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Advanced Plasma Physics

MEFT 2021/22

Problem Class 2

[Solutions]

Clearly present your approximations and enclose all pertinent calculations. Try to solve the problems yourself. Follow the instructions of the Lecturer.

Problem 1. Beam-plasma instability. Consider a cold, homogeneous plasma composed by ions and electrons, where the ions are at rest and the electrons are streaming with velocity $\mathbf{v}_0 = v_0\mathbf{e_x}$. Consider electrostatic perturbations only.

a) Discuss the form of the equilibrium functions $g_{0,e}(v)$ and $g_{0,i}(v)$ and show that the dielectric function for this problem reads

$$\epsilon(k,\omega) = 1 - \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{(\omega-\omega_0)^2}, \tag{1}$$

with $\omega_0 = kv_0$ being the streaming frequency.

From the linearized version of the Vlasov equation, we can show that the dieletric function reads,

$$\epsilon(k,\omega) = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2}{k}\int \frac{g_0'(v)}{\omega - kv}dv = 1 - \sum_\alpha \omega_{p\alpha}^2 \int \frac{g_0(v)}{(\omega-kv)^2}dv,$$

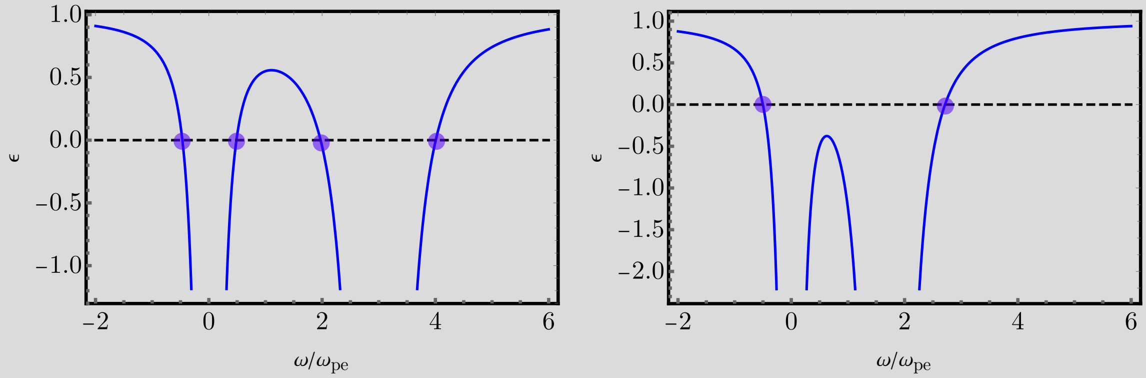
with the last term achieved via an integration by parts. Now, for the situation at hands, we should consider the following equilibria:

$$g_{0,i}(v) = \delta(v), \quad g_{0,e}(v) = \delta(v - v0).$$

By plugging in the dielectric function, the integration over the Dirac-delta functions easily yields the result we want.

b) The computation of the dispersion relation involves a fourth-order polynomial, for which we may expect four real roots. Plot $\epsilon(k,\omega)$ and observe that it only contains two real roots for $\omega_0 < \omega_c$, where ω_c is a certain critical value. Discuss with your colleagues how this relates to the onset of a dynamical instability in the plasma and determine the value of ω_c.

We start by plotting the dielectric function as a function of ω for distinct values of ω_0.



