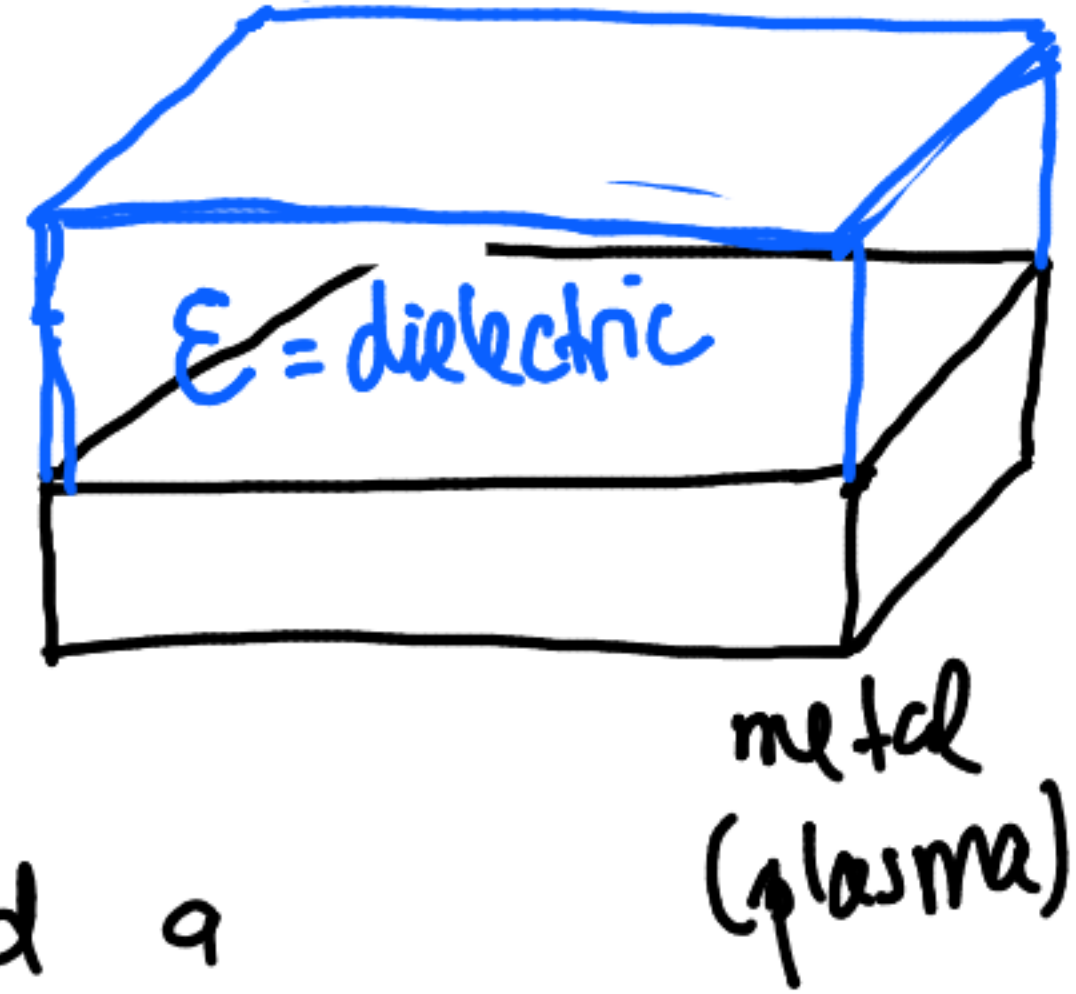
2.6 - Surface plasmon polaritons

Let us look for waves (EM) propagating at the interface of two materials. In our case, a quasi-2D plasma and a dielectric medium.



Such EM surface waves arise via the coupling of the electromagnetic fields to the oscillations of the plasma electrons (plasmons).

Polaritons = photon + plasmon!

Let us start from Maxwell's equations

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\varepsilon_r\vec{E}$$

$$\begin{cases} \vec{\nabla}\times\vec{E} = -\dfrac{\partial\vec{B}}{\partial t} \\ \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\dfrac{\partial\vec{D}}{\partial t} \end{cases}$$

Using the identity $\vec{\nabla}\times\vec{\nabla}\times\vec{E} = \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E}$,

and assuming ε_r = constant, we have

$$\nabla^2\vec{E} - \frac{\varepsilon_r}{c^2}\frac{\partial^2\vec{E}}{\partial t^2} = \mu_0\frac{\partial\vec{J}}{\partial t}$$

The best way to solve this problem is by considering the system in the absence of <u>charges</u> or <u>currents</u>.

Of course, the plasma contains the information of the displacement current $\frac{\partial \vec{J}}{\partial t}$, but we can consider the plasma to be a dielectric medium of dielectric constant $\varepsilon(\omega,k) = \varepsilon(\omega_k)$.

Setting $\vec{J} = 0$, and using the decomposition $\vec{E}(\vec{r},t) = \sum_\omega e^{-i\omega t}\vec{E}(\vec{r})$, we obtain

Helmholtz's equation

$$\left(\vec{\nabla}^2\vec{E} + K_0^2\varepsilon_r\vec{E}\right) = 0, \quad K_0 = \frac{\omega}{c}$$ wavevector in <u>vacuum</u>!

