TÉCNICO LISBOA

27th January 2022
Lecturers: F. Nabais, H. Terças

Advanced Plasma Physics

MEFT 2021/22

Problem Class 6

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. Quantum kinetic equations. Consider the Wigner transport equation governing the (quantum) phase-space density

$$f_e(\mathbf{r}, \mathbf{k}, t) = \int \psi(\mathbf{r} + \mathbf{s}/2, t)^* \psi(\mathbf{r} - \mathbf{s}/2, t) e^{i\mathbf{s}\cdot\mathbf{k}} d\mathbf{s},$$

where $\psi_e(\mathbf{r}, t)$ is the electron wave function. The particle momentum $\mathbf{k}$ is related to the particle velocity as $\mathbf{v} = \hbar\mathbf{k}/m_e$ (in what follows, neglect the motion of the ions, and set $n_i = n_0$, and consider the plasma to be three-dimensional (3D)).

a) The kinetic equation can be formally written as (drop the subscript e for notation simplicity)

$$\frac{\partial f}{\partial t} + \frac{\hbar}{m}\mathbf{k} \cdot \boldsymbol{\nabla} f - \frac{2}{\hbar} f \sin\left(\frac{1}{2} \overleftarrow{\boldsymbol{\nabla}}_{\mathbf{k}} \cdot \overrightarrow{\boldsymbol{\nabla}}\right) U = 0,$$

where the arrows indicate that the operators act on the left ($f(\mathbf{r}, \mathbf{k}, t)$) and on the right ($U(\mathbf{r}, t)$). Show that, in the semi-classical limit, we obtain the Vlasov equation. Comment on that.

b) We can further show (check the notes available at the webpage), that the Wigner equation can be recast in an equivalent form as

$$\frac{\partial f}{\partial t} + \frac{\hbar}{m}\mathbf{k} \cdot \boldsymbol{\nabla} f + \frac{i}{\hbar} \int e^{-i\mathbf{q}\cdot\mathbf{r}} \tilde{U} \left(f_- - f_+\right) d\mathbf{q} = 0,$$

where $\tilde{U} \equiv \tilde{U}(\mathbf{q}, t)$ is the Fourier transform of the potential energy and $f_\pm = f(\mathbf{r}, \mathbf{k} \pm \mathbf{q}/2, t)$ are the displaced Wigner functions. Linearize the kinetic equation, and make use of the usual definition

$$U(\mathbf{r}, t) = -\frac{e^2}{4\pi\epsilon_0} \int \frac{n_i(\mathbf{r}') - n_e(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \Rightarrow U_1(\mathbf{r}, t) = \frac{e^2}{4\pi\epsilon_0} \int \frac{n_1(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' = \frac{e^2}{4\pi\epsilon_0} \iint \frac{f_1(\mathbf{r}', \mathbf{k})}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' d\mathbf{k}$$

to obtain the quantum kinetic dispersion relation,

$$\epsilon(\omega, \mathbf{q}) = 1 - \frac{e^2 n_0}{\hbar \epsilon_0 q^2} \int \frac{g_0(\mathbf{k} - \mathbf{q}/2) - g_0(\mathbf{k} + \mathbf{q}/2)}{\omega - \hbar \frac{\mathbf{k} \cdot \mathbf{q}}{m_e}} d\mathbf{k}.$$

c) Make a change of variables and write the dispersion relation in a more amenable form,

$$\epsilon(\omega, \mathbf{q}) = 1 - \omega_p^2 \int \frac{g_0(\mathbf{k})}{\left(\omega - \frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m_e}\right)^2 - \frac{\hbar^2 q^4}{4 m_e^2}} d\mathbf{k}.$$

d) Consider a quantum degenerate plasma, described by the Fermi distribution in the $T \to 0$ limit,

$$g_0(\mathbf{k}) = \mathcal{N} \Theta(k_F - |\mathbf{k}|),$$

where k_F is the Fermi wavevector and $\mathcal{N}$ is a normalization. Consider that the plasma waves propagate along the $\hat{\mathbf{x}}$ direction. Show that the projected one-dimensional distribution is now given by

$$g_0(k_x) = \frac{3}{4k_F}\left(1 - \frac{k_x^2}{k_F^2}\right).$$

Comment the crucial difference that you find here in comparison with the equilibrium of classical plasmas.