As we can observe, for sufficiently high values of ω_0 (left panel), the dielectric function contains four real roots (the system is stable). On the contrary, for smaller values of ω_0 (right panel), the dielectric function contains only two real roots. As such, the extra missing roots must be complex (the system is unstable). The critical points separating the two situations occurs at a critical streaming frequency $\omega_0 = \omega_c$, which may determine by imposing the following condition,

$$\left.\frac{\partial \epsilon}{\partial \omega}\right|_{\omega=\omega_c} = 0 \quad \Leftrightarrow \quad -\frac{\omega_{pi}^2}{\omega_{pe}^2} = \frac{\omega_c^3}{(\omega_c - \omega_0)^3} \quad \Leftrightarrow \quad \omega_c = \frac{(m_e/m_i)^{1/3}}{1+(m_e/m_i)^{1/3}}\omega_0 \simeq \left(\frac{m_e}{m_i}\right)^{1/3}\omega_0.$$

c) Show that the instability terminates at the cut-off wavevector k_c given by

$$k_c \simeq \frac{\omega_{pe}}{v_0}\left(1+\frac{1}{2}\left(\frac{m_e}{m_i}\right)^{1/3}\right).$$

What happens for modes $k > k_c$?

The answer to this question comes directly from the previous point. We now just need to evaluate for which k point the dielectric function touches zero at the critical point, i.e.

$$\epsilon(k_c, \omega_c) = 0 \quad \Leftrightarrow \quad 1 = \frac{\omega_{pi}^2}{\omega_c^2} + \frac{\omega_{pe}^2}{(\omega_c - \omega_0)^2}.$$

Making use of the result of the previous point, we get

$$k_c = \frac{\omega_{pe}}{v_0}\left[1+\left(\frac{m_e}{m_i}\right)^{1/3}\right]^{1/2} \simeq \frac{\omega_{pe}}{v_0}\left[1+\frac{1}{2}\left(\frac{m_e}{m_i}\right)^{1/3}\right].$$

d) Its is expected that the instability driven in the ion motion happens at a much slower scale than that of the streaming mode, i.e. $\omega \ll \omega_0$ (why?). So, we may look for the most unstable mode, $k_{\max}$, which maximizes the imaginary part of the frequency ($\omega_{i,\max} \equiv \max(\omega_i(k)) = \omega_i(k_{\max})$). Expand Eq. (1) and show that

$$\omega_{i,\max} \simeq \frac{\sqrt{3}}{2^{4/3}}\left(\frac{m_e}{m_i}\right)^{1/3}\omega_{pe}.$$

The most unstable mode k is the one that is resonant with the electrons in the beam, which oscillate at the plasma frequency, $kv_0 \simeq \omega_{pe}$. However, the frequency of this mode also involves the motion of the ion, so the resonant (most unstable mode) should satisfy $kv_0 = \omega_{pe} \gg \omega \sim \omega_{pi}$. As such, we expand the kinetic dispersion relation in powers of $\zeta \equiv \omega/\omega_{pe}$

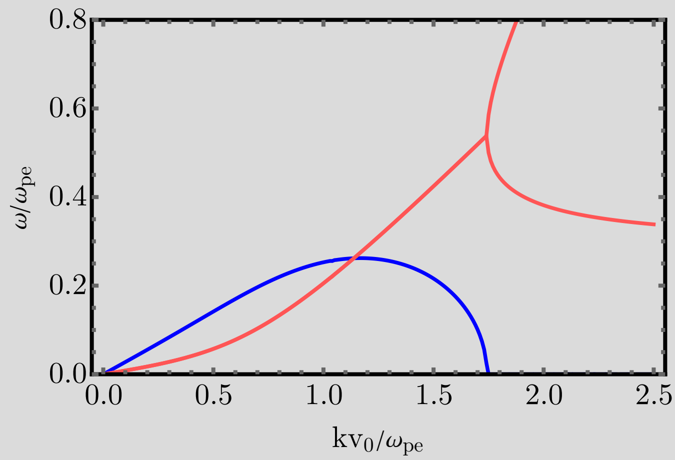
$$\epsilon(\omega_{pe}/v_0, \omega) = 0 \quad \Leftrightarrow \quad 1 = \frac{\omega_{pi}^2}{\omega^2} + \frac{1}{(\zeta - 1)^2} \quad \Leftrightarrow \quad 1 \simeq \frac{\omega_{pi}^2}{\omega^2} + 1 - 2\zeta.$$