We can now approach the solution by imposing boundary conditions at the interface.

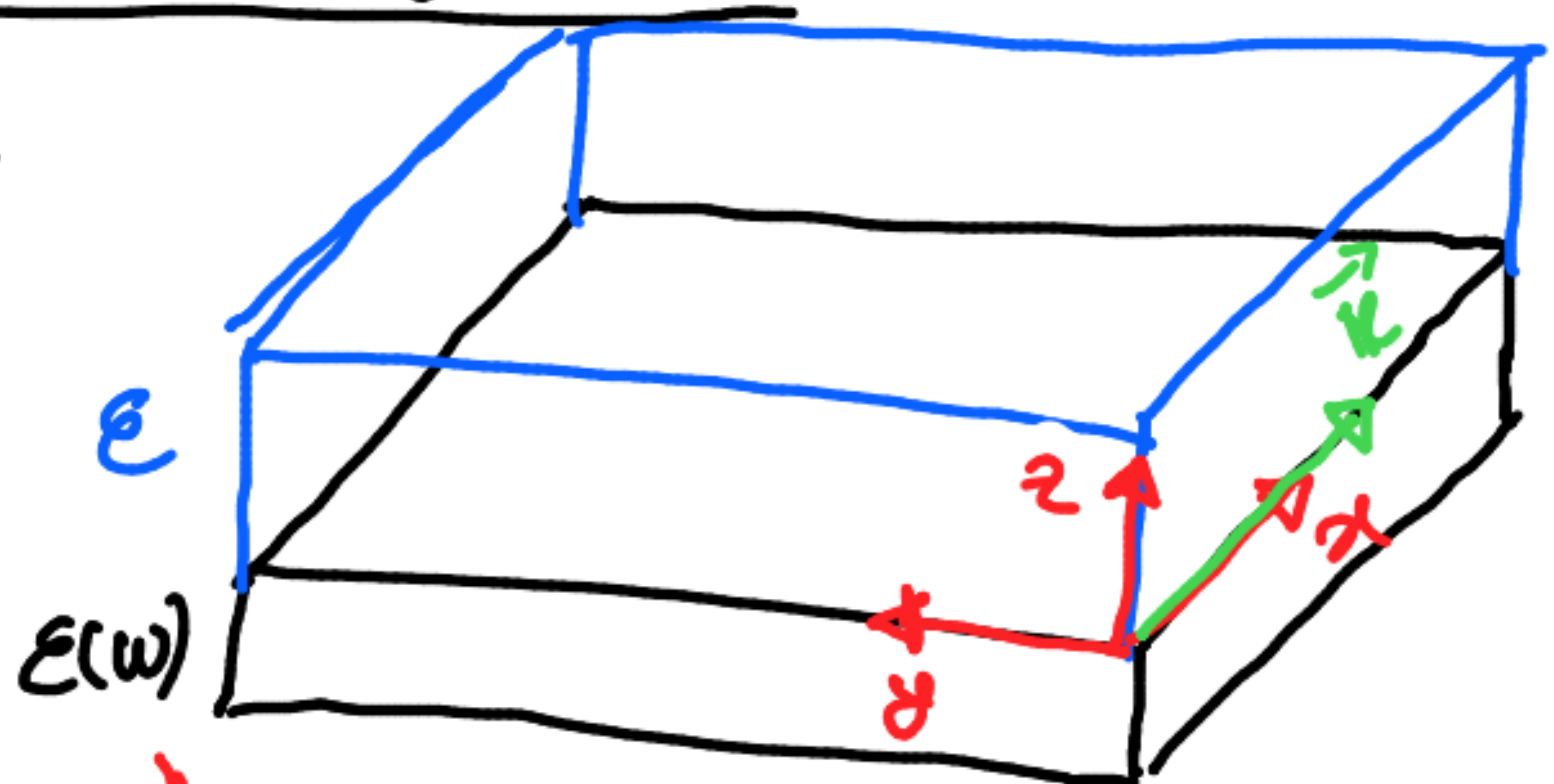
$$\varepsilon_r \begin{cases} \varepsilon, & \text{for the dielectric} \\ \varepsilon(\omega), & \text{for the plasma} \end{cases} = \varepsilon^*(z)$$

Definition of the propagation geometry

Let us consider a wave propagating along x,

$\vec{k} = k\,\vec{e_x}$

$\boxed{\varepsilon = \varepsilon(z)}$ * (discontinuos)



We may further split the field do

$\vec{E}(\vec{r}) = \sum_k e^{ikx} \vec{E}(z,y)$. For symplicity, let us assume <u>linear polarization</u>, $\vec{E} = \vec{E}(z)$

(notice, however, that $\vec{E}(z) = E_x(z)\vec{e_x} + E_y(z)\vec{e_y} + E_z(z)\vec{e_z}$, in principle)

$$\boxed{\frac{\partial^2 \vec{E}(z)}{\partial z^2} + \left(K_0^2 \varepsilon(z) - K^2\right)\vec{E}(z) = 0} \qquad (1)$$

Naturally, a similar equation exists for $\vec{B}$ (or $\vec{H} = \frac{\vec{B}}{\mu_0}$)

