\frac{Z_n}{2n\alpha}\right) = \frac{1}{(2n\alpha)^2} \times V\left[\chi^2_{(2n)}\right] \overset{form}{=} \frac{1}{4n^2\alpha^2} \times 4n = \frac{1}{n\alpha^2}
\end{aligned}
$$

- **Requested proof**

  The application of the definition of convergence in probability and Chebyshev-Bienaymé's inequality leads to

$$
\begin{aligned}
\lim_{n\to+\infty} P\left(|U_n - \alpha^{-1}| > \epsilon\right) &= \lim_{n\to+\infty} P\left[|U_n - E(U_n)| \geq \frac{\epsilon}{\sqrt{V(U_n)}}\sqrt{V(U_n)}\right] \\
&\leq \lim_{n\to+\infty} \frac{1}{\left[\epsilon/\sqrt{1/(n\alpha^2)}\right]^2} \\
&= \frac{1}{\epsilon^2\alpha^2} \lim_{n\to+\infty} \frac{1}{n} \\
&= 0, \quad \forall \epsilon > 0.
\end{aligned}
$$

  Hence, $U_n \xrightarrow{P} \alpha^{-1}$. ✓