Rearranging, we may write $2\omega^3 = -\omega_{pi}^2 \omega_{pe}$, which leads to

$$\omega = \left(-1, -\frac{\sqrt{3}}{2}i, \frac{\sqrt{3}}{2}i\right) \frac{1}{2^{1/3}} \left(\omega_{pi}^2 \omega_{pe}\right)^{1/3}.$$

The last root is imaginary and positive, which is associated with the growth rate of the dynamical instability under investigation. Making use of the relation $\omega_{pi}/\omega_{pe} = (m_e/m_i)^{1/2}$, we finally arrive at the correct result.

e) With the help of Mathematica, solve the kinetic dispersion relation numerically and obtain the $\omega_r(k)$ and $\omega_i(k)$ for a certain value of ω_{pi}/ω_{pe} (or, equivalently, for a certain mass ration m_i/m_e). Identify the features that you estimated analytically in the previous points. Discuss the results with your colleagues.



The plot depicted above represents the numerical solutions $\epsilon(k, \omega) = 0$, obtained with the help of Mathematica for the illustrative situation $\omega_{pi} = 0.1\omega_{pe}$. The imaginary part (blue line) agrees with the analytical estimates, as it indicates that the instability terminates at certain cut-off wavevector k_c. It also depicts a maximum value near the resonant mode $k_{\max} \simeq \omega_{pe}/v_0$, as argued in our estimations. Interestingly, we observe a bifurcation in the real part of the frequency (red line): in the stable regions, two modes exist (a fast e and a slow one), while in the unstable region ($k < k_c$), both modes coalesce. Physically, this means that both fast and slow processes participate in the development of the instability, so they grow together as a whole. Mathematically, this is a mere consequence of the fact that, in conservative systems, complex roots appear in the form of conjugated pairs. This is a typical feature in dynamical instabilities, going well beyond the scope of plasma physics.

Problem 2. The Krook collision integral. Assume that your plasma is sufficiently dense such that collisions start to become important. A way to take them into account is by adding a collision integral within the *relaxation-time (Krook) approximation* to the RHS of the Vlasov equation

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\boldsymbol{\nabla}\right) f_e + \frac{\mathbf{F}}{m_e}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_e = -\nu\left(f_e - f_{0,e}\right),$$

where $f_{0,e}(\boldsymbol{x},\boldsymbol{v},t)$ is the equilibrium distribution function, as usual. We neglect the effect of the ions, which are considered to be at rest.

a) Show that, at first order in $f_e - f_{0,e}$, we may write

$$f_e \approx f_{0,e} - \frac{1}{\nu}\left(\mathbf{v}\cdot\boldsymbol{\nabla} f_0 + \frac{\mathbf{F}}{m_e}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_0\right).$$

By trying to write f_e explicitly, we get

$$f_e = f_{0,e} - \frac{1}{\nu}\left(\frac{\partial f_e}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla} f_e + \frac{\mathbf{F}}{m_e}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_e\right).$$

Keeping things up to first order in $f_e - f_{0,e}$ means that the RHS of the equation must contain $f_{0,e}$ terms only. Assuming $f_0 \equiv f_{0,e}$ to describe equilibrium processes, the partial derivative in time rules out, therefore leading to stated result.

b) Consider that a constant electric field $\mathbf{E}$ is applied to a *homogeneous*, *unmagnetized* plasma. Use the previous result to derive Ohm's law,

$$\mathbf{J}_e = \sigma_e \mathbf{E} \tag{2}$$

where $\sigma_e = e^2 n_0/\nu m_e$ is the electron conductivity and n_0 is the plasma density. If the calculations were repeated in the presence of a transverse magnetic field ($\mathbf{B}\perp\mathbf{E}$), what kind of effect would Eq. (2) be describing (argue without calculations)?