Eq. (1) is the starting point for the general analysis of guided EM modes. From the original Maxwell's equation, we could derive ($E_j = E_j(z)$)

$$\begin{cases} \dfrac{\partial E_z}{\partial y} - \dfrac{\partial E_y}{\partial z} = i\omega B_x \\[2ex] \dfrac{\partial E_x}{\partial z} - \dfrac{\partial E_z}{\partial x} = i\omega B_y \\[2ex] \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} = i\omega B_z \end{cases}
\qquad
\begin{cases} \dfrac{\partial B_z}{\partial y} - \dfrac{\partial B_y}{\partial z} = -\dfrac{i\omega}{c^2} \varepsilon E_x \\[2ex] \dfrac{\partial B_x}{\partial z} - \dfrac{\partial B_z}{\partial x} = -\dfrac{i\omega}{c^2} \varepsilon E_y \\[2ex] \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} = -\dfrac{i\omega}{c^2} \varepsilon E_z \end{cases}$$

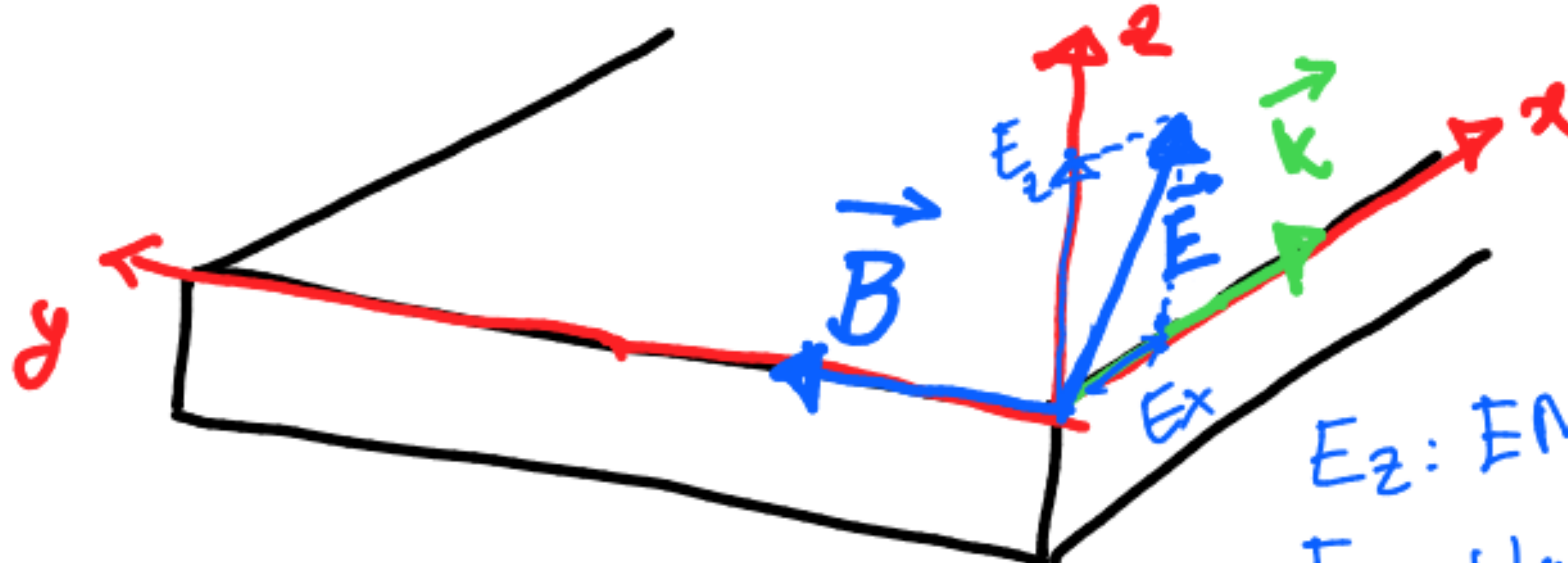
$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \times \vec{B} = +\frac{1}{c^2}\frac{\partial \vec{D}}{\partial t}$$

For propagation along x, $\frac{\partial}{\partial x} = ik$, $\frac{\partial}{\partial y} = 0$, we obtain

$$\begin{cases} \dfrac{\partial E_y}{\partial z} = -i\omega B_x \\ \dfrac{\partial E_x}{\partial z} - ikE_z = i\omega B_y \\ ikE_y = i\omega B_z \end{cases} \qquad \begin{cases} \dfrac{\partial B_y}{\partial z} = \dfrac{i\omega}{c^2} E_x \\ \dfrac{\partial B_x}{\partial z} - i\beta B_z = -\dfrac{i\omega}{c^2} E_y \\ i\beta B_y = -i\dfrac{\omega}{c^2} E_z \end{cases}$$

A) TM (or p) modes

"transverse magnetic"