e) Let us come back to velocity variables, $v = \hbar k_x / m_e$ and write the dispersion as

$$\epsilon(\omega, q) = 1 - \frac{3\omega_p^2}{4 v_F} \int_{-v_F}^{v_F} \frac{1 - \frac{v^2}{v_F^2}}{(\omega - vq)^2 + \hbar^2 q^4 / 4 m_e^2} dv.$$

Show that, in the long-wavelength limit $v \gg \omega_p / k$, the quantum Langmuir (plasmon) dispersion relation reads

$$\omega^2 \simeq \omega_p^2 + \frac{3}{5} v_F^2 q^2 + \frac{\hbar^2 q^4}{4 m_e^2},$$

where we have used that $\langle v^2 \rangle = v_F^2/5$. If you identify the latter with $v_{\rm th}^2$ in classical plasmas, you can identify a purely quantum contribution to the dispersion relation.

Problem 2. Quantum dark solitons. Another way to describe quantum plasmas is based on a Schrödinger-Poisson model (see the lecture notes). The basic equations governing the collective wavefuction $\psi(x,t)$ of a one-dimensional plasma are given by

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m_e}\frac{\partial^2\psi}{\partial x^2} - e\phi\psi + \frac{E_F}{n_0}|\psi|^4\psi,$$

$$\frac{\partial^2\phi}{\partial x^2} = \frac{e}{\epsilon_0}\left(|\psi|^2 - n_0\right),$$

where $E_F = \hbar^2(3\pi n_0)^{2/3}/m_e$ is the Fermi energy. We will look for soliton-like solutions, propagating with a constante velocity v_0.

a) Parametrize your wavefunction as $\psi(x,t) = U(\xi)e^{ik_0x - i\omega_0 t}$, where $\xi = x - v_0 t$ is the Lagrange coordinate. Separe the equation into its real and imaginary part to show that solutions are only possible if

$$\hbar\omega_0 = E_F - \frac{1}{2}m_e v_0^2.$$

Comment on the physical meaning of the quantity $\hbar\omega_0$ and interpret the previous result.

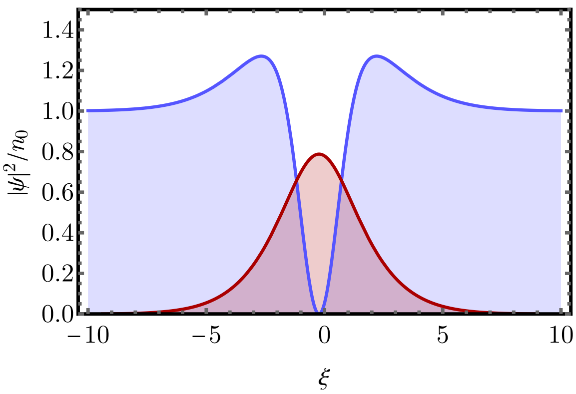
b) Show that the equation governing the envelope $U(\xi)$ and the electrostatic potential $\phi(\xi)$ are given by

$$\frac{\hbar^2}{2m}U'' + e\phi U + E_F U - \frac{E_F}{n_0^2}|U|^4 U = 0,$$

$$\phi'' = \frac{e}{\epsilon_0}\left(|U|^2 - n_0\right),$$

where $A'' = \dfrac{\partial^2 A}{\partial \xi^2}$ for any relevant physical quantity A.

c) Obtain the dimensionless form of the equation, by defining $\tilde{\xi} \equiv \xi/\lambda_F$, $\tilde{\phi} \equiv e\phi/E_F$ and $\tilde{U} \equiv U/\sqrt{n_0}$. Write a small Mathematica code to solve the equations above numerically with the boundary conditions $\tilde{U}(\pm\infty) = 1$ (i.e. the plasma is unperturbed away from the soliton), and $\tilde{U}'(0) = 0$ (of course, similar boundary conditions should be applied to $\tilde{\phi}$). You should obtain something like this,



where the blue line is the electron density $|\tilde{U}|^2$ and the red line is the electrostatic potential $\tilde{\phi}$.