Applying the definition, we have that $\mathbf{J}_e = -e\langle n_e v\rangle = -e\int f_e \mathbf{v} d\mathbf{v}$. As such, we may write

$$\mathbf{J}_e = -e\int\left[f_{0,e} - \frac{1}{\nu}\left(\mathbf{v}\cdot\boldsymbol{\nabla} f_0 + \frac{\mathbf{F}}{m_e}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_0\right)\right]\mathbf{v} d\mathbf{v}.$$

The first term vanishes, as the product $f_0(\mathbf{v})\mathbf{v}$ is an odd function. The same for the second term, considering the plasma to be homogeneous at equilibrium. Therefore,

$$\begin{aligned}\mathbf{J}_e = &-\frac{e^2}{\nu m_e}\int\left(\mathbf{E}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_0\right)\mathbf{v} d\mathbf{v}\\ &-\frac{e^2}{\nu m_e}\int\left(\mathbf{v}\cdot\boldsymbol{\nabla}_{\mathbf{v}} f_0\right) d\mathbf{v}\ \mathbf{E} \equiv \sigma_e\mathbf{E},\end{aligned}$$

where we have used the fact that $\mathbf{E} = \mathbf{E}(\mathbf{r}, t)$ (i.e. the electric field is not a function of the velocity). Finally, integrating by parts, and making use of the definition $f_0(\mathbf{v}) = n_0 g_0(\mathbf{v})$, we obtain the expression for the electric conductivity as stated. If the calculations were performed in the presence of an external magnetic field, σ_e would be given by a tensor quantity. The off-diagonal part of the conductivity would thus relate to the *Hall conductivity* of the electrons in the plasma.

c) Consider now the case of particle transport in such a collisional plasma. For that, neglect the electric field and assume that a temperature gradient $\boldsymbol{\nabla} T$ is present at the terminal of the plasma. You may expect that the system is no longer homogenous (think about the microscopic meaning of "temperature gradient"). Show that the particle current $\mathbf{J} = \mathbf{J}_e/e$ is given by Fick's Law,

$$\mathbf{J} = -\kappa \boldsymbol{\nabla} T, \quad \kappa = \frac{2n_0}{3\nu m_e} C_V,$$

where κ is the *heat conductivity* and $C_V = \partial\langle E\rangle/\partial T$ is the specific heat.

In this case, we make use of the particle current $\mathbf{J} = \langle n_e v\rangle = \int f_e \mathbf{v} d\mathbf{v}$. In the absence of external fields, the only term that survives is the last term

$$\begin{aligned}
\mathbf{J} &= \int \left(-\frac{1}{\nu}\mathbf{v} \cdot \boldsymbol{\nabla} f_0 \mathbf{v} d\mathbf{v}\right). \\
&= -\frac{1}{\nu} \int \left(\mathbf{v}\frac{\partial f_0}{\partial T} \cdot \mathbf{v} d\mathbf{v}\right) \boldsymbol{\nabla} T. \\
&= -\frac{2}{\nu m_e} \int \left(\frac{1}{2} m_e v^2 \frac{\partial f_0}{\partial T} d\mathbf{v}\right) \boldsymbol{\nabla} T \\
&= -\frac{2}{3\nu m_e} \int \left(\frac{1}{2} m_e v^2 \frac{\partial f_0}{\partial T} dv\right) \boldsymbol{\nabla} T.
\end{aligned}$$

In the last step, we have used the isotropy in space to convert the integral along the direction of $\mathbf{v}$ as $1/3$ of the integral along z, for example. Since v does not depend on T (f_0, however, does), we may write

$$\mathbf{J} = -\frac{2n_0}{3\nu m_e} \frac{\partial}{\partial T} \underbrace{\left(\int \frac{1}{2} m_e v^2 g_0 dv\right)}_{\langle E\rangle} \boldsymbol{\nabla} T.$$

The integral is nothing but the average kinetic (internal) energy of the system, and thus $\partial_T\langle E\rangle \equiv C_V$ is the final contribution to the coefficient.