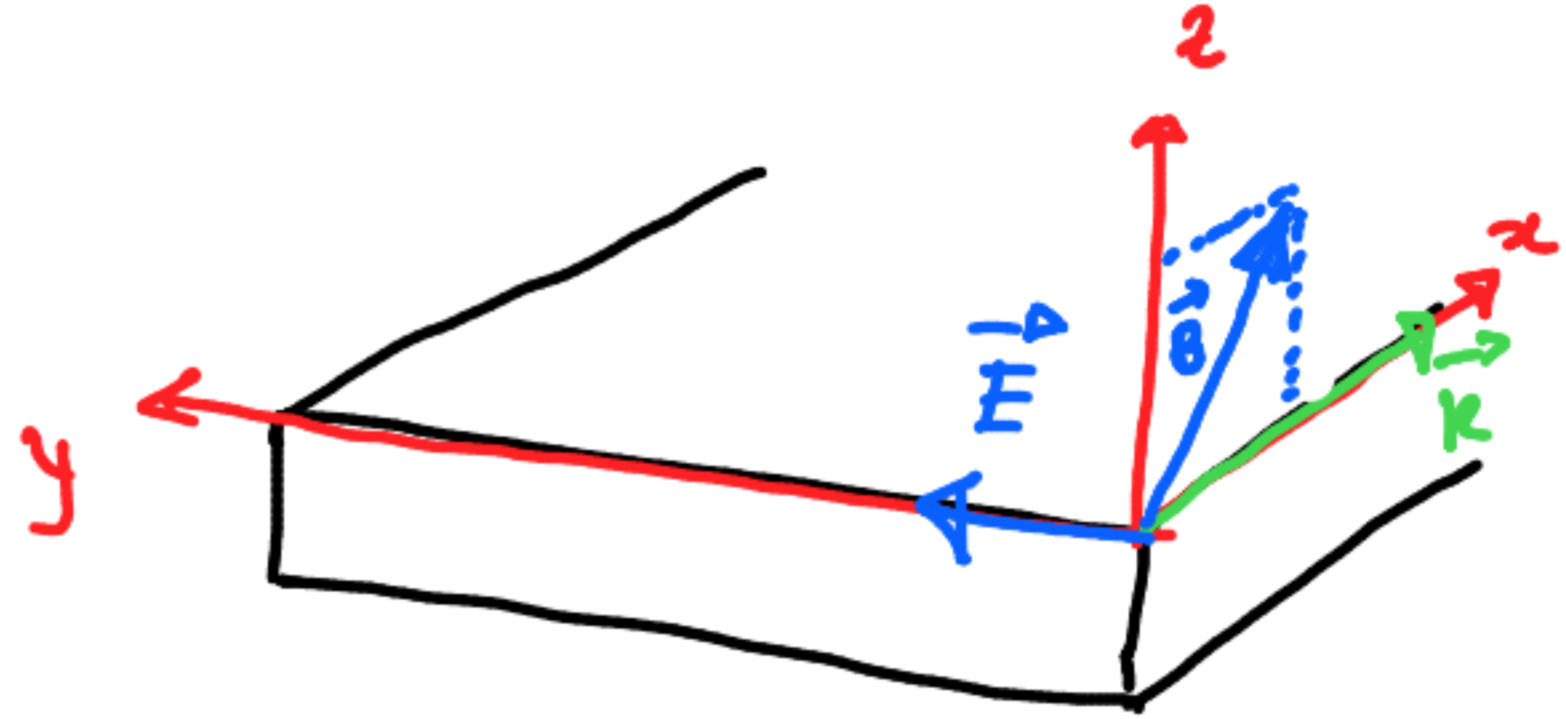
E_z: EM

E_x: plasmons

B) TE (or s) modes

"transverse electric"

Let us consider TM modes

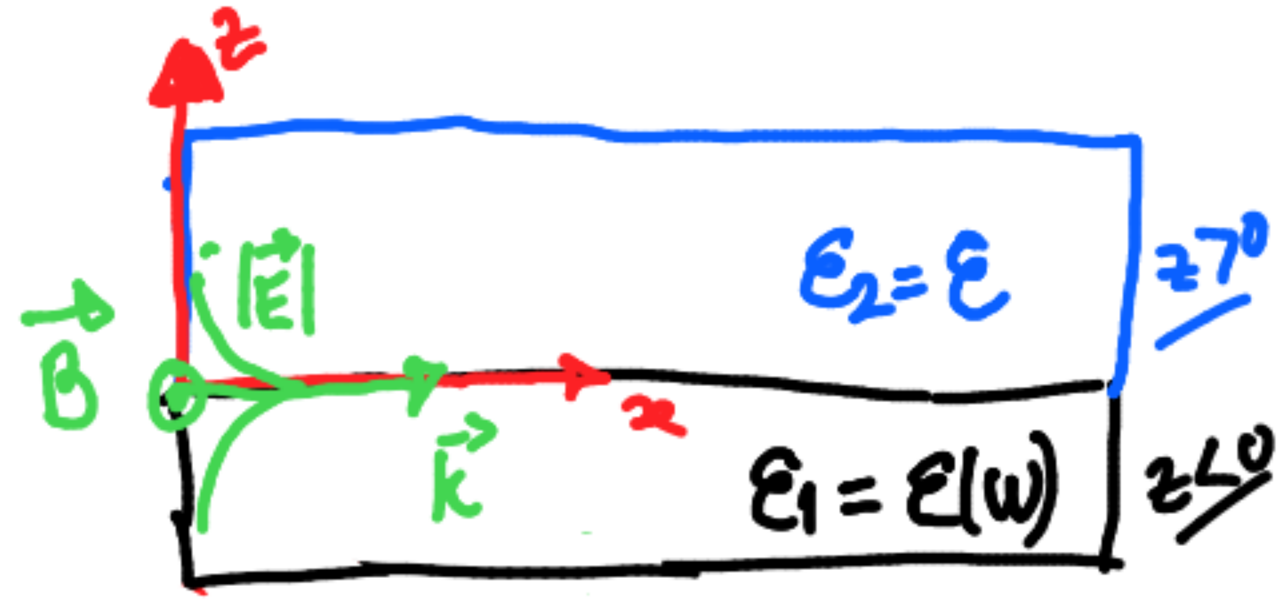


$$\begin{cases} E_x = -\dfrac{ic^2}{\omega\varepsilon}\dfrac{\partial B_y}{\partial z} \\ E_z = -\dfrac{kc^2}{\omega\varepsilon} B_y \end{cases}$$

The wave equation for TM modes then come

$$\frac{\partial^2 B_y}{\partial z^2} + (k_0^2\varepsilon - K^2)\, B_y = 0 \qquad (2)$$

Dielectric $\varepsilon_2 = \varepsilon$ $(\mathrm{Re}[\varepsilon_2] > 0)$



Plasma $\varepsilon_1 = \varepsilon(\omega)$ $(\mathrm{Re}[\varepsilon_1] \leq 0)$

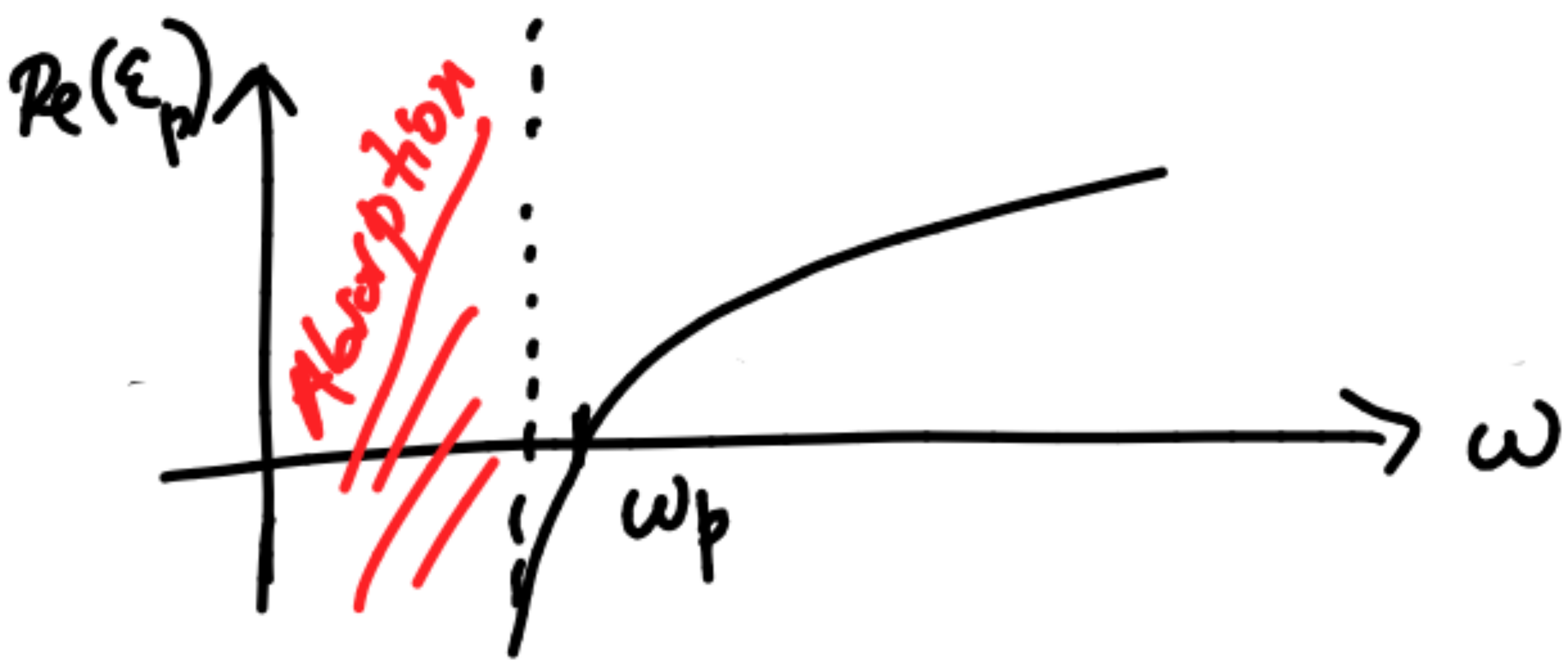
$$\varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega(\omega + i\gamma)} \simeq 1 - \frac{\omega_p^2}{\omega^2}$$

collisions

(Propagation geometry)

(We will come back to this point...)

$\boxed{\omega < \omega_p}$ absorption (evanescent waves in z)



We thus look for solution of the form of evanescent waves, $\sim e^{-k_i |z|}$, where k_i is the inverse of the absorption length, $k_i = \frac{1}{\lambda_i}$ $(i = 1, 2)$

* $z > 0$

$$\begin{cases} B_y(z) = A_2 \, e^{ikx} \, e^{-k_2 z} \\[2ex] E_x(z) = i A_2 \dfrac{c^2 k_2}{\omega \varepsilon_2} e^{ikx} \, e^{-k_2 z} \\[2ex] E_z(x) = -A_2 \dfrac{k c^2}{\omega \varepsilon_2} e^{ikx} \, e^{-k_2 z} \end{cases}$$

* $z < 0$

$$\begin{cases} B_y(z) = A_1 \, e^{ikx} \, e^{k_1 z} \\[2ex] E_x(z) = -i A_1 \dfrac{c^2}{\omega \varepsilon_1} k_1 e^{ikx} \, e^{k_1 z} \\[2ex] E_z(z) = -A_1 \dfrac{k c^2}{\omega \varepsilon_1} e^{ikx} \, e^{k_1 z} \end{cases}$$

Continuity conditions: At the interface,

$$\begin{cases} H_y^{(1)}(0) = H_y^{(2)}(0) \\ D_x^{(1)}(0) = D_x^{(2)}(0) \end{cases} \Rightarrow \begin{cases} A_1 = A_2 \\ \dfrac{k_1}{k_2} = -\dfrac{\varepsilon_1}{\varepsilon_2} \end{cases} \quad (3)$$

We add this subsidiary relations to the wave equation (2)

$$\frac{\partial^2 B_y^{(i)}}{\partial z^2} + (k_0^2 \varepsilon_i - k^2) B_y^{(i)} = 0$$

$$\Rightarrow \begin{cases} k_1^2 = k^2 - k_0^2 \varepsilon_1 \\ k_2^2 = k^2 - k_0^2 \varepsilon_2 \end{cases}$$

(2) + (3)

$$k = k_0 \sqrt{\frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}} = \frac{\omega}{c} \sqrt{\frac{\varepsilon(\omega)\,\varepsilon_2}{\varepsilon(\omega) + \varepsilon_2}} \qquad (4)$$

Solving for ω with the appropriate dielectric model for the plasma, $\varepsilon(\omega)$, we can eventually get the dispersion relation. But before we do that, let us examine what happens to the <u>TE modes</u>. If we repeat the procedure and impose continuity of the longitudinal components of the fields $\vec{D}$ and $\vec{H}$, we would get $\longrightarrow$ (next page)

$$\begin{cases} H_x^{(1)}(0) = H_x^{(2)}(0) \\ D_y^{(1)}(0) = D_y^{(2)}(0) \end{cases} \Longrightarrow A_1(K_1+K_2) = 0 .$$

Since confinement requires $\mathrm{Re}(K_i) > 0$, this condition implies that $A_1 = 0 = A_2$.

$\therefore$ Surface modes do not exist in the TE polarization!

(Only <u>TM - polarized</u> surface modes exist)

Surface plasmon polariton dispersion

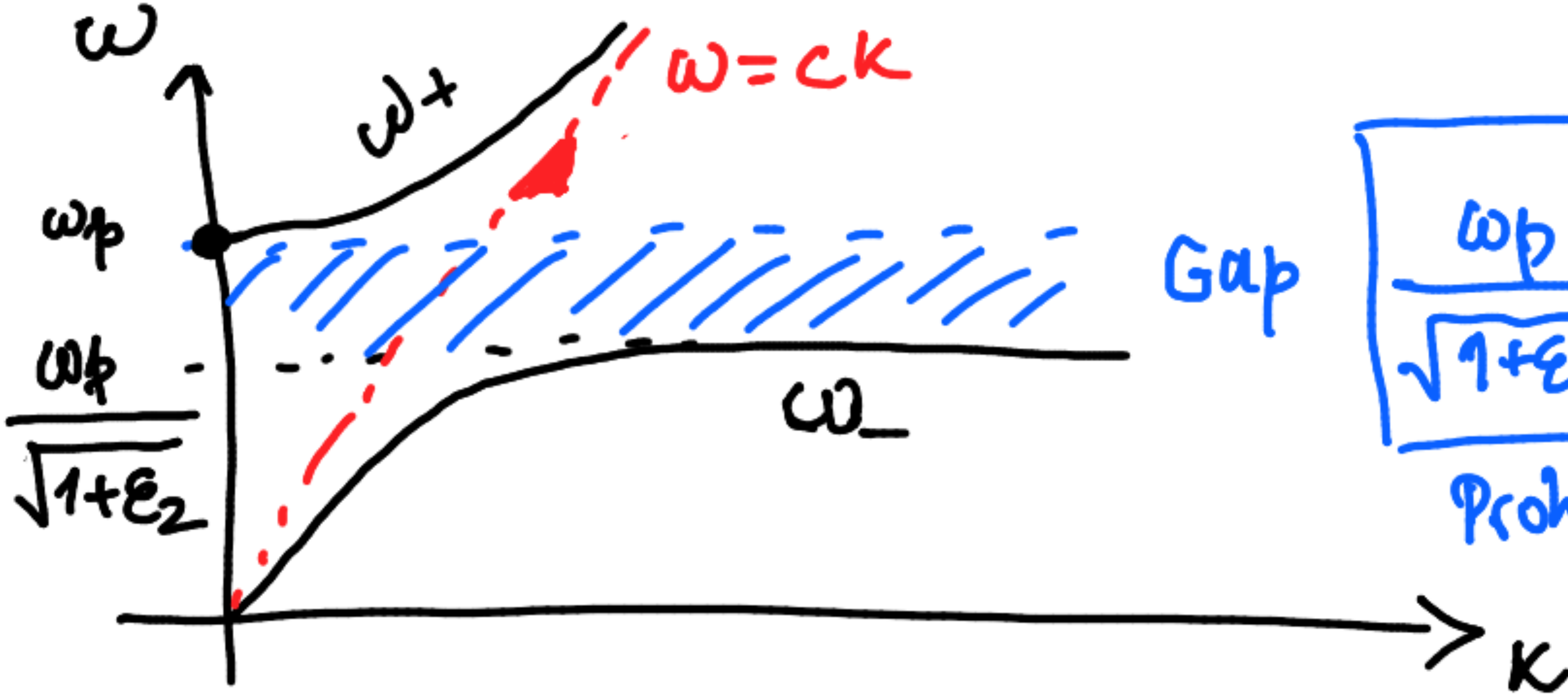
Let us assume local response of the plasma,

$$\varepsilon(\omega,k) = \varepsilon(\omega,0) = 1 - \frac{\omega_p^2}{\omega^2},$$

Otherwise, we have to take into account the full dispersion, $\varepsilon(\omega,k) = 1 - \frac{\omega_p^2}{\omega^2 - c^2k^2}$

Eq. (4) provides

$$k = \frac{\omega}{c}\sqrt{\frac{\varepsilon_2(1-\omega^2/\omega_p^2)}{\varepsilon_2+1-\omega^2/\omega_p^2}} \Rightarrow k^2c^2 = \frac{\varepsilon_2(\omega^2-\omega_p^2)}{(\varepsilon_2+1)-\frac{\omega^2}{\omega_p^2}}$$



$$\boxed{\frac{\omega_p}{\sqrt{1+\varepsilon_2}} < \omega \le \omega_p}$$

Prohibited gap
(no propagation)

- In the short wavelength limit, $kc \gg \omega_p$

$$\omega_- \simeq \frac{\omega_p}{\sqrt{1+\varepsilon_2}} \quad , \quad v_g = \frac{\partial \omega}{\partial k} \to 0 \; ! \quad \text{surface plasmon}$$

- In the long wavelength limit, $kc \ll \omega_p$

$$\omega_- \simeq \frac{ck}{\sqrt{\varepsilon_2}} \quad , \quad v_g = \frac{c}{\sqrt{\varepsilon_2}} \quad \text{surface EM wave}$$

The interest around surface plasmon polaritons (SPP) lies in the fact that the mixture between EM and plasmons allows for propagation of short wavelength (confined light) below the diffraction limit

In vacuum, $\omega = ck$, $\lambda_{vac.} = \frac{2\pi}{k} = \frac{2\pi c}{\omega}$

For $\omega \sim 2\pi \times 10^{14}$ Hz (infrared light), $\lambda_{vac} \simeq 1\,\mu m$

In interfaces, $\omega \to \frac{\omega_p}{\sqrt{1+\varepsilon_2}}$, $\lambda \to 0$

"Superluminal waves", $v_\varphi = \frac{\omega}{k} < \frac{\omega}{\omega_p} c = \sqrt{1+\varepsilon_2}\, c$!

2.6.1 – Surface modes in spherical plasmas

Let us now consider electrostatic oscillations in a metallic sphere.

In fact, the corresponding EM case (leading to the famous Mie Theory[1]) can be constructed from the electrostatic one. We will be interested in the case of electrostatic modes

[1] Bohren & Huffman, (1983)

We consider that inside the sphere we have a plasma. Outside we have a dielectric medium (e.x. air or vacuum).

$$\begin{cases} \dfrac{\partial n_1}{\partial t} + \vec{\nabla}\cdot(n_0 \vec{v}_1) = 0 \\[2ex] \dfrac{\partial \vec{v}_1}{\partial t} = +\dfrac{e}{m}\vec{\nabla}\phi_1 \cancel{- \dfrac{\vec{\nabla} p_1}{mn}} \end{cases}$$

cold plasma

$$\frac{\partial^2 n_1}{\partial t} + \vec{\nabla}\cdot\left[\frac{n_0 e}{m}\vec{\nabla}\phi_1\right] = 0$$

From Poisson's equation, $\nabla^2\phi_1 = \dfrac{e n_1}{\varepsilon_0}$

Fourier transform in time only:

$$-\omega^2 n_1 + \vec{\nabla}\cdot\left[\frac{e n_0}{m_e}\vec{\nabla}\phi_1\right] = 0$$

$$\Leftrightarrow -\omega^2 \frac{\varepsilon_0}{e}\nabla^2\phi_1 + \vec{\nabla}\cdot\left[\frac{e n_0}{m_e}\vec{\nabla}\phi_1\right] = 0$$

$$\Rightarrow \quad \boxed{\vec{\nabla}\cdot\left[\varepsilon(\omega)\vec{\nabla}\phi_1\right] = 0}$$

$$\begin{cases} \vec{\nabla}\cdot\left[\varepsilon(\omega)\vec{\nabla}\phi_1\right] = 0 & , r < R \\ \nabla^2 \varepsilon_2 \phi_1 = 0 & , r > R \end{cases}$$

So, the problem amounts to solving Laplace equation $\nabla^2\phi = 0$ in spherical coordinates

Laplace equation

In spherical coordinates,

$$\phi(r,\theta,\varphi) = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$$\nabla^2 \equiv \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\begin{cases} \dfrac{d^2\Phi}{d\varphi^2} + m^2\,\Phi = 0 \qquad (m = \text{integer}) \\[2ex] \dfrac{1}{\sin\theta}\dfrac{d}{d\theta}\left(\sin\theta\,\dfrac{d\Theta}{d\theta}\right) + \left[\ell(\ell+1) - \dfrac{m^2}{\sin^2\theta}\right]\Theta = 0 \\[2ex] \dfrac{d}{dr}\left(r^2\dfrac{dR}{dr}\right) - \ell(\ell+1)R = 0 \end{cases}$$

The first two equations give rise to the <u>Spherical Harmonics</u>

$$\Theta(\theta)\,\Phi(\varphi) = Y_\ell^m(\theta,\varphi) = e^{im\varphi}\,P_\ell^m(\cos\theta)$$

wher $P_\ell^m(\cos\theta)$ are the <u>Legendre Polynomials</u>

$$\int_{-1}^{1} P_\ell^m(x)\,P_{\ell'}^{m'}(x)\,dx = \delta_{\ell\ell'}\,\frac{2}{2\ell+1}\,\frac{(\ell+m)!}{(\ell-m)!}$$

<u>Radial Equation:</u>

$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} = \ell(\ell+1)R$$

<u>**Ansatz:**</u> $R(r) = \sum_n a_n r^{n+c}$

Plugging in the radial eq., we get:

$$\sum_n (n+c)(n+c-1)\, a_n r^{n+c} + 2\sum_n (n+c)\, a_n r^{n+c}$$

$$-\ell(\ell+1) \sum_n a_n r^{n+c} = 0$$

$$\Leftrightarrow \sum_n \left[(n+c)(n+c+1) - \ell(\ell+1)\right] a_n r^{n+c} = 0$$

$$c = \begin{cases} -(\ell+1)-n \\ \ell - n \end{cases}$$

So, we must add a coefficient for the $r^{-(\ell+1)}$ solution,

$$R(r) = \sum_\ell a_\ell r^\ell + \sum_\ell b_\ell r^{-(\ell+1)}$$

$$\begin{cases} R^{(1)}(r) = \sum_\ell a_\ell r^\ell & \Leftarrow \text{Regular at the origin} \\ R^{(2)}(r) = \sum_\ell b_\ell r^{-(\ell+1)} & \Leftarrow \text{decaying outside.} \end{cases}$$

Boundary conditions

$\phi_1(R) = \phi_2(R)$ (A) , $D_1(R) = D_2(R)$

$\Leftarrow \varepsilon_1 \phi_1'(R) = \varepsilon_2 \phi_2'(R)$ (B)

From A, we get $a_\ell R^\ell = b_\ell R^{-(\ell+1)} \Rightarrow b_\ell = a_\ell R^{2\ell+1}$ //.

From B, we get: $\varepsilon_1 \ell R^{\ell-1} = -\varepsilon_2(\ell+1) R^{-(\ell+2)} \dfrac{b_\ell}{a_\ell}$

$$\Leftrightarrow \varepsilon_1 \ell \cancel{R^{\ell-1}} = -\varepsilon_2(\ell+1)\cancel{R^{\ell-1}} \Leftrightarrow \varepsilon_1 = -\varepsilon_2 \frac{\ell+1}{\ell}$$

Using $\varepsilon_1 = \varepsilon(\omega) = 1 - \dfrac{\omega_p^2}{\omega^2}$, temos

$$1 - \frac{\omega_p^2}{\omega^2} = -\varepsilon_2\frac{(\ell+1)}{\ell} \Leftrightarrow \omega^2 - \omega_p^2 = -\varepsilon_2 \frac{(\ell+1)}{\ell}\omega^2$$

$$\Leftrightarrow \omega^2\left(1 + \frac{\varepsilon_2(\ell+1)}{\ell}\right) = \omega_p^2$$

$$\boxed{\omega = \omega_p \sqrt{\frac{\ell}{\ell + \varepsilon_2(\ell+1)}}}$$

In air, $\varepsilon_2 = 1$

$$\boxed{\omega = \omega_p \sqrt{\frac{\ell}{2\ell+1}}}$$

Mie modes

As one observes,

$$\boxed{\frac{\omega_p}{\sqrt{3}} < \omega < \frac{\omega_p}{\sqrt{2}}}$$

$\therefore$ The frequency lies between the fundamental (Mie) mode and the surface plasmon

$$\frac{\omega_p}{\sqrt{2}}.$$